IEOR 3106: First Midterm Exam, Chapters 1-4, October 4, 2012

There are 3 problems, each with multiple parts.

This exam is open book, but only the Ross textbook.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.

1. Testing for a Disease (35 points)

A laboratory blood test is 90% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 10% of the healthy persons tested. That is, if a healthy person is tested, then the test result will incorrectly indicate that they have the disease with probability 0.10. Suppose that 1% of the population has the disease.

(a) What is the probability that a person who is selected at random from the population and tested for the disease tests positive for the disease? (5 points)

(b) What is the probability that a person who is selected at random from the population and tests positive actually has the disease? (10 points)

(c) Suppose that 1000 people are selected at random to be tested on a given day. Suppose that we plan to carefully examine all people who test positive. What are the mean and variance of the number of people that test positive and need to be carefully examined? (You need not do the final exact calculation of the variance.) (7 points)

(d) In the setting of part (c), what is the *approximate* probability that the number of these people that test positive and need to be carefully examined exceeds 125? (Give a number as well as a formula.) (10 points)

(e) Briefly explain why your approximation in part (d) is justified. (3 points)

2. The Random King (30 points)

A king (chess piece) is placed on one of the corner squares of an empty chessboard (having $8 \times 8 = 64$ squares) and then it is allowed to make a sequence of random moves, taking each of its legal moves in each step with equal probability, independent of the history of its moves up to that time. (Recall that the king can move one square in any direction, horizontally, vertically or diagonally, provided of course that it ends up at one of the other squares on the board. Thus, the king has 3 legal moves from each corner, but 8 legal moves from a square away from an edge of the board.)

(a) What is the probability that the king is back on its initial corner square after two moves? (5 points)

(b) Let $X(n)$ be a random variable indicating the square occupied by the king after n moves. Is the stochastic process $\{X(n) : n \geq 1\}$ a Markov chain? If so, is it an irreducible Markov chain? If so, is it a periodic irreducible Markov chain? (5 points)

(c) What is the long-run proportion of times that the king is on its initial square? Explain?

(10 points)

(d) What is the expected number of moves until the king first returns to its initial square? (5 points)

(e) Show how to compute (but not do not actually do so) the probability that the king reaches the other corner square on its initial row (after several moves) before it reaches any of the other corner squares (not counting the initial corner). Identify what appears in your formula. (5 points)

3. A Finite Markov Chain (35 points)

Consider a Markov chain on the eight states $\{1, 2, \cdots, 8\}$ with transition matrix P given by

$$P = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \begin{pmatrix} 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.3 & 0.0 & 0.1 & 0.1 & 0.2 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.1 & 0.2 & 0.0 & 0.0 & 0.2 & 0.2 & 0.3 \\ 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \end{pmatrix}$$

**Note that we are numbering the rows in the natural order 1, 2, ... , 8, with the columns labeled the same as the rows.

(a) Which states are accessible from state 1? (2 points)

(b) Which states are accessible from state 2? (2 points)

(c) Put the transition matrix in canonical form (showing the original states in their new positions). (5 points)

(d) Identify the open and closed communication classes for this Markov chain. Which states are transient? Which states are recurrent? (5 points)

(e) Compute the two-step transition probability $P^{(2)}_{1,6}$ (2 points)

(f) Compute the two-step transition probability $P^{(2)}_{1,3}$ (2 points)

(g) Compute the four-step transition probability $P^{(4)}_{4,3}$ (2 points)

(h) Starting in state 1, what is the long-proportion of moves spent in state 1? (3 points)

(i) Starting in state 4, what is the long-proportion of moves spent in state 4? (4 points)

(j) Starting in state 2, what is the expected total number of visits to state 5? (4 points)

(k) What is the approximate value of $P^{(25)}_{2,6}$? (4 points)

Half credit on parts (i), (j) and (k) for indicating the correct procedure (formulas). Remaining half credit for the correct numerical answer.