IEOR 3106: First Midterm Exam, Chapters 1-4, October 3, 2013

There are 3 problems, each with multiple parts.

This exam is closed book. You need to show your work. Briefly explain your reasoning.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing, "I have neither given not received improper help on this examination," on your examination booklet and sign your name. You may keep the exam itself. Solutions will eventually be posted on line.

1. A Five-Room Maze for Markov Mouse (35 points, 5 points for each part)

Markov Mouse is placed in room 1 of the 5-room maze below and then moves randomly from room to room through one of the doors (horizontally and vertically) that connect the rooms. On each move, Markov Mouse chooses each of its eligible doors with equal probability, independently of past decisions.

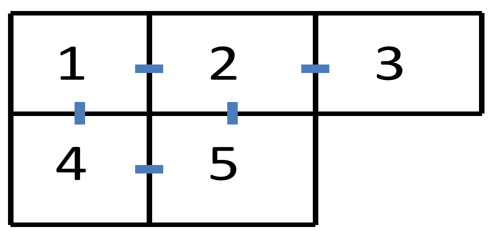

Figure 1: A five-room maze for Markov Mouse.

(a) What is the probability that Markov Mouse is in room 3 after two moves?

(b) What is the probability that Markov Mouse is back in room 1 after two moves?

(c) Does the probability that Markov mouse is in room 1 after n moves converge to a limit as $n \to \infty$? Why or why not?

(d) What is the expected number of moves until Markov Mouse first returns to room 1?

(e) What is the long-run proportion of moves that Markov Mouse spends in room 1?

(f) Give a formula that can be used to calculate the expected total number of visits to room 2 before visiting either room 3 or room 5. Identify all quantities in the formula.

(g) Give a formula that can be used to calculate the probability that Markov Mouse visits room 3 before visiting room 5. Identify all quantities in the formula.

2. Markov Mouse with Memory (40 points, 5 points for each part)

We again consider Markov mouse moving from room to room in the 5-room maze of problem 1. Markov Mouse is placed in room 1 of the 5-room maze and then moves randomly from room to room through one of the doors (horizontally and vertically) that connect the rooms. However, now we assume that Markov Mouse has memory and recognizes where it has been before. (Maybe because of its keen sense of smell.) So now we assume that Markov Mouse

never returns to a room that it occupied before. On each move, Markov mouse chooses each of its eligible doors with equal probability. (Now a door is eligible if it leads to a room that has not been occupied before.) When there are no eligible moves, Markov Mouse stops moving. Hence, Markov Mouse makes at most 4 moves.

(a) What is the probability that Markov Mouse visits all 5 rooms (including the initial room 1)?

(b) What is the conditional probability that Markov mouse visits all 5 rooms given that the last room visited is room 3?

(c) Let X_n be the room occupied by Markov Mouse after n moves. If Markov Mouse stops in room j after k moves, let $X_n = j$ for all $j \geq k$. Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Explain.

(d) If possible, define an absorbing Markov chain representing the movement of Markov Mouse (with the condition that Markov Mouse never returns to a room that it occupied before) and identify the absorbing states.)

(e) What is the mean number of rooms that Markov Mouse visits (including the initial room)?

(f) What is the variance of the number of rooms that Markov Mouse visits (including the initial room)?

(g) If the experiment is repeated 100 times under independent conditions (with 100 different mice of the same type), then what is the approximate probability that the total number of rooms visited by all mice in the 100 experiments exceeds 455? (Make a reasonable rough estimate to within 0.050.)

(h) Explain why your answer in part (g) above is justified.

3. Two Independent Exponential Random Variables (28 points, 4 points for each part)

Consider two independent exponentially distributed random variables X_i with means $m_i \equiv E[X_i] \equiv (1/\lambda_i)$ for $i = 1, 2$ with $m_1 = 1$ and $m_2 = 2$. For $i = 1, 2$, these random variables have probability density functions (pdf's) and cumulative distribution functions (cdf's)

$$f_{X_i}(x) \equiv \lambda_i e^{-\lambda_i x}, \quad x \geq 0, \quad \text{and} \quad F_{X_i}(x) \equiv P(X_i \leq x) = 1 - e^{-\lambda_i x}$$

Let $\min\{X_1, X_2\}$ and $\max\{X_1, X_2\}$ be the minimum and maximum of these two random variables, respectively.

(a) What is $P(X_1 > 3 | X_1 > 1)$?

(b) What is $P(X_1 > 3 | X_2 > 1)$?

(c) What is $Var(X_1 + X_2)$, the variance of the sum?

(d) What is $E[X_1 + X_2 | X_2 = 3]$?

(e) What is $P(\min\{X_1, X_2\} > x)$?

(f) Find the pdf of the sum $f_{X_1+X_2}(x)$.

(g) Find $E[\max\{X_1, X_2\}]$.