IEOR 3106: Second Midterm Exam, Chapters 5-6, November 8, 2012

This exam is closed book. YOU NEED TO SHOW YOUR WORK.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. You may keep the exam itself. Solutions will eventually be posted on line.

1. Kidney Transplants (30 points)

Two individuals, A and B, are candidates for kidney transplants. Suppose that their remaining lifetimes without a kidney are independent exponential random variables with means $1/\mu_A$ and $1/\mu_B$. Suppose that kidneys become available for these two people according to a Poisson process with rate λ. It has been decided that the first kidney will go to A if A is alive when the kidney becomes available. If A is no longer alive, then it will go to B.

(a) What is the probability that A obtains a new kidney? (6 points)

(b) What is the probability that B obtains a new kidney? (6 points)

Suppose that each person survives a kidney operation (independently) with probability p, $0 < p < 1$, and, if so, has an exponentially distributed remaining life with mean $1/\mu$.

(c) What is the expected lifetime of A, assuming that a kidney transplant operation will be performed if A is alive when the kidney is available? (6 points)

(d) Under what condition on the parameters does the possibility of a kidney transplant increase the expected remaining lifetime of A? (6 points)

(e) Suppose that the conditions in part (d) above hold, but A wants to maximize the probability of living beyond a fixed time t (perhaps because that is the date of the wedding of A's daughter). Is a kidney transplant always helpful for that purpose? Why or why not? (6 points)

2. A Model of the Number 1 Subway Line (20 points)

Consider a model of the No. 1 subway line going uptown: Suppose that the uptown subway maintains a fixed deterministic schedule, with a train arriving at each station every 10 minutes, picking up all passengers that want to go uptown. Suppose that all passengers get on and off without delay, without altering the schedule. Let the stations be numbered from 1 to m, increasing as the subway goes uptown from some designated initial station 1. Let the arrivals of passengers to go uptown at the stations be according to independent Poisson processes, with arrival rate λ_i per minute at station i, $1 \leq i \leq m-1$. (Station m is the end of the line; nobody gets on going uptown at station m.) Suppose that each person entering at station i will, independent of everything else, get off at station j with probability $P_{i,j}$, where $\sum_{j=i+1}^{m} P_{i,j} = 1$ for all i. Consider one uptown trip of the subway, starting with station 1, including the usual arrivals there. Let D_j be the number of people that get off at station j, $2 \leq j \leq m$. Suppose that the weights of the people are i.i.d. random variables with mean m pounds and variance σ^2. Let W_j be the total weight of the D_j people that get off at station j.

(a) What are the mean and variance of D_j? (5 points)

(b) What are the mean and variance of W_j? (5 points)

(c) What is $P(D_2 = 2, D_3 = 3)$? (5 points)

(d) What is $P(D_2 = 2 | D_2 + D_3 = 5)$? (5 points)

3. The BAD Barbershop (30 points)

Steven Boyle, Murat Alemdaroglu and Pinhus Dashevsky have joined together to form the Boyle-Alemdaroglu-Dashevsky (BAD) barbershop. They each have one barber chair, but the space is cramped. There is room for only four customers, one waiting and three in service. Suppose that potential customers arrive according to a Poisson process at a rate of 8 per hour. Suppose that potential arrivals finding the barber shop full, with three customers in service and one other customer waiting, will leave and not affect future arrivals. Suppose that successive service times are independent exponential random variables with mean 15 minutes. Suppose that waiting customers have limited patience, being willing to wait only an independent random, exponentially distributed, time with mean 5 minutes before starting service; if they have not started service by that time, then they will abandon, leaving without receiving service.

(a) Let $Y(t)$ be the number of customers in the BAD Barbershop at time t. What kind of stochastic process is $\{Y(t) : t \geq 0\}$? Explain. (5 points)

(b) What proportion of time are all three barbers busy serving customers in the long run? (10 points)

(c) What is the long-run average number of customers in the BAD barbershop? (5 points)

(d) Starting empty, let D_1 be the time until the first customer arrives and then departs. What are the mean and variance of D_1? (5 points)

(e) Starting empty, suppose that 12 potential arrivals come to the barbershop during the first hour. (That includes those that cannot enter, if any.) Under that condition, let X be the number of these customers that arrive (either able to enter or not) in the first fifteen minutes? What are the mean and variance of X? (5 points)

4. Cars in a Highway Segment (20 points)

Suppose that cars enter a highway segment according to a Poisson process with rate $\lambda = 20$ per minute starting at time 0. Assume that different cars do not interact. Suppose that the time each car remains in the highway segment is a random variable uniformly distributed on the interval $[5, 6]$ minutes. Suppose that these random times for different cars are mutually independent. Let $X(t)$ be the number of cars in the highway segment at time t.

(a) Multiple choice (7 points); pick the best answer(and explain):

(i) The stochastic process $\{X(t) : t \geq 0\}$ is a Poisson process.
(ii) The stochastic process $\{X(t) : t \geq 0\}$ is a Markov process.
(iii) Both of the above.
(iv) None of the above.

(b) Give an (exact) expression for $P(X(10) = 20)$. (7 points)

(c) Give an expression for the variance of $X(10) + X(20)$. (6 points)