

**IEOR 3106: Introduction to Operations Research: Stochastic Models**  
**SOLUTIONS to the Second Midterm Exam, November 10, 2011**

**This exam is open book, but only the Ross book and the CTMC notes.**

**You need to show your work. Briefly explain your reasoning.**

**Honor Code:** Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given nor received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will be posted.

**1. The Qin-Lin Tatoo Parlor? (50 points)**

Tim Qin and Yihua Lin have opened the Qin-Lin Tatoo Parlor near campus, specializing in probability tattoos, including images of Pascal's triangle, Galton's quincunx and, their best seller - Gauss's bell curve. The two proprietors each work on one customer at a time. There is a waiting room, which can accommodate two people in addition to the two in service. Here is a model: Potential customers come to the Qin-Lin Tatoo Parlor according to a Poisson process at constant rate of 2 per hour. The required service times are random, since some tattoos are much more complicated and ornate than others. The service times are independent random variables, each with an exponential distribution having a mean of 1 hour. Potential customers arriving when the waiting room is full (when there are two customers in service and two others waiting) leave without getting a tattoo, and without affecting future arrivals. Waiting customers have limited patience, so that will leave without receiving service if they have waited too long. The time each customer is willing to wait before starting service is an exponential random variables with mean 1/2 hour.

**long-run performance (25 points)**

(a) Construct an appropriate stochastic model allowing you to determine the steady-state performance of the tatoo parlor. (10 points)

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The appropriate model is a birth-and-death process, which is a special form of continuous-time Markov chain, as discussed in Chapter 6 of Ross and in §5 of the CTMC notes. The model is constructed by specifying the transition rates. Let  $\lambda = 2$  be the arrival rate,  $\mu = 1$  be the individual service rate and  $\theta = 2$  be the individual customer abandonment rate from queue. (The rate of an exponential random variable is the reciprocal of the mean.) Then the the birth rates are  $\lambda_i = \lambda = 2$  per hour for all  $i$ , while the death rates are  $\mu_1 = \mu = 1$ ,  $\mu_2 = 2\mu = 2$ ,  $\mu_3 = 2\mu + \theta = 4$  and  $\mu_4 = 2\mu + 2\theta = 6$ . These birth and death rates determine the instantaneous transition rate matrix  $Q$ , which in turn specifies the model. The birth and death rates are given in the rate diagram in Figure 1 below.

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(b) What proportion of time are both proprietors simultaneously busy? (5 points)

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Let  $\alpha_j = \lim_{t \rightarrow \infty} P(X(t) = j)$ . Then, for this BD process with 5 states, we have the basic formula

$$\alpha_j = \frac{r_j}{\sum_{k=0}^{k=4} r_k},$$

## Rate Diagram for a Birth-and-Death Process

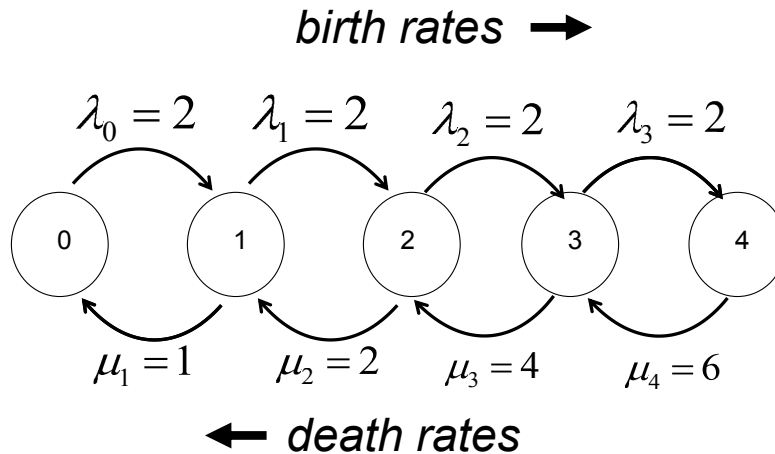


Figure 1: A rate diagram showing the transition rates for the birth-and-death process.

where  $r_0 = 1$  and  $r_j = r_{j-1}\lambda_{j-1}/\mu_j$ . Hence

$$r_0 = 1, \quad r_1 = 2, \quad r_2 = 2, \quad r_3 = 1, \quad r_4 = 1/3.$$

Hence,  $r_0 + r_1 + r_2 + r_3 + r_4 = 19/3$ ,  $\alpha_0 = 3/19$ , and

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3/19, 6/19, 6/19, 3/19, 1/19).$$

Hence, the long-run proportion of time both proprietors are busy in the long run is

$$\alpha_2 + \alpha_3 + \alpha_4 = 10/19.$$

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(c) What is the expected steady-state number of customers in the tattoo parlor? (5 points)

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$$0\alpha_0 + 1\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 31/19.$$


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(d) What is the long-run proportion of all potential arrivals (including ones that are blocked or abandon) that enter and are served? (5 points)

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The answer is the long-run service rate divided by the long-run arrival rate  $\lambda + 2$ . We find the service rate by conditioning on the steady state number in system and multiplying by the service rate in that state. The answer is

$$\frac{\mu\alpha_1 + 2\mu(\alpha_2 + \alpha_3 + \alpha_4)}{\lambda} = \frac{26/19}{2} = \frac{26}{38} = \frac{13}{19}.$$

**starting full or empty (25 points)**

(e) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining time until the first of these customers completes service? (5 points)

We apply the lack of memory property to see that the history does not matter. The expected remaining time for each customer in service is the mean service time 1. Since two customers are in service, the expected time until the first finishes is 1/2 hour. The two initially in service will definitely be served, because abandonment only occurs from waiting customers.

(f) Suppose that new arrivals are not admitted after 7:00 pm, but otherwise the system operates as described above. Suppose that the GM Tattoo Parlor is full at 7:00 pm. What is the expected remaining time until all four customers present at 7:00 pm are gone? (5 points)

This is like the visit to the post office. The time for all students to be gone is the sum of four independent exponential random variables. However, we must consider abandonments as well as service completions. But the rate in each successive state: 4, 3, 2 and 1 are given in the rate diagram. The overall mean time until all four customers are gone is thus

$$\text{Mean time} = \frac{1}{6} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} = \frac{23}{12} \text{ hours.}$$

(g) Suppose that the system starts empty. What is the probability that the first departure occurs before the second arrival? (5 points)

The first arrival has to occur before either of those two events. After the first arrival, the rate of the next arrival is 2, while the rate of the next service completion is 1. The probability that the departure occurs first is

$$\frac{1}{1+2} = \frac{1}{3}.$$

(h) Suppose that the system starts empty. What is the expected time until the first departure occurs? (5 points)

I was thinking that here “departure” refers to a service completion, not counting an abandonment. If abandonments are allowed, then the problem becomes even more complicated, because it is possible that an abandonment occurs before any service completion.

Even interpreting that “departure” means only service completion, this is tricky. First, we can add the expected time until the next event after the first arrival occurs to the expected time until the first arrival. There are two cases: The second arrival occurs before the first departure with probability  $2/3$ . If that event occurs, then two servers are busy and the departure rate becomes faster. Otherwise, there is no extra time to wait.

$$\begin{aligned}
 \text{mean time until first departure} &= \text{mean time until first arrival} \\
 &\quad + \text{mean time until next event (arrival or departure)} \\
 &\quad + \text{prob}(\text{next event departure}) \times 0 \\
 &\quad + \text{prob}(\text{next event arrival}) \times (\text{mean time until one of two depart}) \\
 &= \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{3}\right) 0 + \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) = \frac{7}{6}.
 \end{aligned}$$

(i) Suppose that the system starts empty. Let  $X(t)$  be the number of customers in the tattoo parlor at time  $t$ , including those that are waiting as well as those being served. Give an expression for the probability  $P(X(2) = 3, X(5) = 0, X(9) = 4)$ . (5 points)

We use expressions for the transient behavior of a CTMC, as in (2.8) of the CTMC notes, namely

$$P(X(2) = 3, X(5) = 0, X(9) = 4) = P_{0,3}(2)P_{3,0}(3)P_{0,4}(4),$$

where  $P_{i,j}(t)$  is the  $(i, j)$ <sup>th</sup> entry (from row  $i$  and column  $j$ ) of the transition matrix  $P(t) \equiv (P_{i,j}(t))$ , which can be represented as a matrix exponential,

$$P(t) = e^{Qt} \equiv \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!},$$

as indicated in Theorem 3.2 of the CTMC notes, where  $Q$  is the rate matrix associated with the birth-and-death process, as indicated in §3.2 of the CTMC notes. Alternatively, the transition function  $P(t)$  can be obtained by solving the matrix ordinary differential equation  $\dot{P}(t) = QP(t)$  with  $P(0) = I$ , where  $I$  is the identity matrix. Here  $Q_{i,i+1} = \lambda_i$ ,  $Q_{i,i-1} = \mu_i$ ,  $Q_{i,i} = -(Q_{i,i+1} + Q_{i,i-1}) = -(\lambda_i + \mu_i)$ , with all other entries being 0.

## 2. The BS Insurance Company (50 points)

Emile Barraza and Dasmer Singh discovered that they both recently had a common experience; they had both only narrowly avoided serious injury from being hit by a speeding restaurant delivery man on a bicycle near campus. Seeing opportunity in these near disasters, Emile and Dasmer decide to start the Barraza-Singh (BS) Insurance Company to insure people against damages from being hit by a bicycle.

In order to carefully plan this new business venture, Emile and Dasmer recognize that it can be helpful to apply stochastic models to analyze the performance. They make up the following model: Since there will be a growing number of insured people, accidents resulting in claims occur according to a nonhomogeneous Poisson process at rate  $\lambda(t) = 2t$  per month after the starting time, which is taken to be time 0. There is a random delay between the time each accident occurs and the payment is made. These claims not yet paid are said to be “outstanding claims.” Suppose that these delays between occurrence of the accident and

payment are independent and identically distributed random variables, uniformly distributed between 0 and 4 months. Also suppose that the dollar value of the claims are independent and identically distributed random variables, having a gamma distribution with mean 100 (in units of one thousand dollars) and variance 30,000. It is your lucky day; you get to help analyze this model.

**accidents (20 points)**

(a) What is the expected number of accidents occurring in the first 10 months? (5 points)

Let  $N(t)$  be the number of accidents in the interval  $[0, t]$ . The stochastic process  $\{N(t) : t \geq 0\}$  is a nonhomogeneous Poisson process with arrival rate  $\lambda(t) = 2t$ , as discussed in §5.4.1 of the book. The expected number of accidents in the interval  $[0, t]$  is

$$E[N(t)] \equiv m(t) = \int_0^t \lambda(s) ds.$$

In our case, we want

$$E[N(10)] \equiv m(10) = \int_0^{10} 2s ds = 100.$$

(b) Give an expression for the probability that 2 accidents occur during the first month and 4 accidents occur during the second month. (5 points)

These are independent Poisson random variables with mean  $m(1) - m(0) = m(1) = 1$  and  $m(2) - m(1) = 4 - 1 = 3$ . Hence,

$$P(N(1) = 2, N(2) - N(1) = 4) = \left( \frac{e^{-1}1^2}{2!} \right) \left( \frac{e^{-3}3^4}{4!} \right)$$

(c) Let  $T_1$  be the time of the first accident. What is  $P(T_1 > 1 \text{ month})$ ? (4 points)

Let  $T_1$  be the time of the first accident. For any  $t \geq 0$ ,

$$P(T_1 > t) = P(N(t) = 0) = \frac{e^{-m(t)}m(t)^0}{0!} = e^{-m(t)},$$

where

$$m(t) = \int_0^t \lambda(s) ds = \int_0^t 2s ds = t^2.$$

Hence

$$P(T_1 > t) = e^{-t^2} \quad \text{and} \quad P(T_1 > 1) = e^{-1}.$$

(d) Give an expression for the probability distribution of  $T_1$ . (4 points)

This was worked out in part (a). We give the cdf.

$$P(T_1 \leq t) = 1 - P(T_1 > t) = 1 - e^{-t^2}, \quad t \geq 0.$$

Because  $t^2$  appears instead of  $t$  in the exponent, this distribution is *not* exponential.

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(e) Does the probability distribution of  $T_1$  have the lack of memory property? (Explain.) (2 points)

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No, because the distribution of  $T_1$  is *not* exponential. That is so because  $-t^2$  appears in the exponent of the exponential instead of simply  $-t$  or  $-ct$  for some positive constant  $c$ . The exponential distribution is the *only* probability distribution with the lack of memory property. The lack of memory property states that

$$P(T > a + b | T > a) = P(T > b)$$

for all  $a > 0$  and  $b > 0$  or, equivalently,

$$P(T > a + b) = P(T > a)P(T > b).$$

Here that would be claiming that

$$e^{-(a+b)^2} = e^{-a^2}e^{-b^2},$$

but, by a basic property of the exponential function,

$$e^{-a^2}e^{-b^2} = e^{-(a^2+b^2)}.$$

In order for these two functions to be equal, the exponents would have to be equal. We would need to have

$$(a + b)^2 = a^2 + b^2 \quad \text{for all } a > 0, b > 0,$$

but that clearly fails.

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**total dollar values (15 points)**

(f) Compute the expected total dollar value of all claims associated with accidents that occur during the first 10 months. (5 points)

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We total dollar value of all claims over all accidents that have occurred in the interval  $[0, t]$  is a compound Poisson process. It can be represented as a random sum

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where  $Y_i$  is the random dollar value of the claim associated with the  $i^{\text{th}}$  accident. The compound Poisson process is discussed in §5.4.2 of the book. Here we have a nonhomogeneous Poisson process. In general,

$$E[X(t)] = m(t)E[Y_1], \quad \text{where } m(t) = E[N(t)]$$

Here then  $E[N(10)] = 10^2 = 100$  and

$$E[X(10)] = 100E[Y_1] = 100 \times 100 = 10,000 \quad (\text{in units of thousands of dollars})$$

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(g) Compute the variance of the total dollar value of all claims associated with accidents that occur during the first 10 months. (5 points)

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The corresponding general formula for the variance is

$$\text{Var}[X(t)] = m(t)E[(Y_1)^2], \quad \text{where } m(t) = E[N(t)]$$

As above,  $E[N(10)] = 10^2 = 100$ . Here we need the second moment of  $Y_1$ . We use the formula

$$E[(Y_1)^2] = \text{Var}(Y_1) + (E[Y_1])^2 = 30,000 + (100)^2 = 40,000$$

which implies that

$$\text{Var}[X(10)] = 100E[(Y_1)^2] = 100 \times 40,000 = 4,000,000$$

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(h) What is the approximate probability that the total dollar value of all claims associated with accidents that occur during the first 10 months exceeds 14,000 (again in units of one thousand dollars)? (5 points)

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We find the approximate probability by applying a normal approximation. As often is the case, it is justified by the central limit theorem. The important property of the stochastic process  $\{X(t) : t \geq 0\}$  is that it has stationary independent increments. That explains why the distribution of  $X(t)$ , suitably normalized, approaches the normal distribution as  $t \rightarrow \infty$ .

We thus need the mean and the standard deviation. The mean has been determined in part (f). Note that the standard deviation is easy to compute from the variance computed in part (g), i.e.

$$SD(X(10)) = \sqrt{\text{Var}(X(10))} = \sqrt{4,000,000} = 2,000 \quad (\text{in units of thousands of dollars})$$

Hence, we have the usual normal approximation,

$$\begin{aligned} P(X(10) > 14,000) &= P\left(\frac{X(10) - E[X(10)]}{SD(X(10))} > \frac{14,000 - E[X(10)]}{SD(X(10))}\right) \\ &= P\left(\frac{X(10) - E[X(10)]}{SD(X(10))} > \frac{14,000 - 10,000}{2,000}\right) \\ &= P\left(\frac{X(10) - E[X(10)]}{SD(X(10))} > 2.0\right) \\ &\approx P(N(0, 1) > 2.0) = 1 - 0.9772 = 0.223 \end{aligned}$$

using the table on page 82 of the book.

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**outstanding claims (15 points)**

(i) What is the mean number of outstanding claims at time 10? (5 points)

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This part of the problem is like the question about the Columbia Space Company on the 2005 exam, discussed in class. The process describing the number of outstanding claims here

has the structure of the number of busy servers in an infinite-server queue with a nonhomogeneous Poisson arrival process. Here the mean number of outstanding claims has the general formula

$$m(t) = \int_0^t \lambda(s)G^c(t-s) ds,$$

where  $\lambda(s)$  is the arrival rate at time  $s$  and  $G$  is the service-time cdf, while  $G^c(x) \equiv 1 - G(x)$ . Here, because the service time distribution is uniform on the interval  $[0, 4]$ , we have

$$G^c(s) \equiv 1 - G(s) = 1 - \frac{s}{4}, \quad 0 \leq s \leq 4, \quad \text{and} \quad G^c(s) = 0, \quad s > 4.$$

As a consequence, we have

$$\begin{aligned} m(10) &= \int_6^{10} 2s \left(1 - \frac{10-s}{4}\right) ds \\ &= \int_6^{10} 2s \left(\frac{s-6}{4}\right) ds \\ &= \frac{1}{2} \int_6^{10} (s^2 - 6s) ds \\ &= \frac{1}{2} \left(\frac{1000}{3} - 300 - \frac{216}{3} + 108\right) = \frac{104}{3} \end{aligned}$$

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(j) What is the variance of the number of outstanding claims at time 10? (4 points)

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Since the number of outstanding claims has a Poisson distribution, the variance equals the mean. The variance equals the mean computed in part ((i)).

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(k) Is the number of outstanding claims at time 9 independent of the number of outstanding claims at time 10? Why? (3 points)

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No, these two random variables are dependent. Observe that some of the claims outstanding at time 9 can also be outstanding at time 10. Even though the distribution of the number of outstanding claims at time  $t$  is Poisson for each  $t$  (used in part (j)), the process as a function of  $t$  is not a Poisson process. In particular, it need not have independent increments. One way to quickly see that the process is not a Poisson process is to observe that the number of outstanding claims need not be nondecreasing, as is required of the sample paths of a Poisson process.

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(l) Is the number of outstanding claims at time 10 independent of the number of paid claims at time 10? Why? (3 points)

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Yes, these random variables *are* independent, surprising as it may seem. This primarily follows from the Poisson thinning property. This specific property is discussed in the paper, "The Physics of The Mt/G/infty Queue," discussed in class on October 25. The independence can be understood by using the representation in terms of a Poisson measure in the plane,



with arrivals on the horizontal  $x$  axis, and the random service times as points in the vertical  $y$  axis above these arrival times. The number of claims outstanding falls in the region above the  $45^\circ$  line in Figure 1 there, whereas, the number of paid claims falls in the region below that line. Thus these are disjoint sets in the Poisson measure representation. Disjoint sets are independent under a Poisson random measure.

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