IEOR 3106: Introduction to Operations Research: Stochastic Models

Second Midterm Exam, Chapters 5-6, November 10, 2011

This exam is open book, but only the Ross book and the CTMC notes.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will be posted.

1. The Qin-Lin Tatoo Parlor? (50 points)

Tim Qin and Yihua Lin have opened the Qin-Lin Tattoo Parlor near campus, specializing in probability tattoos, including images of Pascal's triangle, Galton's quincunx and, their best seller - Gauss's bell curve. The two proprietors each work on one customer at a time. There is a waiting room, which can accommodate two people in addition to the two in service. Here is a model: Potential customers come to the Qin-Lin Tattoo Parlor according to a Poisson process at constant rate of 2 per hour. The required service times are random, since some tattoos are much more complicated and ornate than others. The service times are independent random variables, each with an exponential distribution having a mean of 1 hour. Potential customers arriving when the waiting room is full (when there are two customers in service and two others waiting) leave without getting a tattoo, and without affecting future arrivals. Waiting customers have limited patience, so that will leave without receiving service if they have waited too long. The time each customer is willing to wait before starting service is an exponential random variables with mean $1/2$ hour.

long-run performance (25 points)

(a) Construct an appropriate stochastic model allowing you to determine the steady-state performance of the tatoo parlor. (10 points)

(b) What proportion of time are both proprietors simultaneously busy? (5 points)

(c) What is the expected steady-state number of customers in the tatoo parlor? (5 points)

(d) What is the long-run proportion of all potential arrivals (including ones that are blocked or abandon) that enter and are served? (5 points)

starting full or empty (25 points)

(e) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining time until the first of these customers completes service? (5 points)

(f) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining time until all four customers present at 7:00 pm are gone? (5 points)

(g) Suppose that the system starts empty. What is the probability that the first departure occurs before the second arrival? (5 points)

(h) Suppose that the system starts empty. What is the expected time until the first departure occurs? (5 points)

(i) Suppose that the system starts empty. Let $X(t)$ be the number of customers in the tatoo parlor at time t, including those that are waiting as well as those being served. Give an expression for the probability $P(X(2) = 3, X(5) = 0, X(9) = 4)$ (5 points)

2. The BS Insurance Company (50 points)

Emile Barraza and Dasmer Singh discovered that they both recently had a common experience; they had both only narrowly avoided serious injury from being hit by a speeding restaurant delivery man on a bicycle near campus. Seeing opportunity in these near disasters, Emile and Dasmer decide to start the Barraza-Singh (BS) Insurance Company to insure people against damages from being hit by a bicycle.

In order to carefully plan this new business venture, Emile and Dasmer recognize that it can be helpful to apply stochastic models to analyze the performance. They make up the following model: Since there will be a growing number of insured people, accidents resulting in claims occur according to a nonhomogeneous Poisson process at rate $\lambda(t) = 2t$ per month after the starting time, which is taken to be time 0. There is a random delay between the time each accident occurs and the payment is made. These claims not yet paid are said to be "outstanding claims." Suppose that these delays between occurrence of the accident and payment are independent and identically distributed random variables, uniformly distributed between 0 and 4 months. Also suppose that the dollar value of the claims are independent and identically distributed random variables, having a gamma distribution with mean 100 (in units of one thousand dollars) and variance 30,000. It is your lucky day; you get to help analyze this model.

accidents (20 points)

(a) What is the expected number of accidents occurring in the first 10 months? (5 points)

(b) Give an expression for the probability that 2 accidents occur during the first month and 4 accidents occur during the second month. (5 points)

(c) Let T_1 be the time of the first accident. What is $P(T_1 > 1 \quad \text{month})$? (4 points)

(d) Give an expression for the probability distribution of T_1. (4 points)

(e) Does the probability distribution of T_1 have the lack of memory property? (Explain.) (2 points)

total dollar values (15 points)

(f) Compute the expected total dollar value of all claims associated with accidents that occur during the first 10 months. (5 points)

(g) Compute the variance of the total dollar value of all claims associated with accidents that occur during the first 10 months. (5 points)

(h) What is the approximate probability that the total dollar value of all claims associated with accidents that occur during the first 10 months exceeds 14,000 (again in units of one thousand dollars)? (5 points)

outstanding claims (15 points)

(i) What is the mean number of outstanding claims at time 10? (5 points)

(j) What is the variance of the number of outstanding claims at time 10? (4 points)

(k) Is the number of outstanding claims at time 9 independent of the number of outstanding claims at time 10? Why? (3 points)

(l) Is the number of outstanding claims at time 10 independent of the number of paid claims at time 10? Why? (3 points)