SOLUTIONS

IEOR 3106: Second Midterm Exam, Chapters 5-6, November 8, 2012

This exam is closed book. YOU NEED TO SHOW YOUR WORK.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. You may keep the exam itself. Solutions will eventually be posted on line.

1. Kidney Transplants (30 points)

Two individuals, A and B, are candidates for kidney transplants. Suppose that their remaining lifetimes without a kidney are independent exponential random variables with means $1/\mu_A$ and $1/\mu_B$. Suppose that kidneys become available for these two people according to a Poisson process with rate λ. It has been decided that the first kidney will go to A if A is alive when the kidney becomes available. If A is no longer alive, then it will go to B.

(a) What is the probability that A obtains a new kidney? (6 points)

Parts (a) and (b) of this problem make up problem 5.34, assigned in Homework 7. The event that A obtains a new kidney happens when the first kidney arrives before A dies; i.e., with T_k denoting the arrival time of the k^{th} kidney, if $T_1 < T_A$, so

$$P(\text{A obtains a new kidney}) \quad = \quad P(T_1 < T_A) = \frac{\lambda}{\lambda + \mu_A}.$$

That is, use the formulas in §1 and §2 of the formula sheet.

(b) What is the probability that B obtains a new kidney? (6 points)

To obtain the probability of B obtains a new kidney, we will condition on the first event (and then uncondition), i.e., which happens first: a kidney arrives, A dies or B dies. Thus

$$\begin{aligned}
&P(\text{B obtains a new kidney}) \\
&\quad = P(\text{B obtains a new kidney}|T_1 = \min\{T_1, T_A, T_B\})P(T_1 = \min\{T_1, T_A, T_B\}) \\
&\qquad + P(\text{B obtains a new kidney}|T_A = \min\{T_1, T_A, T_B\})P(T_A = \min\{T_1, T_A, T_B\}) \\
&\qquad + P(\text{B obtains a new kidney}|T_B = \min\{T_1, T_A, T_B\})P(T_B = \min\{T_1, T_A, T_B\}) \\
&\quad = P(T_2 < T_B)P(T_1 = \min\{T_1, T_A, T_B\}) + P(T_1 < T_B)P(T_A = \min\{T_1, T_A, T_B\}) + 0 \\
&\quad = \Big(\frac{\lambda}{\lambda + \mu_B}\Big)\Big(\frac{\lambda}{\lambda + \mu_A + \mu_B}\Big) + \Big(\frac{\lambda}{\lambda + \mu_B}\Big)\Big(\frac{\mu_A}{\lambda + \mu_A + \mu_B}\Big) \\
&\quad = \Big(\frac{\lambda}{\lambda + \mu_B}\Big)\Big(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B}\Big).
\end{aligned}$$

We again use the formulas in §1 and §2 of the formula sheet.

Suppose that each person survives a kidney operation (independently) with probability p, $0 < p < 1$, and, if so, has an exponentially distributed remaining life with mean $1/\mu$.

(c) What is the expected lifetime of A, assuming that a kidney transplant operation will be performed if A is alive when the kidney is available? (6 points)

We consider the time until either A dies or the kidney arrives, whichever is first. The mean time until that is $1/(\mu_A + \lambda)$. A will of course live until that time. We now consider the expected remaining time A will live after that time. The conditional probability that A will live until the kidney arrives is $\lambda/(\lambda + \mu_A)$, independent of the time itself. Let L_A be the lifetime of A. Given that the kidney operation is performed, the expected remaining life is p/μ. Thus

$$E[L_A] = \frac{1}{\lambda + \mu_A} + \left(\frac{\lambda}{\lambda + \mu_A}\right)\left(\frac{p}{\mu}\right) = \frac{\mu + \lambda p}{(\lambda + \mu_A)\mu}.$$

(d) Under what condition on the parameters does the possibility of a kidney transplant increase the expected remaining lifetime of A? (6 points)

From part (c), we see that the condition is

$$\frac{\mu + \lambda p}{(\lambda + \mu_A)\mu} > \frac{1}{\mu_A}.$$

By simple algebra, we see that this holds if and only if $p\mu_A > \mu$ or if and only if

$$\frac{p}{\mu} > \frac{1}{\mu_A}.$$

Note that, by the lack-of-memory property of the exponential distribution, the left side is the conditional remaining life just as the operation is about to be performed, while the right side is the expected remaining life from any time, given that no kidney is provided.

(e) Suppose that the conditions in part (d) above hold, but A wants to maximize the probability of living beyond a fixed time t (perhaps because that is the date of the wedding of A's daughter). Is a kidney transplant always helpful for that purpose? Why or why not? (6 points)

No, the kidney transplant does NOT always help with this alternative objective, because there is a fixed positive probability $1 - p$ of not surviving the operation. If the kidney arrives before t, but just before time t (sufficiently close to t), then the probability of surviving beyond t will be greater without the operation. Given that the operation will be performed, the probability of living beyond t approaches $1 - p < 1$ as the time of the kidney arrival and the operation gets closer and closer to t. In contrast, the probability of living beyond t without the operation approaches 1 as the time of the kidney arrival time approaches t itself approaches 1. Thus, when the kidney arrival time is before t, but sufficiently close to t, the objective will be better met by not having the operation.

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2. A Model of the Number 1 Subway Line (20 points)

Consider a model of the No. 1 subway line going uptown: Suppose that the uptown subway maintains a fixed deterministic schedule, with a train arriving at each station every 10 minutes, picking up all passengers that want to go uptown. Suppose that all passengers get on and off without delay, without altering the schedule. Let the stations be numbered from 1 to m, increasing as the subway goes uptown from some designated initial station 1. Let the arrivals of passengers to go uptown at the stations be according to independent Poisson processes, with arrival rate λ_i per minute at station i, $1 \leq i \leq m-1$. (Station m is the end of the line; nobody gets on going uptown at station m.) Suppose that each person entering at station i will, independent of everything else, get off at station j with probability $P_{i,j}$, where $\sum_{j=i+1}^{m} P_{i,j} = 1$ for all i. Consider one uptown trip of the subway, starting with station 1, including the usual arrivals there. Let D_j be the number of people that get off at station j, $2 \leq j \leq m$. Suppose that the weights of the people are i.i.d. random variables with mean m pounds and variance σ^2. Let W_j be the total weight of the D_j people that get off at station j.

(a) What are the mean and variance of D_j? (5 points)

Recall that independent splitting of Poisson processes produces independent Poisson processes, as stated within §4 of the formula sheet. Let $N_{i,j}(t)$ be the number of people that get on at station i in the interval $[0, t]$ planning to get off at j. Then $\{N_{i,j}(t) : t \geq 0\}$ is a Poisson process with rate $\lambda_i P_{i,j}$. Moreover, these Poisson processes for different i and different j are mutually independent. Since the sum (superposition) of independent Poisson processes is an independent Poisson process, it is evident that the random variables D_j are independent Poisson random variables with means

$$E[D_j] = \sum_{i=1}^{j-1} 10\lambda_i P_{i,j}, \quad 2 \leq j \leq m.$$

Reasoning more slowly, we can write down D_j explicitly in terms of the Poisson processes. In particular, given that the subway arrival time at station j associated with D_j is at time t,

$$D_j = \sum_{i=1}^{j-1} N_{i,j}(t - 10(j-i)) - N_{i,j}(t - 10(j-i+1)),$$

which is the sum of independent Poisson random variables. Hence, $E[D_j] = \sum_{i=1}^{j-1} E[N_{i,j}(10)]$, where $= E[N_{i,j}(10)] = 10\lambda_i P_{i,j}$, and we get the mean above.

Since the distribution is Poisson,

$$Var[D_j] = E[D_j].$$

(b) What are the mean and variance of W_j? (5 points)

Since D_j has a Poisson distribution, W_j has a compound Poisson distribution. Thus,

$$E[W_j] = E[D_j]m \quad \text{and} \quad Var[W_j] = E[D_j](m^2 + \sigma^2),$$

for $E[D_j]$ given above. Note that we are using the second moment of the individual random weight, which is $m^2 + \sigma^2$, just as in the formula given in §6 of the formula sheet.

(c) What is $P(D_2 = 2, D_3 = 3)$? (5 points)

These are independent Poisson random variables, by the independence assumption and the independent splitting. Thus,

$$P(D_2 = 2, D_3 = 3) = \left(\frac{e^{-E[D_2]}E[D_2]^2}{2!}\right)\left(\frac{e^{-E[D_3]}E[D_3]^3}{3!}\right)$$

where the means $E[D_2]$ and $E[D_3]$ are given above.

(d) What is $P(D_2 = 2|D_2 + D_3 = 5)$? (5 points)

$$P(D_2 = 2|D_2 + D_3 = 5) = \frac{P(D_2 = 2, D_2 + D_3 = 5)}{P(D_2 + D_3 = 5)} = \frac{P(D_2 = 2, D_3 = 3)}{P(D_2 + D_3 = 5)}$$
$$= \frac{P(D_2 = 2)P(D_3 = 3)}{P(D_2 + D_3 = 5)} = b(2; 5, p) = \frac{5!p^2(1-p)^3}{2!3!} \quad \text{(binomial)}$$
$$= 10p^2(1-p)^3 \quad \text{for} \quad p \equiv E[D_2]/(E[D_2] + E[D_3]).$$

The binomial distribution is naturally obtained by writing out the explicit Poisson distributions and canceling. The exponential terms drop out and the rest reduces to the binomial probability with those parameters.

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3. The BAD Barbershop (30 points)

Steven Boyle, Murat Alemdaroglu and Pinhus Dashevsky have joined together to form the Boyle-Alemdaroglu-Dashevsky (BAD) barbershop. They each have one barber chair, but the space is cramped. There is room for only four customers, one waiting and three in service. Suppose that potential customers arrive according to a Poisson process at a rate of 8 per hour. Suppose that potential arrivals finding the barber shop full, with three customers in service and one other customer waiting, will leave and not affect future arrivals. Suppose that successive service times are independent exponential random variables with mean 15 minutes. Suppose that waiting customers have limited patience, being willing to wait only an independent random, exponentially distributed, time with mean 5 minutes before starting service; if they have not started service by that time, then they will abandon, leaving without receiving service.

(a) Let $Y(t)$ be the number of customers in the BAD Barbershop at time t. What kind of stochastic process is $\{Y(t) : t \geq 0\}$? (5 points)

The stochastic process is a **birth-and-death (BD) process**, which in turn is a special case of a CTMC. It is a BD process because all transition are either $+1$ or -1. It is a CTMC because it a markov property with an integer state space and has transitions in continuous time. It has the Markov property because the probability of a future event conditional on the present state and past events depends only on the present state. That in turn is true essentially because of the lack-of-memory property of the exponential distribution.

(b) What proportion of time are all three barbers busy serving customers in the long run? (10 points)

We first need to specify the birth and death rates of the BD process. This can be done by a rate diagram. Then we calculate the steady-state probabilities. Measuring time in hours, the birth rates are $\lambda_k = 8$, $0 \leq k \leq 3$, while the death rates are $\mu_1 = 4$, $\mu_2 = 8$ $\mu_3 = 12$ and $\mu_4 = 24$. The formula for the steady-state distribution is

$$\alpha_j \equiv \lim_{t \to \infty} P(X(t) = j) = \frac{r_j}{\sum_{k=0}^{4} r_k},$$

where $r_0 = 1$, $r_1 = \lambda_0/\mu_1 = 8/4 = 2$, $r_2 = \lambda_0\lambda_1/\mu_1\mu_2 = 2$, $r_3 = \lambda_0\lambda_1\lambda_2/\mu_1\mu_2\mu_3 = 4/3$ and $r_4 = \lambda_0\lambda_1\lambda_2\lambda_3/\mu_1\mu_2\mu_3\mu_4 = 4/9$. Hence, $r_0+r_1+r_2+r_3+r_4 = (9+18+18+12+4)/9 = 61/9$. Finally, we have

$$\alpha_0 = \frac{9}{61}, \quad \alpha_1 = \alpha_2 = \frac{18}{61}, \quad \alpha_3 = \frac{12}{61} \quad \text{and} \quad \alpha_4 = \frac{4}{61}.$$

To answer the question, the proportion of time that all three barbers are busy serving customers in the long run is

$$\alpha_3 + \alpha_4 = \frac{12}{61} + \frac{4}{61} = \frac{16}{61}.$$

(c) What is the long-run average number of customers in the BAD barbershop? (5 points)

$$\sum_{k=0}^{4} k\alpha_k = \frac{9 \times 0}{61} + \frac{18 \times 1}{61} + \frac{18 \times 2}{61} + \frac{12 \times 3}{61} + \frac{4 \times 4}{61} = \frac{106}{61}.$$

(d) Starting empty, let D_1 be the time until the first customer arrives and then departs. What are the mean and variance of D_1? (5 points)

Since D_1 is the sum of two independent exponential random variables, the first with mean $1/\lambda = 1/8$ and the second with mean $1/\mu = 1/4$,

$$E[D_1] = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \quad \text{and} \quad Var[D_1] = \left(\frac{1}{8}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{1}{64} + \frac{1}{16} = \frac{5}{64}.$$

(e) Starting empty, suppose that 12 potential arrivals come to the barbershop during the first hour. (That includes those that cannot enter, if any.) Under that condition, let X be the number of these customers that arrive (either able to enter or not) in the first fifteen minutes? What are the mean and variance of X? (5 points)

Conditional on the number of arrivals in the first hour being 12, the actual arrival times during that hour are distributed as 12 i.i.d. random variables, each uniformly distributed over the hour. Thus each one arrives in the first 15 minutes, which equals 1/4 hour, with probability 1/4. Let N be the number that arrive in the first 15 minutes. That number is Poisson with mean and variance

$$E[N] = np = 12 \times (1/4) = 3 \quad \text{and} \quad Var(n) = np(1-p) = 12(1/4)(3/4) = 9/4.$$

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4. Cars in a Highway Segment (20 **points)**

Suppose that cars enter a highway segment according to a Poisson process with rate $\lambda = 20$ per minute starting at time 0. Assume that different cars do not interact. Suppose that the time each car remains in the highway segment is a random variable uniformly distributed on the interval $[5, 6]$ minutes. Suppose that these random times for different cars are mutually independent. Let $X(t)$ be the number of cars in the highway segment at time t.

(a) Multiple choice (7 points); pick the best answer:

(i) The stochastic process $\{X(t) : t \geq 0\}$ is a Poisson process.
(ii) The stochastic process $\{X(t) : t \geq 0\}$ is a Markov process.
(iii) Both of the above.
(iv) None of the above.

This is an example of an infinite-server queue, as discussed in class on October 25; see the "physics" paper there and Chapter 5 of Ross. For this question, the correct answer is (iv) NONE OF THE ABOVE. The distribution of $X(t)$ IS Poisson for each t, but the process is NOT a Poisson process. It fails to have many of the key properties of a Poisson process, as given in Section 5; e.g., the sample paths are not nondecreasing. Since the length of time each car remains in the highway segment is NOT exponential, the process $\{X(t) : t \geq 0\}$ is also NOT a Markov process. All the cars in the segment at time t that arrived just before t will remain in the segment shortly after time t. On the other hand, cars in the segment that arrived between 5 and 6 minutes before time t will soon be gone after time t. Thus, the future after time t depends on the past prior to time t in addition to the state at time t. So that the Markov property does NOT hold.

(b) Give an (exact) expression for $P(X(10) = 20)$. (7 points)

Since this is the $M/GI/\infty$ infinite-server queue, the number $X(10)$ has a Poisson distribution with some mean $m(10)$. Thus,

$$P(X(10) = 20) = \frac{e^{-m(10)} m(10)^{20}}{20!}$$

so that it remains only to specify the mean. Let S be the random time each car remains in the segment, and let G be its cdf. For any service-time cdf G, the formula for the mean at time t is

$$E[X(t)] = \int_0^t \lambda(t-s)(1 - G(s))\, ds = \lambda \int_0^t (1 - G(s))\, ds, \quad t \geq 0,$$

as stated on the formula sheet; also see (5.18). However,

$$1 - G(t) = P(S > t) = 1 \quad \text{for} \quad t < 5, \quad 1 - G(t) = 0 \quad \text{for} \quad t > 6,$$

and

$$1 - G(t) = 6 - t \quad \text{for} \quad 5 \leq t \leq 6.$$

Thus, for all t with $t \geq 6$,

$$E[X(t)] = \lambda \int_0^5 1\, ds + \lambda \int_5^6 (6 - s)\, ds = \lambda(5 + (1/2)) = 5.5\lambda = 110.$$

In this case, we could anticipate the value, because it coincides with the simple steady-state value which is $E[X(\infty)] = \lambda E[S] = 20 \times 5.5 = 110$. Since the service times are bounded, all being less than 6, the system steady state is reached by time 6, and is in effect thereafter.

(c) Give an expression for the variance of $X(10) + X(20)$. (6 points)

Since no car in the segment at time 10 can be in the segment at time 20, and no car in the segment at time 20 could have been in the segment at time 10, because the length of time spent in the segment falls in the interval $[5, 6]$, the random variables $X(10)$ and $X(20)$ are necessarily independent Poisson random variables. In the figure in the physics paper, the regions corresponding to these random variables are disjoint. Looking at the last part carefully, we see that the distribution of $X(t)$ is the same for all $t \geq 6$. Since the distributions are Poisson, the variances equals the means. Thus,

$$\begin{aligned} Var(X(10) + X(20)) &= Var(X(10)) + Var(X(20)) \quad \text{(by independence)} \\ &= E[X(10)] + E[X(20)] = 2E[X(10)] \quad \text{(by the Poisson property)} \\ &= 2 \times 110 = 220 \quad \text{(from part (b))}. \end{aligned}$$