A Concise Summary

Everything you need to know about exponential and Poisson

Exponential Distribution

Assume that $X \sim \exp(\lambda)$, by which we mean that X has an exponential distribution with *rate* λ. Then X has mean $1/\lambda$; i.e., $EX = 1/\lambda$. Also the variance is $Var(X) = (EX)^2 = 1/\lambda^2$. In addition, assume that $Y \sim \exp(\mu)$ and $X_i \sim \exp(\lambda_i)$ for $i = 1, \cdots, n$, where all these exponential random variables are independent.

1. *Lack of memory*: $P(X > s+t | X > s) = P(X > t)$ for all $s > 0$ and $t > 0$.
   (check the computation)
2. *Minimum*: $\min\{X, Y\} \sim \exp(\lambda + \mu)$
   (check the computation)
   and hence
   $\min\{X_1, \cdots, X_n\} \sim \exp(\lambda_1 + \cdots + \lambda_n)$ without computation.
3. *Maximum*: $X+Y = \min\{X, Y\} + \max\{X, Y\}$ tells us an easy way to compute $E[\max\{X, Y\}]$.)
4. *More on Minimum*: $P(X = \min\{X, Y\}) = P(X < Y) = \frac{\lambda}{\lambda+\mu}$: (check the computation)
   and hence
   $P(X_k = \min\{X_1, \cdots, X_n\}) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}$ without computation.
5. *Even more on Minimum*:The events $\{X = \min\{X, Y\}\}$ and $\{\min\{X, Y\} > t\}$ are independent for all t.
   and hence
   the events $\{X_k = \min\{X_1, \cdots, X_n\}\}$ and $\{\min\{X_1, \cdots, X_n\} > t\}$ are independent for all t.

Poisson Processes

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ, referred to by $N(t)(\lambda)$. In addition, consider Poisson processes $N_j(t)(\lambda_j)$, $1 \leq j \leq m$.

1. *Interarrival Times*: The interarrival times of $N(t)(\lambda)$ are IID $exp(\lambda)$.
2. *Thinning (Type classification)* : When arrivals occur in the Poisson process $N(t)$, they are classified randomly and *independently* (the successive classifications are done independently according to the same probabilities) into classes (indexed by j) with probability $p_1, \cdots, p_m$. Let $N_j(t)$ be the input (arrival) process for class j (obtained with probability p_j). These newly created counting processes $N_j(t)$ are independent Poisson processes with rates λp_j.
3. *Superposition (Type aggregation)* : When *independent* Poisson arrivals $N_j(t)$ occur with rates λ_j, the total number of arrivals is a Poisson($\lambda_1 + \cdots + \lambda_m$).
4. *Conditioning* : When we know $N(t) = n$, the occurrence time of n arrivals are distributed as independent random variables, each uniform on the interval $[0, t]$.