IEOR 3106: First Midterm Exam, Chapters 1-4, October 3, 2013

SOLUTIONS

1. A Five-Room Maze for Markov Mouse (35 points, 5 points for each part)

Markov Mouse is placed in room 1 of the 5-room maze below and then moves randomly from room to room through one of the doors (horizontally and vertically) that connect the rooms. On each move, Markov Mouse chooses each of its eligible doors with equal probability, independently of past decisions.

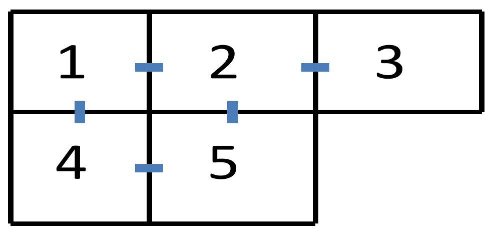

Figure 1: A five-room maze for Markov Mouse.

(a) What is the probability that Markov Mouse is in room 3 after two moves?

$$P_{1,3}^{(2)} = P_{1,2}P_{2,3} = (1/2)(1/3) = (1/6)$$

(b) What is the probability that Markov Mouse is back in room 1 after two moves?

$$P_{1,1}^{(2)} = P_{1,2}P_{2,1} + P_{1,4}P_{4,1} = (1/2)(1/3) + (1/2)(1/2) = (1/6) + (1/4) = 5/12$$

(c) Does the probability that Markov mouse is in room 1 after n moves converge to a limit as $n \to \infty$? Why or why not?

No, because the Markov chain is **periodic**. Starting in an odd numbered room, it can be in an odd numbered rooms only after an even number of steps.

(d) What is the expected number of moves until Markov Mouse first returns to room 1?

Let Z_i be the number of moves until first returning to room i, starting in room i. The expected number of moves to return to room 1 is $E[Z_1] = 1/\pi_1$, where π_1 is the long-run proportion of moves ending in room 1. Since $\pi_1 = 1/5$ by the next part,

$$E[Z_1] = 5.$$

(e) What is the long-run proportion of moves that Markov Mouse spends in room 1?

This is a simple application of a random walk on a weighted graph, where all the weights are 1. The long-run proportion of moves spent in room i, π_i, is proportional to the number of doors out of room i. Thus,

$$\pi_1 = \frac{2}{2+3+1+2+2} = \frac{2}{10} = \frac{1}{5}.$$

(f) Give a formula that can be used to calculate the expected total number of visits to room 2 before visiting either room 3 or room 5. Carefully identify all quantities in the formula.

We use the theory for absorbing chains. We want $N_{1,2}$, the $(1,2)^{\text{th}}$ element of the fundamental matrix

$$N = (1-Q)^{-1},$$

where 1 is the initial state and 2 is another transient states, while rooms 3 and 5 are absorbing states. In particular, we can re-order the rooms to make the transition matrix be of the block-matrix form

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

, where I is a 2×2 identity matrix, R is 3×2 and Q is the 3×3 transition matrix for the 3 transient states. In particular, we write P as

$$P = \begin{array}{c} 3 \\ 5 \\ 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 & 0.0 & 0.0 \\ 0.0 & 1/2 & 1/2 & 0.0 & 0.0 \end{pmatrix}$$

**Note that we are labeling the states with the columns labeled the same as the rows. Hence,

$$Q = \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 0.0 & 1/2 & 1/2 \\ 1/3 & 0.0 & 0.0 \\ 1/2 & 0.0 & 0.0 \end{pmatrix}$$

and

$$I - Q = \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 1.0 & -1/2 & -1/2 \\ -1/3 & 1.0 & 0.0 \\ -1/2 & 0.0 & 1.0 \end{pmatrix}$$

Finally, N is the inverse, i.e., $N = (I-Q)^{-1} = inv(I-Q)$.

(g) Give a formula that can be used to calculate the probability that Markov Mouse visits room 3 before visiting room 5. Carefully identify all quantities in the formula.

Now we want $B_{1,3}$, the $(1,3)$ element of the matrix B, where

$$B = NR,$$

with $N = (1-Q)^{-1}$ for Q given in the previous part and

$$
R = \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \begin{pmatrix} 0.0 & 0.0 \\ 1/3 & 1/3 \\ 0.0 & 1/2 \end{pmatrix}
$$

with the columns labeled first 3 and then 5.

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2. Markov Mouse with Memory (40 points, 5 points for each part)

We again consider Markov mouse moving from room to room in the 5-room maze of problem 1. Markov Mouse is placed in room 1 of the 5-room maze and then moves randomly from room to room through one of the doors (horizontally and vertically) that connect the rooms. However, now we assume that Markov Mouse has memory and recognizes where it has been before. (Maybe because of its keen sense of smell.) So now we assume that Markov Mouse **never returns to a room that it occupied before**. On each move, Markov mouse chooses each of its eligible doors with equal probability. (Now a door is eligible if it leads to a room that has not been occupied before.) When there are no eligible moves, Markov Mouse stops moving. Hence, Markov Mouse makes at most 4 moves.

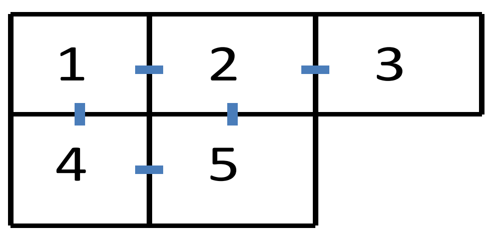


Figure 2: A five-room maze for Markov Mouse.

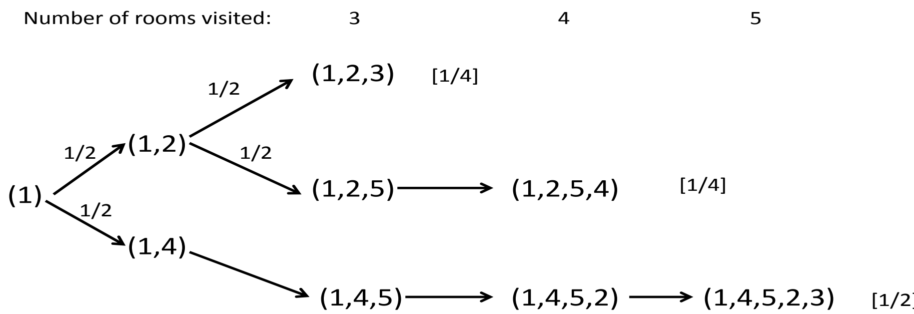
(a) What is the probability that Markov Mouse visits all 5 rooms (including the initial room 1)?

It is good to start with an insightful figure. Here we draw a **probability tree**, just as in the lecture notes for the first class. The nodes are labeled by the rooms visited so far, in order.

There are only two possible random choices. There is a random choice on the first move and then another random choice in the second move, if the first move is to room 2. There are just 3 possible sequences of moves. The probability that Markov Mouse with Memory (M^3) visits all 5 rooms is thus $\mathbf{1/2}$. This happens if and only if the initial move is to room 4.

(b) What is the conditional probability that Markov mouse visits all 5 rooms given that the last room visited is room 3?

Let A be the event that M^3 visits *all* 5 rooms, corresponding to the sequences of rooms $(1, 4, 5, 2, 3)$. Let B be the event that the last room visited is room 3, which corresponds to



the two sequences $(1,2,3)$ and $(1,4,5,2,3)$, which has probability $3/4$. Then

$$P(A|B) \equiv \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/2}{3/4} = 2/3.$$

(c) Let X_n be the room occupied by Markov Mouse after n moves. If Markov Mouse stops in room j after k moves, let $X_n = j$ for all $j \geq k$. Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Explain.

No. The stochastic process $\{X_n : n \geq 0\}$ is *not* a Markov chain. It fails to have the Markov property. The Markov property states that the transition probability given the entire history is the same as the transition probability given only the current state occupied. Here the current state is the current room (by definition above). For example, the transition probability from room 5 depends on the history of previous rooms visited.

(d) If possible, define an absorbing Markov chain representing the movement of Markov Mouse (with the condition that Markov Mouse never returns to a room that it occupied before) and identify the absorbing states.

Just as in Example 4.4 of the textbook and homework exercise 4.2, we can obtain a Markov process if we include some of the history into our definition of the states. The key is to define *new* states. In this case, we should include the rooms visited previously and the order they are visited. The appropriate states are the shown in the nodes of the probability tree drawn for part (a). There are 9 states: (1), $(1,2)$, $(1,4)$, $(1,2,3)$, $(1,2,5)$, $(1,4,5)$, $(1,2,5,4)$, $(1,4,5,2)$ and $(1,4,5,2,3)$. The transition probabilities are as specified in the probability tree of part (a). In particular,

$$
P = \begin{array}{c} (1) \\ (1,2) \\ (1,4) \\ (1,2,3) \\ (1,2,5) \\ (1,4,5) \\ (1,2,5,4) \\ (1,4,5,2) \\ (1,4,5,2,3) \end{array}
\begin{pmatrix}
0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0
\end{pmatrix}
$$

**Note that we are labeling the states with the columns labeled the same as the rows.

The three absorbing states are: $(1,2,3)$, $(1,2,5,4)$ and $(1,4,5,2,3)$. The transition probabilities from these states to these states above are 1.

(e) What is the mean number of rooms that Markov Mouse visits (including the initial room)?

Let N be the number of rooms visited by M^3. From the probability tree in part (a), we see that

$$P(N = 3) = 1/4, \quad P(N = 4) = 1/4 \quad \text{and} \quad P(N = 5) = 1/2,$$

so that

$$E[N] = (3 \times \frac{1}{4}) + (4 \times \frac{1}{4}) + (5 \times \frac{1}{2}) = \frac{17}{4} = 4.25$$

(f) What is the variance of the number of rooms that Markov Mouse visits (including the initial room)?

Using the same notation, $Var(N) = E[N^2] - (E[N])^2$, where

$$E[N^2] = (9 \times \frac{1}{4}) + (16 \times \frac{1}{4}) + (25 \times \frac{1}{2}) = \frac{75}{4} = \frac{300}{16}.$$

Hence,

$$Var(N) = \frac{300}{16} - \left(\frac{17}{4}\right)^2 = \frac{11}{16}.$$

(g) If the experiment is repeated 100 times under independent conditions (with 100 different mice of the same type), then what is the approximate probability that the total number of rooms visited by all mice in the 100 experiments exceeds 455? (Make a reasonable rough estimate, to within 0.050)

Let N_n be the number of rooms visited in experiment n, $1 \le n \le 100$. Let $S_n \equiv N_1 + \cdots N_n$. Thus we want the approximate probability $P(S_{100} > 455)$. We use a normal approximation. We need the mean and variance of S_n:

$$E[S_n] = nE[N_1] = 4.25n \quad \text{and} \quad Var(S_n) = nVar(N_1) = (11/16)n,$$

so that

$$E[S_{100}] = 100E[N_1] = 425 \quad \text{and} \quad Var(S_{100}) = 11Var(N_1) = 1100/16 = 68.75$$

We will need only a rough estimate of the variance. Hence, we use a normal approximation. Let $N(0,1)$ denote a random variable with the standard normal distribution (mean 0 and variance 1). Then, adding and dividing on both sides as usual:

$$\begin{aligned} P(S_{100} > 455) &= P\left(\frac{S_{100} - E[S_{100}]}{\sqrt{Var(S_{100})}} > \frac{455 - E[S_{100}]}{\sqrt{Var(S_{100})}}\right) \\ &\approx P\left(N(0,1) > \frac{455 - E[S_{100}]}{\sqrt{Var(S_{100})}}\right) \\ &= P\left(N(0,1) > \frac{455 - 425]}{\sqrt{68.75}}\right) \\ &= P\left(N(0,1) > \frac{30}{8.29}\right) \approx P(N(0,1) > 3.61) \approx 0.0001 \approx 0, \end{aligned}$$

where we apply the table of the normal distribution.

It is important to recognize that it is *not* necessary to compute $\sqrt{68.75}$ exactly! A rough estimate would be based on the elementary observation that $8 = \sqrt{64} < \sqrt{68.75} < \sqrt{81} = 9 < 10$, so that the probability satisfies

$$P(S_{100} > 455) \le P(N(0,1) > 3) = 0.0013 < 0.050.$$

So a rough estimate is $P(S_{100} > 455) \approx P(N(0,1) > 3) \approx 0.0013 \approx 0$.

(h) Explain why your answer in part (g) above is justified.

The normal approximation is justified by **the central limit theorem**.

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3. Two Independent Exponential Random Variables (28 points, 4 points for each part)

Consider two independent exponentially distributed random variables X_i with means $m_i \equiv E[X_i] \equiv (1/\lambda_i)$ for $i = 1,2$ with $m_1 = 1$ and $m_2 = 2$. For $i = 1,2$, these random variables have probability density functions (pdf's) and cumulative distribution functions (cdf's)

$$f_{X_i}(x) \equiv \lambda_i e^{-\lambda_i x}, \quad x \ge 0, \quad \text{and} \quad F_{X_i}(x) \equiv P(X_i \le x) = 1 - e^{-\lambda_i x}$$

Let $\min\{X_1, X_2\}$ and $\max\{X_1, X_2\}$ be the minimum and maximum of these two random variables, respectively.

First, we chose this problem because it leads into our next topic, Chapter 5. In particular, see §5.2 in the book. You should be able to answer these questions using basic probability, as in Chapters 1-3 of the textbook. see (2.7) and §2.3.2 in the book.

(a) What is $P(X_1 > 3 | X_1 > 1)$?

Use the definition of conditional probability:

$$\begin{aligned} P(X_1 > 3 | X_1 > 1) &= \frac{P(X_1 > 3 \quad \text{and} \quad X_1 > 1)}{P(X_1 > 1)} \\ &= \frac{P(X_1 > 3)}{P(X_1 > 1)} = \frac{e^{-3}}{e^{-1}} \\ &= e^{-(3-1)} = e^{-2} = P(X_1 > 2). \end{aligned}$$

This is an example of the **lack-of-memory property** of the exponential distribution. We can rewrite it as

$$P(X_1 > x + y | X_1 > y) = P(X_1 > x) \quad \text{for all} \quad x, y > 0.$$

See (5.2) in the book.

(b) What is $P(X_1 > 3 | X_2 > 1)$?

Since X_1 and X_2 are independent, the conditioning has no impact:

$$P(X_1 > 3 | X_2 > 1) = P(X_1 > 3) = e^{-3}$$

(c) What is $Var(X_1 + X_2)$, the variance of the sum?

Since the random variables are independent,

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2).$$

Since X_i is exponentially distributed, $Var(X_i) = m_i^2$. Hence,

$$Var(X_1 + X_2) = m_1^2 + m_2^2 = 1^2 + 2^2 = 5.$$

We remark that in general

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2),$$

where $Cov(X_1, X_2) \equiv E[X_1 X_2] - (E[X_1]E[X_2])$ is the covariance between X_1 and X_2. Independence implies that $Cov(X_1, X_2) = 0$. (See §2.5.2.)

(d) What is $E[X_1 + X_2 | X_2 = 3]$?

The expectation of a sum is always the sum of the expectations, even for conditional expectations. Hence,

$$E[X_1 + X_2 | X_2 = 3] = E[X_1 | X_2 = 3] + E[X_2 | X_2 = 3] = E[X_1] + 3 = 1 + 3 = 4.$$

(e) What is $P(\min\{X_1, X_2\} > x)$?

Observe that the minimum is greater than x if and only if both random variables are greater than x. Hence,

$$\begin{aligned} P(\min\{X_1, X_2\} > x) &= P(X_1 > x, X_2 > x) = P(X_1 > x)P(X_2 > x) = e^{-\lambda_1 x}e^{-\lambda_2 x} \\ &= e^{-(\lambda_1 x + \lambda_2 x)} = e^{-(\lambda_1+\lambda_2)x} = e^{-(1+(1/2))x} = e^{-1.5x} \end{aligned}$$

(f) Find the pdf of the sum $f_{X_1+X_2}(x)$.

The density of the sum of two independent random variables is the convolution of the component densities, so that

$$\begin{aligned} f_{X_1+X_2}(x) &= \int_0^x f_{X_1}(u) f_{X_2}(x-u)\, du \\ &= \int_0^x \lambda_1 e^{-\lambda_1 u}\lambda_2 e^{-\lambda_2(x-u)}\, du \\ &= \lambda_1\lambda_2 e^{-\lambda_2 x}\int_0^x e^{-\lambda_1 u}e^{+\lambda_2 u}\, du \\ &= \lambda_1\lambda_2 e^{-\lambda_2 x}\int_0^x e^{-(\lambda_1-\lambda_2)u}\, du \\ &= \frac{\lambda_1\lambda_2 e^{-\lambda_2 x}(1-e^{-(\lambda_1-\lambda_2)x})}{\lambda_1-\lambda_2} \\ &= \frac{1(1/2)e^{-0.5x}(1-e^{-0.5x})}{1-(0.5)} \\ &= e^{-0.5x} - e^{-x}. \end{aligned}$$

See §5.2.4 in the book.

(g) Find $E[\max\{X_1, X_2\}]$.

From part (e), we see that $\min\{X_1, X_2\}$ has an exponential distribution with a rate equal to the sum of the rates. Hence,

$$E[\min\{X_1, X_2\}] = \frac{1}{\lambda_1+\lambda_2} = \frac{1}{1+(1/2)} = \frac{2}{3}.$$

Next, observe that

$$\max\{X_1, X_2\} + \min\{X_1, X_2\} = X_1 + X_2,$$

so that

$$\max\{X_1, X_2\} = X_1 + X_2 - \min\{X_1, X_2\}$$

and

$$E[\max\{X_1, X_2\}] = E[X_1] + E[X_2] - E[\min\{X_1, X_2\}] = 1 + 2 - \frac{2}{3} = \frac{7}{3}.$$
