IEOR 3106: Introduction to Operations Research: Stochastic Models

SOLUTIONS for First Midterm Exam, October 4, 2012

There are 3 problems, each with multiple parts.

This exam is open book, but only the Ross textbook.

You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.

1. Testing for a Disease (35 points)

A laboratory blood test is 90% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 10% of the healthy persons tested. That is, if a healthy person is tested, then the test result will incorrectly indicate that he has the disease with probability 0.10. Suppose that 1% of the population has the disease.

(a) What is the probability that a person who is selected at random from the population and tested for the disease tests positive for the disease? (5 points)

It may help to draw a probability tree here. (See the lecture notes for the first lecture on September 4.) Let D be the event of having the disease and let Pos be the event of testing positive. Then

$$\begin{aligned} P(Pos) &= P(D)P(Pos|D) + P(D^c)P(Pos|D^c) \\ &= (0.01)(0.90) + (0.99)(0.10) = 0.009 + 0.099 = 0.108 \end{aligned}$$

(b) What is the probability that a person who is selected at random from the population and tests positive actually has the disease? (10 points)

This is easy to see from the probability tree. We apply Bayes theorem from Chapter 1:

$$P(D|Pos) = \frac{P(D \cap Pos)}{P(Pos)} = \frac{0.009}{0.009 + 0.099} = \frac{0.009}{0.108} = \frac{1}{12}.$$

(c) Suppose that 1000 people are selected at random from the population to be tested on a given day. Suppose that we plan to carefully examine all people who test positive. What is the mean and what is the variance of the number of people that test positive and need to be carefully examined? (You need not do the final exact calculation.) (7 points)

The outcome for each person can be regarded as a Bernoulli random variable, assuming the value 1 if the test if positive, and assuming the value 0 otherwise. The total number that test positive is thus binomial with parameters $n = 1000$ and $p = 0.108$ from part (a). The mean is $np = 1000 \times 0.108 = 108$ and the variance is $np(1-p) = 1000 \times 0.108 \times 0.892 \approx 96.4 \approx 100$.

(d) In the setting of part (c), what is the *approximate* probability that the number of these people that test positive and need to be carefully examined exceeds 125? (Give a number as well as a formula.) (10 points)

Use the normal approximation for the binomial distribution, using the approximation 10 for the standard deviation. (It is quite evident that the standard deviation must be in the interval $[0.9, 0.11]$. Any approximation in that interval will give a good answer.) Let T be the total number of people that test positive. Hence

$$\begin{aligned}
P(T > 125) &= P\left(\frac{T - ET}{\sqrt{Var(T)}} > \frac{125 - ET}{\sqrt{Var(T)}}\right) \approx P\left(N(0,1) > \frac{125 - ET}{\sqrt{Var(T)}}\right) \\
&= P\left(N(0,1) > \frac{125 - 108}{\sqrt{96}}\right) \approx P(N(0,1) > (17/9.80) = 1.734) \approx 0.041 \\
&= P\left(N(0,1) > \frac{125 - 108}{10}\right) \approx P(N(0,1) > 1.7) \approx 0.045
\end{aligned}$$

using the table for the normal distribution on page 82. If we use 9, 10 and 11 for the standard deviation, then we get respectively

$$\begin{aligned}
P\left(N(0,1) > \frac{125 - 108}{9}\right) &\approx P(N(0,1) > (17/9) = 1.89) \approx 0.030 \\
P\left(N(0,1) > \frac{125 - 108}{10}\right) &\approx P(N(0,1) > 1.70) \approx 0.045 \\
P\left(N(0,1) > \frac{125 - 108}{11}\right) &\approx P(N(0,1) > (17/11) = 1.545) \approx 0.060
\end{aligned}$$

Without a calculator or lengthy calculation, you should be able to see that the answer is about 0.045 ± 0.020.

(e) Briefly explain why your approximation in part (d) is justified. (3 points)

The normal approximation is justified by the **Central Limit Theorem**. See the lecture notes for Tuesday, September 11. See Section 2.8.

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2. The Random King (30 **points)**

A king (chess piece) is placed on one of the corner squares of an empty chessboard (having $8 \times 8 = 64$ squares) and then it is allowed to make a sequence of random moves, taking each of its legal moves in each step with equal probability, independent of the history of its moves up to that time. (Recall that the king can move one square in any direction, horizontally, vertically or diagonally, provided of course that it ends up at one of the other squares on the board. Thus, the king has 3 legal moves from each corner, but 8 legal moves from a square away from an edge of the board.)

(a) What is the probability that the king is back on its initial corner square after two moves? (5 points)

The king can make three legal first moves. After 2 of these, the king has 5 moves of which he must select the initial corner square. For 1 of these initialmoves, the king has 8 moves of which he must select the initial corner square. (A square on the side is different from a square on the interior.) Hence the probability of being back at the initial square after two moves is

$$\left(\frac{2}{3} \times \frac{1}{5}\right) + \left(\frac{1}{3} \times \frac{1}{8}\right) = \frac{2}{15} + \frac{1}{24} = \frac{16+5}{120} = \frac{21}{120} = \frac{7}{40}.$$

(b) Let $X(n)$ be a random variable indicating the square occupied by the king after n moves. Is the stochastic process $\{X(n) : n \geq 1\}$ a Markov chain? If so, is it an irreducible Markov chain? If so, is it a periodic irreducible Markov chain? (5 points)

The stochastic process $\{X(n) : n \geq 1\}$ **is** a Markov chain, and it is an **irreducible Markov chain**, but it is **not periodic**, because the king can return in either two steps or three. Hence the greatest common divisor of the times that the Markov chain can be back in its initial state is necessarily 1. (This example is different from Markov mouse, where the period is 2; see the lecture notes of September 18.)

(c) What is the long-run proportion of times that the king is on its initial square? Explain? (10 points)

Since this is an irreducible Markov chain, we can find the long-run proportion of times the Markov chain visits each state by solving the matrix equation $\pi = \pi P$, but this corresponds to a system of 64 linear equations, which would not be easy to solve. However, this is an example of a **reversible** irreducible Markov chain, as discussed in §4.8 and in class on September 27. Hence, it suffices to count the number of possible moves from each square. There are 3 possible moves from each of the 4 corner squares. There are 5 possible moves from each of the 24 side squares. There are 8 possible moves from each of the 36 interior squares. Thus the sum of all these moves is $12 + 120 + 288 = 420$. The long run proportion of moves after which the king ends up at its initial square, which we refer to as state 1, is thus

$$\pi_1 = \frac{3}{420} = \frac{1}{140}.$$

(d) What is the expected number of moves until the king first returns to its initial square? (5 points)

Let N be the number of moves until the king first returns to its initial square. Then

$$E[N] = \frac{1}{\pi_1} = \frac{1}{1/140} = 140$$

See the end of the lecture notes for September 27. The successive returns can be regarded as renewals, so that we can easily derive this formula from renewal theory, as we will see in Chapter 7 of the book.

(e) Show how to compute (but not do not actually do so) the probability that the king reaches the other corner square on its initial row (after several moves) before it reaches any of the other corner squares (not counting the initial corner). Identify what appears in your formula. (5 points)

Even though this Markov chain is an irreducible Markov chain, we can apply the theory for absorbing Markov chains. To find the answer, we let the other three corner square be absorbing states. We then put the transition matrix in canonical form, which we can denote by

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix},$$

where the first three rows are the absorbing states corresponding to the three other corner squares, I is a 3×3 identity matrix, giving the transition probabilities among these absorbing states, 0 is a 3×61 matrix of 0's giving the transition probabilities from the absorbing states to the transient states, R is a 61×3 transition matrix giving the one-step probabilities of being absorbed in one of the absorbing states starting in each of the 61 transient states, and Q is the 61×61 square transition matrix among the remaining 61 states, now regarded as transient states. If the square initially occupied by the king corresponds to the first row of Q after relabeling, while the other corner on the same row corresponds to the first absorbing state, and thus row 1 of I, then the desired probability is

$$B_{1,1} = (NR)_{1,1} = \sum_{j=1}^{61} N_{1,j} R_{j,1},$$

where $N \equiv (I - Q)^{-1}$ is the **fundamental matrix** of absorbing Markov chains, constructed from Q (involving the inverse). The first 1 subscript in $B_{1,1}$ refers to the initial corner square, which we have designated as the first transient state, while the second subscript 1 refers to the first absorbing state, which we have designated as the other corner on the king's initial row.

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3. A Finite Markov Chain (35 **points)**

Consider a Markov chain on the eight states $\{1, 2, \cdots, 8\}$ with transition matrix P given by

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.3 & 0.0 & 0.1 & 0.1 & 0.2 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.1 & 0.2 & 0.0 & 0.0 & 0.2 & 0.2 & 0.3 \\ 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.7 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.5 & 0.0 \end{pmatrix}$$

**Note that we are numbering the rows in the natural order 1, 2, ... , 8, with the columns labeled the same as the rows.

(a) Which states are accessible from state 1? (2 points)

States 1 and 6 are accessible from 1.

(b) Which states are accessible from state 2? (2 points)

All states are accessible from state 2.

(c) Put the transition matrix in canonical form (showing the original states in their new positions). (5 points)

We reorder the states, putting the recurrent states (states in closed communication classes) first, keeping the states in the same communication class together. We order the communication classes by size, putting the smallest ones first. We then put the transient states (states in open communication classes) last. We order the open communication classes, putting the ones that can be reached from other open classes above those, if there are such. Here there are three closed communication classes, $\{3\}$, $\{1, 6\}$ and $\{4, 7, 8\}$, and only one open communication class, $\{2, 5\}$. The canonical form is

$$P = \begin{array}{c} 3 \\ 1 \\ 6 \\ 4 \\ 7 \\ 8 \\ 2 \\ 5 \end{array} \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.7 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.7 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.1 & 0.1 & 0.0 & 0.2 & 0.0 & 0.2 & 0.1 \\ 0.2 & 0.0 & 0.2 & 0.0 & 0.2 & 0.3 & 0.1 & 0.0 \end{pmatrix}$$

(d) Identify the open and closed communication classes for this Markov chain. Which states are transient? Which states are recurrent? Is this chain irreducible? (5 points)

The recurrent states are in the closed classes, while the transient states are in the open classes; see §4.3 and the lecture notes for September 25. The **closed classes** are $\{3\}$, $\{1, 6\}$ and $\{4, 7, 8\}$, while the single **open class** is: $\{2, 5\}$. Since there is more than one communication class, the chain is **reducible**, not irreducible.

(e) Compute the two-step transition probability $P^{(2)}_{1,6}$ (2 points)

$$P^{(2)}_{1,6} = P_{1,1}P_{1,6} + P_{1,6}P_{6,6} = (0.7)(0.3) + (0.3)(0.7) = 0.42$$

(f) Compute the two-step transition probability $P^{(2)}_{1,3}$ (2 points)

States 1 and 3 are in different closed classes. Therefore, $P^{(2)}_{1,3} = 0$.

(g) Compute the four-step transition probability $P^{(4)}_{4,3}$ (2 points)

States 4 and 3 are in different closed classes. Therefore, $P^{(4)}_{4,3} = 0$.

(h) Starting in state 1, what is the long-proportion of moves spent in state 1? (3 points)

Since state 1 is a recurrent state, it suffices to solve $\pi = \pi P$ for the little 2×2 subchain containing only the states 1 and 6:

$$P = \begin{array}{c} 1 \\ 6 \end{array} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

However, since this chain is doubly stochastic (the column sums are all 1 as well as the row sums), the steady state probabilities are discrete uniform, i.e.,

$$\pi_1 = \pi_6 = 1/2$$

Thus the long-run proportion is $1/2$.

(i) Starting in state 4, what is the long-proportion of moves spent in state 4? (4 points)

Since state 4 is a recurrent state, it suffices to solve $\pi = \pi P$ for the 3×3 subchain containing only the states 4, 7 and 8:

$$P = \begin{array}{c} 4 \\ 7 \\ 8 \end{array} \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.5 & 0.5 & 0.0 \end{pmatrix},$$

which is especially easy to solve, sine there are so many 0 elements. We can write $\pi = (x, y, z)$ with the understanding that these are nonnegative numbers such that $x + y + z = 1$. We then get three equations: The first equation is $x = 0.5z$ and the third equation is $z = y$. Since $x + y + z = 1$, we must have $0.5z + z + z = 1$ or $2.5z = 1$ or $z = 2/5$. Hence we get

$$\pi = (1/5, 2/5, 2/5),$$

so that the long-proportion of moves spent in state 4 is $1/5$.

(j) Starting in state 2, what is the expected total number of visits to state 5? (4 points)

These last two parts j and k are harder. Here we need to apply the theory for absorbing chains. We now focus on the 2×2 subchain associated with the transient states 2 and 5:

$$Q = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.0 \end{pmatrix}.$$

With this notation, we want $N_{2,5}$, where $N = (I - Q)^{-1}$. first, we see that

$$I - Q = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 0.8 & -0.1 \\ -0.1 & 0.0 \end{pmatrix},$$

which we write as fractions. Then we calculate the inverse as we learned in linear algebra, getting

$$N = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 100/79 & 10/79 \\ 10/79 & 80/97 \end{pmatrix}.$$

We thus see that $N_{2,5} = 10/79$. Of course, significant partial credit is given for the correct formula, provided that the notation is explained,

We now provide full details on **the inverse calculation:** In particular, we successively (i) multiply rows by constants and then (ii) add a multiple of one row to the other in both $I - Q$ and I until we convert $I - Q$ into I and I into N. In the first step we multiply the first row of both by $(10/8)$. Hence $I - Q$ becomes

$$N_1 = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 1 & -1/8 \\ -1/10 & 1 \end{pmatrix},$$

while I becomes

$$I_1 = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 10/8 & 0 \\ 0 & 1 \end{pmatrix},$$

We next add $(1/10)$ of the first row to the second in N_1 and I_1 to obtain

$$N_2 = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 1 & -1/8 \\ 0 & 79/80 \end{pmatrix},$$

so that I becomes

$$I_2 = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 10/8 & 0 \\ 1/8 & 1 \end{pmatrix},$$

We next multiply the second row of N_2 and I_2 by $80/79$ to get

$$N_3 = \begin{array}{c} 2 \\ 5 \end{array} \begin{pmatrix} 1 & -1/8 \\ 0 & 1 \end{pmatrix},$$

so that I becomes

$$I_3 = \begin{array}{c} 2 \\ 5 \end{array} \left(\begin{array}{cc} 10/8 & 0 \\ 10/79 & 80/79 \end{array} \right),$$

Finally, we add $1/8$ of the second row of N_3 and I_3 to the first row in order to obtain $N_4 = I$ and $I_4 = N$, as given above.

We have two easy checks on N if we get this far. First, these entries must all be nonnegative because they are expected values of nonnegative random variables. Second, we can directly verify that $N(I-Q) = I$, which holds if and only if $N = (I-Q)^{-1}$.

(k) What is the approximate value of $P_{2,6}^{(25)}$? (4 points)

We combine the absorbing theory for the open communication class $\{2,5\}$, using part (j) with the asymptotic theory for the closed communication class $\{1,6\}$, using part (h), both parts already done.

For the absorbing part, we will be applying the formula $B = NR$, but we have to set this up. To do so, we collapse all three closed communication classes into single states, and add the probabilities of absorption from the transient states. Thus we reduce states 1 and 6 into a single state, and consider the probability of ever getting absorbed in this absorbing state starting in state 2. We get an absorbing chain of the usual form (in block matrix form)

$$P = \left(\begin{array}{cc} I & 0 \\ R & Q \end{array} \right),$$

where I is a 3×3 identify matrix, corresponding to the three absorbing classes constructed, while

$$Q = \begin{array}{c} 2 \\ 5 \end{array} \left(\begin{array}{cc} 0.2 & 0.1 \\ 0.1 & 0.0 \end{array} \right),$$

just as given in part (j), and R is the 2×3 matrix of one-step absorption probabilities (with the columns labeled in the order of the closed communication classes, first $1 \equiv \{3\}$, then $2 \equiv \{1,6\}$ and finally $3 \equiv \{4,7,8\}$, yielding

$$R = \begin{array}{c} 2 \\ 5 \end{array} \left(\begin{array}{ccc} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.5 \end{array} \right).$$

We get R from P above by adding the appropriate elements in the final two rows for the transient states 2 and 5.

In this setting, we want $B_{2,2}$ the probability starting in transient state 2 being absorbed in absorbing state $2 \equiv \{1,6\}$. We get

$$B_{2,2} = (NR)_{2,2} = \sum_{j=1}^{2} N_{2,j} R_{j,2} = (100/79)(0.2) + (10/79)(0.2) = (20/79) + (2/79) = 22/79.$$

Thus, the probabiliy starting in (transient state 2 of eventually getting absorbed in the closed communication class $\{1,6\}$ is $22/79$. As an approximation, we assume that absorption happens well before 25 steps if it does, which is very likely.

Once we are in the closed communication class $\{1,6\}$, as a further approximation, we use the steady state probability of being in state 6, which we have found is $1/2$. Hence,

$$P_{2,6}^{(25)} \approx B_{2,2}\pi_6 = (22/79) \times (1/2) = 11/79.$$

where $B_{2,2}$ and π_6 have the special meaning above.

Half credit on parts (i), (j) and (k) for indicating the correct procedure (formulas). Remaining half credit for the correct numerical answer.