

MARKOV CHAIN MODELS TO ESTIMATE THE PREMIUM FOR EXTENDED HEDGE FUND LOCKUP

by

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Abstract

A lockup period for investment in a hedge fund is a time period after making the investment during which the investor cannot freely redeem his investment. It is routine to have a one-year lockup period, but recently the requested lockup periods have grown longer. Assuming that the investor will rebalance his portfolio of hedge funds on a yearly basis, if permitted, we define the annual lockup premium as the difference between the expected return per year from an investment in a hedge fund with a nominal one-year lockup period and the expected return per year from an investment in a hedge fund with an extended lockup period, as a function of the length of that extended lockup period. We develop Markov chain models to estimate this lockup premium function. By solving systems of equations, we fit the Markov chain transition probabilities to three directly observable hedge fund performance measures: the persistence of return, the variance of return and the hedge-fund death rate. The model quantifies the way the lockup premium increases as a function of both the persistence of return and the variance of return, but decreases as a function of the hedge-fund death rate. Increasing death rate lowers the lockup premium because investors can redeem their investment when the hedge fund fails, even when a lockup condition is in force.

1. Introduction

A *lockup period* for investment in a hedge fund is a time period after making the investment during which the investor cannot freely redeem his investment. It is routine to have a one-year lockup period, but recently the requested lockup periods have grown longer. It is reasonable for an investor in a hedge fund to expect compensation for the restricted investment opportunities imposed by an extended lockup condition, with the compensation increasing as the length of the lockup period increases. We regard that compensation as a *lockup premium*, and we ask: *What should that lockup premium be as a function of the length of the lockup period?*

We assume that the investor will rebalance his portfolio of hedge funds on a yearly basis, as permitted. Thus, we *define* the (annual) lockup premium as the difference between the expected return per year from an investment in a hedge fund with a nominal one-year lockup period and the expected return per year from an investment in a hedge fund with an extended lockup period, as a function of the length of that extended lockup period. Our definition accounts for lost gains due to rebalancing the investment portfolio in hedge funds, but not for other lost investment opportunities, so we provide a conservative estimate of the lockup premium.

With that definition specified, we develop mathematical models to estimate the lockup premium function as a function of key hedge-fund performance measures. Specifically, we develop both discrete-time and continuous-time Markov chain models for that purpose. Our main contribution is to take a modelling approach, but there also are significant challenges in deciding what modelling approach to use. We want a good model, one which is easy to understand, properly reflects the specific lockup conditions, has predictive power, can be effectively analyzed and can be fit to available data.

These requirements lead us to propose relatively simple three-state Markov chain models. By introducing models with relatively few parameters, we have fewer parameters to fit to data. We do not directly fit the natural model parameters, which are the Markov chain transition probabilities, but instead we *indirectly fit the model* to more directly observable hedge fund performance measures, specifically, the persistence of return, the variance of return and the hedge-fund death rate. This indirect approach requires that we solve systems of equations to determine the required model parameters.

Hedge fund lockup is an instance of a classical liquidity problem. It is well known that an illiquid investment, which limits the holder's ability to redeem, usually offers higher return

than a liquid investment, which can be redeemed freely. It is common to regard the spread (the difference in yield rates) as the *liquidity premium*. A popular example is a certificate of deposit (CD). Unlike a usual savings account, a CD restricts its owner from redeeming his money until the CD matures. As a consequence, banks offer higher interest rates for CD's than for regular savings accounts. Experience shows that the liquidity premium increases as the length of the time period increases, but increases more slowly as the time increases, so that overall the premium function is concave.

In practice, the liquidity premium for CD's is determined by market forces, and is usually taken as given. Stock option prices are also determined by market forces, but because of the complex dynamics, it has proven useful to have the Black-Scholes option pricing model and other related models to estimate what the price should be. In the same spirit, in our hedge fund context, we introduce models to help understand what the lockup premium should be.

While hedge fund lockup is a liquidity problem like a CD and many other liquidity problems, it has its own special character. There is a complication with hedge funds, because investors may actually have an early opportunity to redeem their investment. If the hedge fund performs very poorly, so that it ceases operating, then a significant portion of the investment is returned to investors, even if the lockup period has not expired. Thus, it might appear that there should be no liquidity problem at all, but the two extreme alternatives are not the only possibilities: Hedge fund performance may be weak, so that returns are low and future prospects are dim, even though the fund does not cease operating. The lockup prevents the investor from moving his investment away from such "sick" funds. This special way hedge fund lockup is treated makes the liquidity premium more complicated, providing motivation for more careful analysis.

Our proposed model directly responds to this special feature of hedge fund investments: *We consider three possible states for a hedge fund: good, sick and dead, and we assume that transitions among these states occur randomly according to a Markov chain.* In a dead state, the investor suffers a low return, but at the next yearly reinvestment opportunity the state changes to a good state, because the investor gets his money back and can invest in a new fund, which we take to be in the good state. There is no extra penalty from the lockup associated with a dead fund, but there is from a sick fund. With only nominal one-year hedge fund lockup, we assume that investors will reinvest in a good fund at the next yearly reinvestment opportunity whenever any fund they have invested in becomes sick. In contrast, with the extended lockup period, no reinvestment is possible until the end of the lockup period. In the meantime, the sick fund may perform poorly, and produce low returns, but there also

is a chance that it may rebound and become a good fund. Clearly, some care is needed to properly account for the various good and bad possibilities, which inevitably must be regarded as random events. The Markov chain models can capture the behavior described above, so should provide insight.

Of course, it remains to specify the three Markov chain states. We propose classifying the funds according to their returns. We say that a fund is in a: *good* state when its relative return is higher than U percent, *sick* state when its relative return is between L and U percent, and *dead* state when its relative return is less than L percent. We leave U and L as model parameters in general. Figure 1 illustrates how states might be defined.

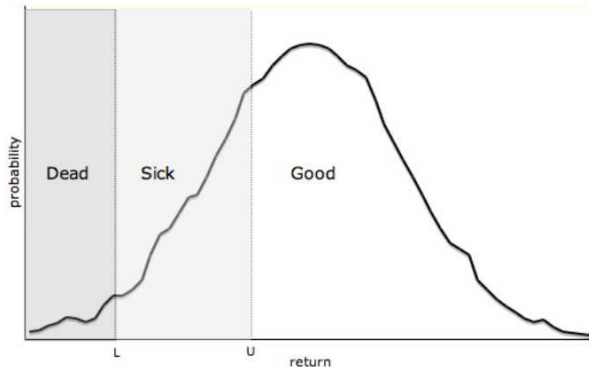


Figure 1: A hypothetical distribution of hedge fund annual returns with levels L and U dividing the funds into the three states G , S and D .

A fundamental principal guides our analysis: the *persistence hypothesis*. We postulate that there is a persistence in hedge fund performance within a particular hedge fund strategy category: We assume that above-average funds will tend to continue doing well, while below-average funds will tend to continue faltering. A persistence of γ means that *for every 1% you earn above the average in the current year, you expect to earn $\gamma\%$ above the average in the next year*. We estimate the persistence by doing a regression analysis on the hedge-fund-return data from the Tremont Advisory Shareholders Services (TASS) data, and find evidence to support the persistence hypothesis for some strategy categories of hedge funds.

As we will explain in §3, there is a consensus among researchers that there is persistence, but there is controversy about its extent. There are questions about the quality of the data and the proper way to analyze it. We describe our data-analysis procedure in Appendix. There are eleven strategy categories of hedge funds in the TASS data. We found positive persistence for all of them, using data from 2000 to 2005, but some estimated persistence factors were very

low. We found five strategy categories of funds with significant persistence: (i) convertible, (ii) dedicated short bias, (iii) fixed income, (iv) fund of fund, and (v) others. Figure 2 is the scatter plot of two consecutive relative returns and the associated least-squares-fit with zero intercept for four of these fund categories. For these fund categories, the least-squares-fit slope, γ , ranges from 0.33 to 0.49, but there evidently is considerable randomness in the data. The persistence plays a big role in determining our Markov chain transition probabilities and, thus, our estimate of the lockup premium. We work hard to show how the lockup premium should depend on the persistence, and not on determining precisely what the persistence is. It is important to note, though, that the fund category is not the only element affecting persistence. Persistence can be measured for fund manager's tenure, asset size, fee structure, and so on, depending on the investor's judgement. As long as persistence is found or anticipated, our Markov chain models can be applied.

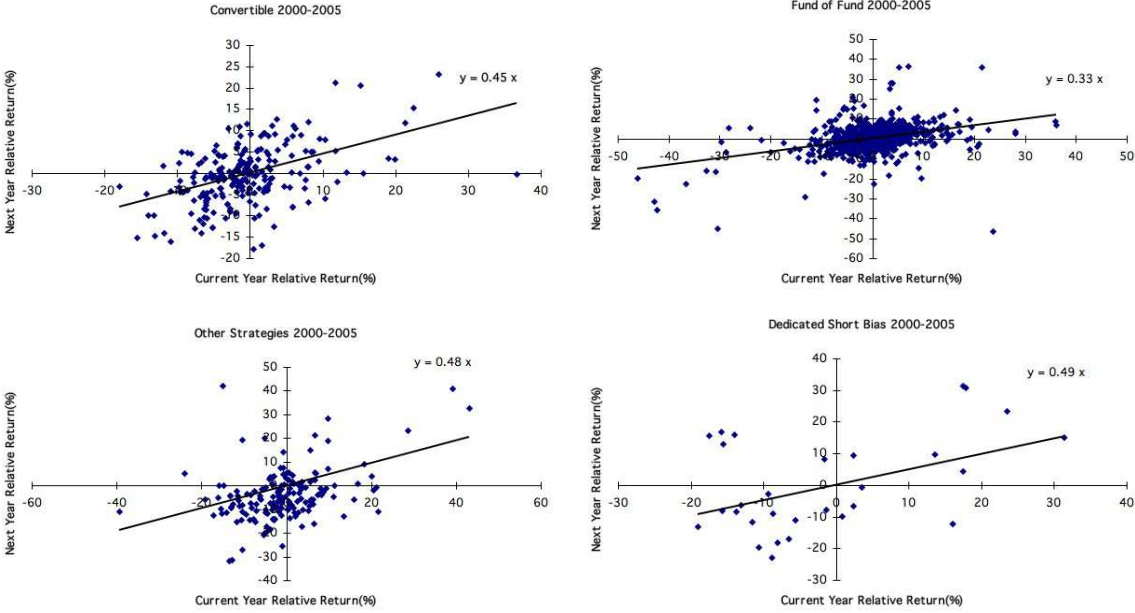


Figure 2: Scatter plots and associated least-squares lines for relative hedge fund returns in successive years from 2000 to 2005.

Having decided to use a Markov chain model, we must specify how the transitions take place over time. Given that hedge funds operate continually, it is natural to use continuous-time Markov chains (CTMC's), which allow transitions from one state to another in continuous time, even though we assume that reinvestment opportunities are restricted to being yearly. And that is what we do. However, discrete-time Markov chains (DTMC's) tend to be easier to work with, so we start with DTMC's with yearly transitions. With yearly transitions, we

let state changes occur at the potential reinvestment times.

After we carry out the analysis for DTMC's, we show that it is also possible to carry out the corresponding analysis with CTMC's, but the analysis is more involved. We use nonlinear programming to efficiently solve the equations for model fitting. With a CTMC, state changes occur in continuous times, but reinvestment opportunities still occur yearly in discrete time. We thus use the transient one-year transition probabilities of the CTMC in an associated DTMC to describe what happens at the yearly reinvestment times. In addition to being more realistic, the CTMC model has the advantage that it can be fit to a wider range of hedge-fund performance measures.

We show that the Markov chain models can be used to estimate how the lockup premium depends on the hedge-fund performance measures. Consistent with intuition, we show that the lockup premium is increasing in both the persistence of the return and the variance of the return. What is less obvious, but consistent with intuition upon reflection, is that the lockup premium is decreasing in the hedge fund death rate. Of course the models do more: The models quantify the effect of these observable hedge fund performance measures on the lockup premium.

In §4 we present a simple analysis of the lockup premium based on persistence alone, without any Markov chains, which is appropriate when no hedge funds die. The more elaborate analysis in this paper is thus primarily intended to determine the effect of the death rate, denoted by δ . Just as for the persistence, there is controversy about what is the actual death rate of hedge funds, with estimates ranging from 3 – 12%. Higher estimates follow from estimates of the median life of a hedge fund, as we explain in §4.1. That leads us to conclude that the death rate might be as high as $\delta = 0.09$. Just as with the persistence, we work hard to show how the lockup premium should depend on the death rate, and not on determining precisely what the death rate is.

From our CTMC model, we conclude that a death rate of $\delta = 0.09$ per year makes the lockup premium about *half* of what it would be for $\delta = 0$. Figure 3 shows the lockup premium function for four values of δ . We conclude that the death rate is potentially a significant factor.

Organization of the paper. We start in §2 by reviewing the related literature on liquidity, including premiums for hedge fund lockup. In §3 we discuss persistence of hedge fund returns, reviewing the literature and analyzing data from the TASS data. In §4 we present the simple

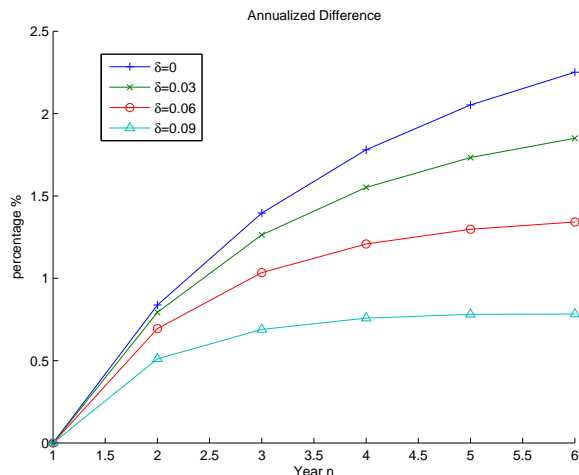


Figure 3: The lockup premium as a function of the extended lockup period, n , based on the CTMC model for four values of δ : 0.00, 0.03, 0.06 and 0.09. The parameter values are in Table 2.

analysis of lockup premium based on persistence alone, without any Markov chains, which is based on no dying funds. This simple analysis provides a useful reference case, because it yields a simple formula. In §4.1 we discuss hedge-fund persistence and death rate in more detail. In §5 we introduce and analyze our three-state DTMC model. In §7 we consider the corresponding CTMC model. In section 6 we show how the model parameters and the lockup premium depend on basic hedge fund performance measures. Finally, in §8 we draw conclusions. We present additional material in an appendix.

2. Liquidity Literature Review

There is a substantial literature on liquidity, including hedge fund lockup, but it mostly has a different character.

Liquidity premiums in asset pricing. The liquidity premium is well recognized as an important factor in asset pricing, but it is commonly measured by transaction cost; e.g, see Amihud and Mendelson (1986), Pastor and Stambaugh (2003), Chordia et al. (2001), and Eleswarapu and Reinganum (1993). For example, in the stock market, bid-ask spread is one measure of the liquidity premium. Amihud and Mendelson (1986) showed that there exists an increasing and concave relationship between the asset return and the bid-ask spread. Darar et al. (1998) confirmed this result, using the reciprocal of the stock turnover rate to measure the liquidity premium. More recently, Vayanos (2004) considered liquidity in an equilibrium

model. He considered the liquidity premium in asset pricing with different transaction costs. He showed that as assets become more volatile, the required excess return from a riskless asset is increasing in the transaction costs.

Studies of liquidity have also been performed for the bond market; e.g., Amihud and Mendelson (1991), Warga (1992), Krishnamurthy (2002) and Longstaff (2004). For bonds, it is argued that there should exist a clear premium for liquidity, separate from the credit risk premium. Most-recently-issued U.S. Treasury bonds are considered the most liquid bonds available, among all bonds with similar conditions. Since US Treasury bonds are assumed to be riskless, they provide a natural way to measure the liquidity premium, without having to consider credit risk. The papers above study the liquidity premium by comparing the price of most-recently-issued US Treasury bonds (on the run) to the price of the bonds issued three months previously (off the run).

There are a few papers that are more closely related to what we do here, namely, Longstaff (1995, 2001) and Brown et al. (2003). These papers also view the liquidity premium as arising from the investor's inability to rebalance his portfolio in a timely way. Specifically, they define the liquidity premium as the additional required fixed return to compensate for the loss of the investor's utility from the inability to rebalance the investor's portfolio. They calculate the required liquidity premium as a function of the degree of risk averseness in the utility function, the market growth rate, and the liquidity restriction period. They rely heavily on mathematical models and mathematical analysis for this purpose. Unlike these references, we do not use utility functions. Our use of expected value corresponds to a linear utility function, which may be roughly appropriate for a fund of funds, which is a typical investor in hedge funds.

Empirical studies on hedge fund lockup. There is a growing literature on hedge funds, e.g., see Agarwal and Naik (2005), but only a few researchers have focused on hedge fund lockup. Liang (1999) found that the average hedge fund returns are related positively to the lockup periods from the analysis of Hedge Fund Research, Inc. (HFR) database. Aragon (2007) quantified the lockup premium for hedge funds empirically. He compared the hedge fund performance with and without extended lockup conditions. He estimated that the average difference in the annual returns is around 4 – 7%.

There also are empirical studies on the liquidity premium for funds other than hedge funds. For example, Ippolito (1989) conducted a similar study for mutual funds. There is a load-type

mutual fund, which assesses sales charges. Ippolito (1989) found that the load-type mutual funds make approximately 3.5% higher return than no-load mutual funds.

In summary, from our investigation of the literature, we find that only a few papers - Longstaff (1995, 2001) and Brown et al. (2003) - have interpreted liquidity premium as quantification of the cost of a restricted rebalance opportunity. We found no previous papers employing models calibrated to data in order to estimate the liquidity premium. And none of the papers have used Markov chains, with the exception of Derman (2006), which is a preliminary account of the research reported here.

3. Persistence of Hedge Fund Returns

We specify how hedge funds perform by looking at the *relative return* of a fund, which is the difference between the annual return of the fund and the average annual return of all hedge funds in that hedge fund strategy category, where return is measured as a percentage. We do that to factor out the performance of the market as a whole. We evaluate hedge funds the way we might evaluate hedge fund managers, trying to identify whether or not their funds perform better than average. We estimate the persistence by doing a regression analysis on the hedge-fund-return data over several years from the TASS data.

The persistence literature. Before discussing our own analysis of data, we discuss the literature on performance persistence. Researchers have tried to take advantage of the two main hedge fund databases - TASS and HFR. In doing so, researchers have discovered that it is difficult to make unbiased estimates because reporting is voluntary, and some funds stop reporting, especially those performing poorly; e.g., see Jagannathan et al. (2006).

Despite the difficulty with biases in the hedge fund data, researchers have conducted studies. Although some researchers did not find evidence of performance persistence, others did. Brown et al. (1999) used a simple two-state categorization - win or lose - to measure performance persistence, recording a win if the fund beats the median return, but they did not find evidence of persistence. Boyson and Cooper (2004) carried out a similar analysis and still did not find evidence of persistence.

However, several papers found performance persistence for shorter periods ranging from a quarter to three years. Koh et al. (2003) used the method of Brown et al. (1999) for Asian hedge funds and found strong persistence in short horizons from monthly to quarterly. Agarwal and Naik (2000) and Jagannathan et al. (2006) used linear regression, like we do,

as well as the previous two-way classifications. Agarwal and Naik (2000) did not provide regression slope explicitly but showed that depending on the strategy category of hedge fund, the percentage of funds which have statistically significant positive slope in regression ranges from 5 to 34 percent, where most of the strategy categories have around 20 percent. Using the same parametric linear regression and non-parametric two-way classifications, Agarwal and Naik (2000) claimed that the evidence of persistence is strongest for the shorter quarterly time periods. On the other hand, Edwards and Caglayan (2001) found strong persistence in over 1-2 years from the Managed Account Reports (MAR/Hedge) data. Furthermore, Jagannathan et al. (2006) found a significantly high performance persistence for a three-year period in his empirical study with HFR data. Jagannathan et al. (2006) carefully took account bias from voluntary reports and did regression of relative return for three consecutive years. Using generalized method of moment (GMM) estimation, they found a statistically significant persistence factor of 0.56 for three-year period.

There also exists indirect evidence of performance persistence from the study of hedge-fund liquidation or survival. Brown et al. (2001) indirectly supported performance persistence when they found that a negative aggregated return over the previous two years increases the probability that a fund will liquidate. Furthermore, ter Horst et al. (2001) concluded that hedge-fund survival is strongly related to historical performance. Baquero et al. (2005) conducted probit regression analysis of hedge-fund liquidation. They found that funds with high returns are much less likely to liquidate than funds with low returns from quarterly return data, which again indirectly supports persistence. We lastly remark that from the study of bid-ask spread in the stock market, Roll (1984) claimed that bid-ask spread increases as the price change is more serially correlated. As mentioned in §2, bid-ask spread is one of the representative measures of liquidity. Thus, we expect that the extended lockup period, which makes the hedge fund less liquid, would increase the serial correlation of performance, which increases performance persistence.

Our regression analysis. We conduct linear autoregression analysis with the TASS hedge fund performance data to find the best linear regression line between two consecutive year's relative returns. Specifically, letting the current year's relative return be denoted by R_c and the next year's relative return be denoted by R_n , we find the slope γ for the line $R_n = \gamma \cdot R_c$, which produces the minimum sum of squared errors.

The actual data analysis procedure is somewhat complicated; the key features are described

in Appendix. There are eleven strategy categories of hedge funds in the TASS data. We found positive persistence for all of these, but some estimated persistence factors were very low. (We think that better data should show higher persistence.)

We found five strategy categories of funds with significant persistence: (i) convertible, (ii) dedicated short bias, (iii) fixed income, (iv) fund of fund, and (v) others. Figure 2 is the scatter plot of two consecutive relative returns and the associated least-squares-fit with zero intercept for four of these fund categories. For these fund categories, the least-squares-fit slope, γ , ranges from 0.328 to 0.488.

A different way to estimate the persistence factor is to look at the ratio of the next-year average returns to the current-year average return, restricting attention to the returns that are positive in the current year. You get the same estimate when you repeat that procedure, but instead restrict attention to the returns that are negative in the current year. See Appendix for the details.

4. Estimating the Lockup Premium from Persistence Alone

In this section we show how persistence alone, without any Markov chains, can be used to generate an estimate of the lockup premium. This simple analysis depends on four assumptions:

1. There is a single persistence factor γ .
2. We can ignore the phenomenon of hedge funds dying.
3. The returns each year are normally distributed with a fixed variance σ^2 .
4. The performance of a fund is considered good if its annual return exceeds the average annual return.

The first two assumptions imply that the expected relative returns over time evolve *linearly*, enabling us to derive a simple *no-death lockup premium* as a function of the expected excess return of a good fund. The last two assumptions enable us to determine the expected excess return of a good fund. The third assumption can be weakened, but some sort of *ceteris paribus assumption* is needed. The fourth assumption is just one possible case; it can easily be varied without altering the rest of the analysis.

The no-death lockup premium. We assume that the hedge fund starts off in a good state having just earned a relative return $Y_G > 0$, to be specified below. Let R_n be the expected

relative return in the n^{th} year. The assumed γ persistence implies that the expected relative return at the end of the first year is $R_1 = \gamma Y_G$. The notion of γ persistence, with no funds dying, implies that we can *recursively* determine the expected relative return in successive years by

$$R_n = \gamma \cdot R_{n-1} = \gamma^n \cdot Y_G, \quad n \geq 1. \quad (4.1)$$

As a consequence of (4.1), the total expected relative return up to the n^{th} year is

$$\sum_{i=1}^n R_i = \frac{\gamma Y_G (1 - \gamma^n)}{1 - \gamma}. \quad (4.2)$$

Based on this simple analysis, we can compare the expected relative return from an n -year lockup with the expected relative return from a 1-year lockup in order to calculate the lockup premium. Under 1-year lockup, investors have a chance to replace all sick funds with good funds at the end of each year. If they do, the expected return each year is the same as in the first year: $R_1 = \gamma Y_G$. Thus, at the end of the n^{th} year, the total expected relative return is simply $n\gamma Y_G$. On the other hand, under an n -year lockup, the fund just evolves without replacement up to the n^{th} year, as in (4.2). We assume that after the n^{th} year, the funds with 1-year and n -year lockups are both replaced by funds with the same 1-year lockup, so that there necessarily will be no difference in a fund's return after the n^{th} year.

Letting C_n be the total cumulative difference in expected return up through year n , we thus have

$$C_n = \sum_{i=1}^n (R_1 - R_i) = nR_1 - \sum_{i=1}^n R_i = \gamma Y_G \left(n - \frac{1 - \gamma^n}{1 - \gamma} \right). \quad (4.3)$$

The lockup premium, denoted by $A_n \equiv A_n(\gamma)$, is the average annual difference. By (4.3), the *no-death lockup premium formula* is

$$A_n \equiv \frac{C_n}{n} = \gamma Y_G \left(1 - \frac{1 - \gamma^n}{n \cdot (1 - \gamma)} \right), \quad n \geq 1, \quad (4.4)$$

which is a concave increasing function in n for each γ , $0 < \gamma < 1$, and a concave function of γ for each $n \geq 1$. The lockup premium $A_n(\gamma)$ is *not* an increasing function of γ overall; e.g., for $n = 2$, $A_n(\gamma) = Y_G \gamma (1 - \gamma)/2$, which is increasing for $0 < \gamma < 1/2$, but decreasing for $1/2 < \gamma < 1$. However, the lockup premium function $A_n(\gamma)$ is increasing in γ for all sufficiently small γ , for each $n \geq 1$.

From (4.4), we see that $A_1 = 0$, $A_n \rightarrow \gamma Y_G$ as $n \rightarrow \infty$, and we have the bounds

$$\gamma Y_G \left(1 - \frac{1}{n(1 - \gamma)} \right) \leq A_n \leq \gamma Y_G \left(1 - \frac{1}{n} \right), \quad n \geq 1, \quad (4.5)$$

which yield convenient approximations. For large n or small γ , the lower bound is an accurate approximation.

The excess return from a good fund. The no-death lockup premium function $A_n(\gamma)$ clearly shows how the lockup premium depends on the three quantities: the length n of the extended lockup period, the persistence factor γ and the expected excess return of a good fund, Y_G . Clearly, n is directly observable, and we have seen how we can estimate γ , but it remains to specify Y_G .

However, if we define Y_G as the expected excess return of a good fund and apply the last two assumptions, then we can calculate Y_G . Letting $N(m, \sigma^2)$ denote a normally distributed random variable with mean m and variance σ^2 , we have

$$Y_G = E[N(0, \sigma^2) | N(0, \sigma^2) > 0] = E[|N(0, \sigma^2)|] = \sigma E[|N(0, 1)|] = \sqrt{2/\pi} \sigma \approx 0.8\sigma . \quad (4.6)$$

We can combine (4.4) and (4.6) to obtain the following general no-death lockup premium function

$$A_n(\gamma, \sigma) = 0.8\sigma\gamma \left(1 - \frac{1 - \gamma^n}{n \cdot (1 - \gamma)} \right), \quad n \geq 1 . \quad (4.7)$$

With assumptions 3 and 4, we see that the no-death lockup premium should be directly proportional to the standard deviation σ . Assumption 4 plays a key role in getting the simple formula (4.6), but we can generalize for arbitrary boundary point U , using the following formula for the conditional expectation of a normal random variable:

$$E[N(m, \sigma^2) | a \leq N(m, \sigma^2) \leq b] = m + \sigma \frac{[\phi((a - m)/\sigma) - \phi((b - m)/\sigma)]}{[\Phi((b - m)/\sigma) - \Phi((a - m)/\sigma)]} \quad (4.8)$$

for $-\infty \leq a < b \leq +\infty$; e.g., see Proposition 18.3 of Browne and Whitt (1995). From formula (4.8), we see that Y_G will *not* be proportional to σ if we change the upper boundary point U .

We emphasize that, even under assumption 4 above, having A_n be directly proportional to σ depends critically on the third ceteris-paribus assumption made above. Since we are free to choose the monetary units, we can choose to define all returns relative to the standard deviation σ , which must be in the same units as the returns. In that sense, the lockup premium is *automatically* proportional to σ . The proportionality conclusion becomes more meaningful when we assume that the distribution of returns depends on σ as a simple scale factor, as provided by assumption 3 above. We need to impose a strong condition on the way the return distribution changes with σ in order to deduce the desired proportionality conclusion. The normality is only used to compute the precise value of the mean.

Relating to the calibration by Markov chains. We remark that the Markov chain model calibration will also produce its own estimates of the excess return Y_G , but we will find that

analysis yields similar conclusions. Indeed, our main numerical example has $Y_G = 0.67\sigma$. We remark that we can obtain exactly that value if we take Y_G to be the *median* of the positive relative returns, because the median of the random variable $|N(0, 1)|$ is 0.67.

Anticipating our future numerical examples with Markov chains, we plot our estimate for the lockup premium in Figure 4 for the case $\gamma = 0.5$, $\sigma = 0.1$ and $Y_G = 0.067$. Our estimate without death appears as the upper curve in Figure 4. We see that the lockup premium increases toward the limit $Y_G/2 = 0.0335$ as n increases.

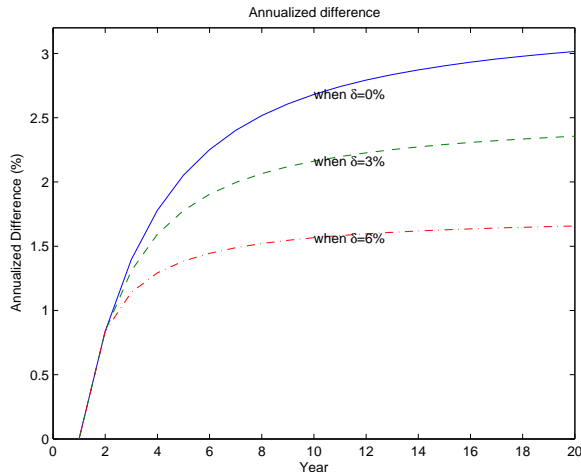


Figure 4: The lockup premium function for the DTMC model for three values of the hedge-fund death rate δ . The remaining model parameters are $Y_G = 0.067$, $Y_S = -0.15$, $Y_S = -0.20$ and $\gamma_G = \gamma_S = \gamma = 0.5$.

We also plot two curves for positive death rates δ , obtained using the DTMC model in §5. The plotted cases for $\delta = 0.03$ and $\delta = 0.06$ show the importance of going beyond the no-death model. Consistent with Figure 4, we will see that the lockup premium is decreasing in the hedge fund death rate with our Markov chain model. Consequently, formulas (4.4) and (4.7) in this section, derived under the assumption of zero death rate, provide upper bounds on our estimated lockup premium with positive δ .

4.1. Important Hedge-Fund Performance Measures

Our Markov chain model will depend critically on the persistence of returns and the hedge-fund death rate. So we discuss these performance measures further now.

Two persistence factors: γ_G and γ_S . In equations (5.5) and (5.6) below we will introduce two state-dependent persistent factors γ_G and γ_S , instead of just the single γ , as we had in

§4. Clearly, this generalization is important if the persistence factors for the two states do in fact differ significantly. To illustrate what actually may happen, Figure 5 shows the results of an regression analysis applied two consecutive-year relative returns for positive and negative parts of the current relative-return data separately. Figure 5 shows a significant difference in the slope of regression line for several fund categories, suggesting that it may be important to use separate state-dependent persistence factors.

The stationary death rate δ . We calibrate our models by specifying the overall annual death rate, denoted by δ . Unfortunately, estimating the death rate from the TASS data is difficult, in part because poorly performing funds often stop reporting, but funds also stop reporting for other reasons, e.g., because they have completed a successful merger-and-acquisition closure with another fund.

After checking the reasons for funds being terminated in the HFR data, Rouah (2006) concluded that, after removing these biases, 3 – 5% of the hedge funds leave the database each year because of failure. As noted in §3, Park (2006) estimated that the fund death rate is only 3.1 %, even though the total attrition rate from the TASS data was 8.7 % , based on her analysis from 1995 to 2004.

The death rate is closely related to the survival probability and median life of the fund. Clearly, as the death rate increases, the survival probability and the median life decrease. Since median life is more easily observable, it is convenient to verify the death rate of our model through the median life in the hedge fund data.

One way to check the validity of the model is to calculate the survival probability curve produced by the model. In terms of the transition matrix P to be introduced in (5.1). the *probability of surviving n years* is $S_n = P_{G,G}^n + P_{G,S}^n$ for $n \geq 1$.

Figure 6 shows the survival probability curve for the DTMC model when $\delta = 0.03$ and 0.06. When $\delta = 0.03$, about 90% survive for 5 years, whereas the survival probability goes down to around 80% when $\delta = 0.06$. If we increase δ above 0.07, then r goes below 0 and we are unable to fit a DTMC model. Thus, under fixed $Y_G = 0.067, Y_S = -0.15, Y_D = -0.20$, $\delta = 0.07$ is the maximum range.

Studies estimating the median survival time of hedge funds were discussed in §3. In addition, Gregoriou (2002) estimated that median survival time of all hedge funds is 5.5 years, depending on factors such as millions managed, performance fee, leverage, minimum purchase and also on redemption period. More recently, Rouah (2006) reported estimates of median

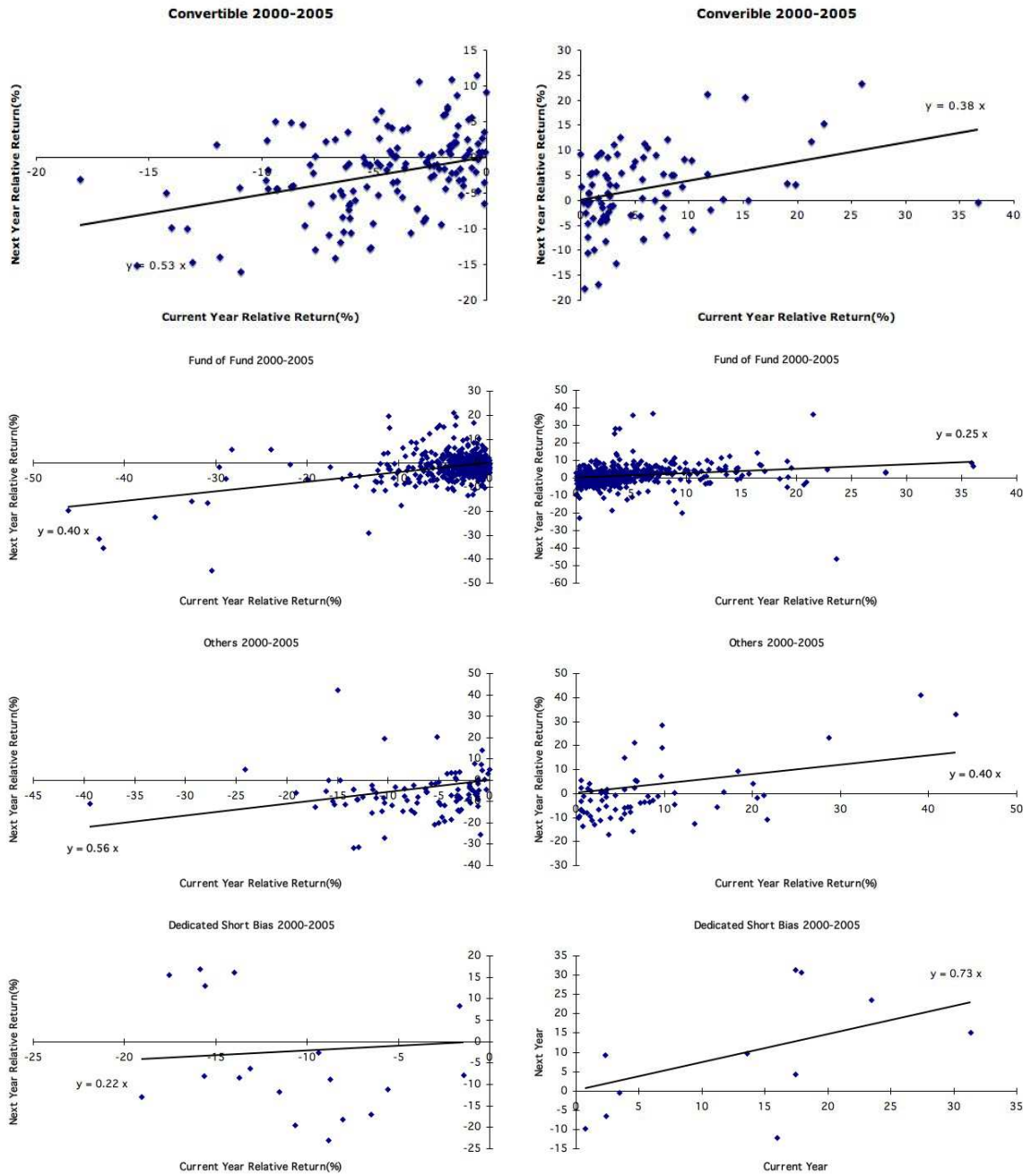


Figure 5: Scatter plots and least-squares lines for positive current relative returns and negative current relative returns of hedge funds from 2000 to 2005 in four categories: (i) convertible bond, (ii) fund of fund, (iii) others, and (iv) dedicated short bias.

survival time due before liquidation as ranging from 5.8 to 7.4 years based on the HFR data and from 7.2 to 17.4 years based on the TASS data. This last observation by Rouah (2006) suggests that our model with $\delta = 0.06$ may reasonably approximate the fund's performance.

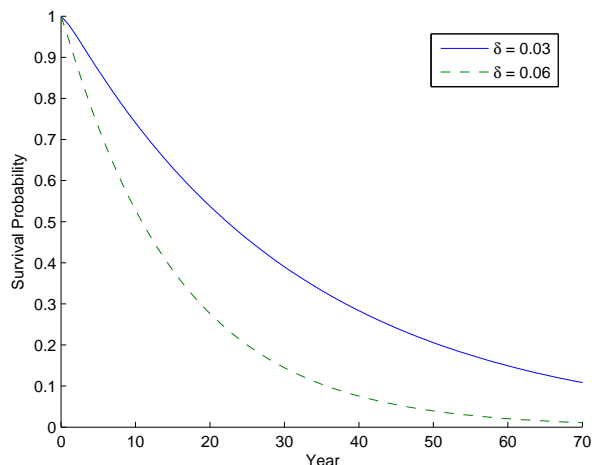


Figure 6: The survival probability for the DTMC model when $\delta = 0.03$ and 0.06 , for parameter values of p, q, r, Y_g, Y_S , and Y_D values given in the Table 1.

5. The Discrete-Time-Markov-Chain Model

We start in §5.1 by defining the basic three-state DTMC model, which has six parameters. Next in §5.2 we introduce four equations that the six parameters must satisfy, based on standard hedge fund performance measures. In §5.3 we develop explicit formulas for the three parameters appearing in the DTMC transition probabilities. In §5.4 we show how to calculate all the parameters after specifying two of the relative returns. We present numerical examples in §5.5. Finally, we show how to calculate the lockup premium in §5.6. Paralleling our treatment in the Appendix, we introduce a related two-state DTMC based on the assumption of zero death rate. That simplification is appealing because the formulas are more elementary.

5.1. The Basic DTMC Model

As indicated in the introduction, we let our Markov chain models have three states: good, sick and dead. We model the changing fund state over time as a DTMC, as in Chapter 4 of Ross (2003). We let time be discrete, with the unit of time representing one year. The initial DTMC is an *absorbing Markov chain*, with the D state being the sole absorbing state; once a fund becomes dead, it remains dead forever. We consider a transition matrix depending on three parameters: p, q and r :

$$P = \begin{matrix} G \\ S \\ D \end{matrix} \begin{pmatrix} p & 1-p & 0 \\ q & r & 1-q-r \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.1)$$

In the displayed transition matrix P , we have only labelled the rows. The columns are assumed to be labelled in the same order. We have assumed that it is impossible to transition from good to dead in a single year, thus eliminating one parameter. The parameters p , q and r represent good events in alphabetic order: $p = G \rightarrow G$, $q = S \rightarrow G$, $r = S \rightarrow S$.

We now move on to consider an associated *ergodic Markov chain*, having a non-degenerate limiting steady-state distribution, by assuming that a new hedge fund appears in the good state to replace a dead hedge fund right after it dies. This can be done with the new three-state DTMC transition matrix

$$P = \begin{matrix} G \\ S \\ D \end{matrix} \begin{pmatrix} p & 1-p & 0 \\ q & r & 1-q-r \\ p & 1-p & 0 \end{pmatrix}. \quad (5.2)$$

In (5.2), the transition probabilities from a dead state are the same as from a good state, because a dead fund is immediately replaced by a good fund.

From the basic theory of DTMC's, as in Theorem 4.1 of Ross (2003), we obtain the steady-state probability vector $\pi \equiv (\pi_G, \pi_S, \pi_D)$ by solving $\pi = \pi P$ under the condition that $\pi_G + \pi_S + \pi_D = 1$. The stationary probability vector π for the transition matrix P in (5.2) is

$$\pi_G = \frac{q + p(1 - q - r)}{2 - p - r}, \quad \pi_S = \frac{1 - p}{2 - p - r}, \quad \pi_D = \frac{(1 - p)(1 - q - r)}{2 - p - r}. \quad (5.3)$$

Our DTMC model uses both transition matrices. We use the absorbing transition matrix in (5.1) when we compute the expected return of a fund, while we use the ergodic transition matrix in (5.2) when we calculate the steady-state death rate and performance variance.

We will act as if the fund earns a state-dependent fixed (average relative return) in each state. We must specify these relative returns. Let Y_G , Y_S and Y_D denote the expected relative returns in the states G , S and D , respectively. Overall, we have six parameters: p , q , r , Y_G , Y_S and Y_D .

5.2. The Four Model-Fitting Equations

We first consider the death rate, which is defined as the proportion of live funds (in a good or sick state) that die during one transition period, which we take to be one year. For the transition matrix in (5.1), only sick funds can die in one transition. Thus, the death rate equals the product of the steady-state probability that a fund is sick times the transition probability from sick to dead. By (5.1) and (5.3), the *death rate* is

$$\delta = \pi_S \cdot P_{S,D} = \frac{1 - p}{2 - p - r}(1 - q - r) = \pi_D. \quad (5.4)$$

We now introduce two equations determined by the persistence. For greater model flexibility, we allow different persistence in states G and S . The two *DTMC-persistence* equations are:

$$\gamma_G \cdot Y_G = p \cdot Y_G + (1 - p) \cdot Y_S \quad (5.5)$$

and

$$\gamma_S \cdot Y_S = q \cdot Y_G + r \cdot Y_S + (1 - q - r) \cdot Y_D . \quad (5.6)$$

We explain these DTMC-persistence equations as follows: In equation (5.5), the fund starts with state G ; in equation (5.6) the fund starts with state S . The left side describes expected return in the next period calculated using the relevant persistence factor, whereas the right side calculates expected return in the next period using the transition probabilities of the DTMC in (5.1).

Our fourth equation is for the steady-state variance of the annual returns. Since we are working with returns relative to the mean, the variance of the steady-state return coincides with the second moment. Thus, the *variance* equation is

$$\sigma^2 = \pi_G \cdot Y_G^2 + \pi_S \cdot Y_S^2 + \pi_D \cdot Y_D^2 . \quad (5.7)$$

5.3. Explicit Formulas for the Transition Probabilities

We now derive formulas for the DTMC transition probability parameters p , q and r in terms of Y_G , Y_S , Y_D , γ_G , γ_S and δ using the three equations (5.4), (5.5) and (5.6).

The three formulas. Assuming that γ_G , γ_S , δ , Y_G , Y_S and Y_D are specified, the three equations in (5.4), (5.5), and (5.6) produce three equations in the three unknowns p , q and r . We first observe that the variable p can be solved from the single equation in (5.5), because that is a single equation for the single unknown variable p . The solution is

$$p = \frac{\gamma_G \cdot Y_G - Y_S}{Y_G - Y_S} . \quad (5.8)$$

Having found the explicit expression for p in (5.8), we substitute in for p to obtain two equations in the remaining two unknowns q and r . Indeed, given p , we can rewrite each of the two remaining equations to express q directly as a function of r . First, from (5.4), we get

$$q \equiv q(r) = 1 - r - \frac{\delta(2 - p - r)}{1 - p} = 1 - \delta \left(\frac{2 - p}{1 - p} \right) - r \frac{(1 - p - \delta)}{(1 - p)} . \quad (5.9)$$

Since $\delta < 1 - p$ by (5.4), the function $q(r)$ in (5.9) is necessarily strictly decreasing in r .

Next, (5.6) can be rewritten as

$$q \equiv q(r) = \frac{\gamma_S \cdot Y_S - Y_D - r(Y_S - Y_D)}{Y_G - Y_D} = \frac{(\gamma_S - r)Y_S - (1 - r)Y_D}{Y_G - Y_D}. \quad (5.10)$$

Combining the two equations (5.9) and (5.10), we get an explicit expression for r , first in terms of p and then in terms of the basic model parameters, namely,

$$r = \frac{\left(\frac{1-p-\delta(2-p)}{1-p}\right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D}\right)}{\left(\frac{1-p-\delta}{1-p}\right) - \left(\frac{Y_S - Y_D}{Y_G - Y_D}\right)} = \frac{\left(\frac{(1-\delta)(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G}\right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D}\right)}{\left(\frac{(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G}\right) - \left(\frac{Y_S - Y_D}{Y_G - Y_D}\right)} \quad (5.11)$$

To be feasible, we of course need $0 \leq q \leq 1 - r$ and $0 \leq r \leq 1$. Formulas (5.9) and (5.11) simplify when $\delta = 0$.

Commentary. We now want to determine what parameter values can occur. Figure 7 shows the three parameters as a function of δ in the base case with $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$ and $\gamma_G = \gamma_S = 0.5$.

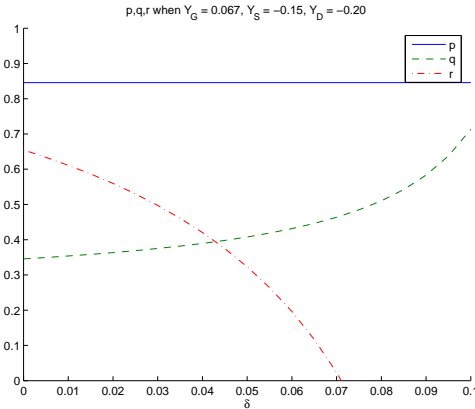


Figure 7: The DTMC parameter values p , q and r as a function of δ when $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$ and $\gamma_G = \gamma_S = 0.5$

From (5.8) we see that p is a linear function of γ_G with positive slope $Y_G/(Y_G - Y_S)$. If $Y_S \leq 0$, then we necessarily have $\gamma_G < p < 1$. The minimum possible value of p , attained when $\gamma_G = 0$, is $|Y_S|/(Y_G + |Y_S|)$. For example, if $Y_G = 0.05 > 0 > Y_S = -0.15$, then the minimum value of p is 0.75 (at $\gamma_G = 0$) and the slope is 0.25. On the other hand, if $Y_G > Y_S > 0$, then we must have $p \leq \gamma_G$. If, instead, $Y_G > Y_S > 0$, then we require that $\gamma_G \cdot Y_G > Y_S$.

Under the general condition that $Y_G > Y_S > Y_D$, we see that $q \equiv q(r)$ via (5.10) is a strictly decreasing function of r . The largest possible value of q occurs for $r = 0$, which is $(\gamma_S \cdot Y_S - Y_D)/(Y_G - Y_D)$. In order for q to be feasible (nonnegative), we must have that largest

possible value be nonnegative. Hence to have a feasible nonnegative value of q , we must have $\gamma_S \cdot Y_S \geq Y_D$. That is always satisfied provided that $Y_D \leq 0$ (given that $Y_G > Y_S > Y_D$).

From (5.10) alone, we can find an upper bound on r in terms of γ_S , Y_S and Y_D . If $0 > Y_S > Y_D$, then we must have $(1-r)|Y_D| \geq (r-\gamma_S)|Y_S|$, so that

$$r < \frac{|Y_D/Y_S| + \gamma_S}{|Y_D/Y_S| + 1} < 1 \quad \text{for } 0 < \gamma_S < 1, \quad (5.12)$$

where $|Y_D/Y_S| > 1$. On the other hand, if $Y_S \geq 0 > Y_D$, then we have

$$r < \frac{(|Y_D|/Y_S) - \gamma_S}{(|Y_D|/Y_S) + 1} < 1, \quad (5.13)$$

where now $|Y_D|/Y_S$ can assume a wide range of values.

When $Y_G > 0 \geq Y_S > Y_D$, r has the form $r = (a - B)/(A - b)$, where $a < A$ and $b < B$, so that we always have $r < 1$. We then have $r > 0$ if and only if either $a > B$ or $A < b$; r is negative otherwise. To have $r > 0$, we must have

$$a - B \equiv \left(\frac{(1-\delta)(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G} \right) - \left(\frac{\gamma_S \cdot Y_S - Y_D}{Y_G - Y_D} \right) > 0 \quad (5.14)$$

or

$$b - A \equiv \left(\frac{Y_S - Y_D}{Y_G - Y_D} \right) - \left(\frac{(1-\gamma_G)Y_G - \delta(Y_G - Y_S)}{(1-\gamma_G)Y_G} \right) > 0. \quad (5.15)$$

Examination of (5.11) shows that there can be difficulties in r as $\gamma \uparrow 1$, because the term $\delta(Y_G - Y_S)/(1-\gamma)Y_G$ appearing in the terms a and A blows up as $\gamma \uparrow 1$. The difficulty occurs approximately for γ such that

$$\frac{\delta(Y_G - Y_S)}{(1-\gamma)Y_G} = 1 - \frac{Y_S - Y_D}{Y_G - Y_D}. \quad (5.16)$$

In summary, from this analysis, we see that there is an upper limit on how high the death rate δ can be before the parameter r becomes negative. For the other parameters we consider, this limit is $\delta = 0.07$. We will see that the CTMC model allows higher values of δ , up to $\delta = 0.13$ for these parameter values.

5.4. Determining All Model Parameters

We now put everything together to develop an algorithm for computing all the model parameters.

An iterative algorithm. There are several ways we may proceed. We choose to specify Y_S and Y_D in addition to δ , γ_G , γ_S and σ . (This decision is supported by the fact that the model parameters are less sensitive to Y_S and Y_D than to Y_G , as we will see in §6.) Specifying these two quantities determines all the parameters. We then calculate the model parameters iteratively. We do so by guessing Y_G , which enables us to directly calculate the DTMC parameters p , q and r . and then the steady-state probability vector π . Given π , we can then calculate σ from (5.7). We then iterate until the calculated σ agrees with the initially specified value of σ .

Although it is not entirely evident from the equations, because π depends on Y_G too, experience indicates that σ is an increasing function of Y_G , so it is easy to find the appropriate value of Y_G , e.g., by performing bisection search. A simple plot of σ versus Y_G verifies this property, and reveals the appropriate value of Y_G . We illustrate in Figure 8 below for the special case in which $Y_S = -0.15$, $Y_D = -0.20$, $\gamma_G = \gamma_S = 0.5$ and $\delta = 0.03$.

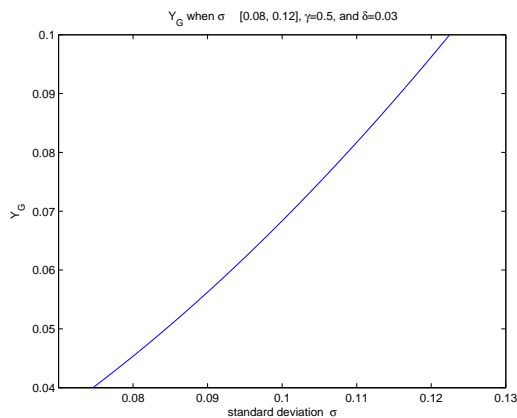


Figure 8: The standard deviation of relative return σ versus Y_G when $Y_S = -0.15$, $Y_D = -0.20$, $\gamma_G = \gamma_S = 0.5$, and $\delta = 0.03$.

Denominating in terms of σ . For additional insight, it is helpful to express our returns in units of the standard deviation σ . We can divide through by σ^2 in (5.7) to obtain

$$1 = \pi_G \cdot (Y_G/\sigma)^2 + \pi_S \cdot (Y_S/\sigma)^2 + \pi_D \cdot (Y_D/\sigma)^2 . \quad (5.17)$$

Observe that the steady-state probability vector π in (5.3) and the death rate δ in (5.4) depend only on DTMC parameters p , q and r , while the equations (5.8), (5.10) and (5.11) for p , q and r are invariant under scale multiples of Y_G , Y_S and Y_D .

Paralleling Figure above, it is useful to see how Y_G/σ behaves as a function of σ when we fix Y_S/σ and Y_D/σ in addition to δ and γ . It turns out that Y_G/σ is nearly constant after fixing

Y_S/σ and Y_D/σ . For the special case in which $Y_S/\sigma = -1.5$, $Y_D/\sigma = -2.0$, $\gamma_G = \gamma_S = 0.5$, and $\delta = 0.03$, $Y_G/\sigma \approx 0.685$ for σ ranging from 0.07 to 0.13.

5.5. Numerical Examples

We now consider some numerical examples. Our base case is $\delta = 0$, $\gamma_G = \gamma_S = \gamma = 0.5$, $\sigma = 0.1$, $Y_S = -0.15$, and $Y_D = -0.20$. If we try $Y_G = 0.067$, then we get $p = 0.8456$, $q = 0.3456$, $r = 0.6544$, $\pi_G = 0.6912$, $\pi_S = 0.3088$, $\pi_D = 0$ and $\sigma = 0.1002$.

Table 1 shows parameter values for various δ , γ_G , γ_S , with Y_S , Y_D and σ fixed as above, the return Y_G is calculated iteratively by the method above. The last line of the Table 1 shows that r is negative. Our numerical analysis shows that r reaches 0 and becomes negative when δ is above 0.07. The CTMC model is more flexible, providing a solution for $\delta \leq 0.13$.

Table 1: Parameter value sets

δ	γ_G	γ_S	σ	Y_G	Y_S	Y_D	Calculated σ	p	q	r
0	0.5	0.5	0.1	0.067	-0.15	-0.20	0.1002	0.8456	0.3456	0.6544
0.03	0.5	0.5	0.1	0.0685	-0.15	-0.20	0.1001	0.8432	0.3719	0.5030
0.06	0.5	0.5	0.1	0.070	-0.15	-0.20	0.1001	0.8409	0.4207	0.2282
0.07	0.5	0.5	0.1	0.075	-0.15	-0.20	0.1001	0.8401	0.4474	0.0796
0	0.6	0.4	0.1	0.076	-0.15	-0.20	0.1000	0.8655	0.3982	0.6018
0.03	0.6	0.4	0.1	0.077	-0.15	-0.20	0.1002	0.8643	0.4320	0.4069
0.06	0.6	0.4	0.1	0.0775	-0.15	-0.20	0.1000	0.8637	0.5068	-0.0127

5.6. The Lockup Premium Calculation

The lockup premium calculation is essentially the same as in §4, following (4.3) and (4.4), but using the DTMC in (5.2) to compute the expected relative returns for the n -year lockup. In particular,

$$A_n \equiv \frac{C_n}{n} \equiv R_1 - \frac{1}{n} \sum_{i=1}^n R_i = \gamma Y_G - \frac{1}{n} \sum_{i=1}^n (P_{G,G}^i \cdot Y_G + P_{G,S}^i \cdot Y_S + P_{G,D}^i \cdot Y_D) , \quad (5.18)$$

where P^i is the i^{th} power of the transition matrix P in (5.2).

For example, if we set $Y_G = 0.067$, $Y_S = -0.15$, $Y_D = -0.20$, $\gamma_G = \gamma_S = \gamma = 0.5$ and $\delta = 0.03$, we get $p = 0.846$, $q = 0.375$ and $r = 0.497$ from §5.4. The difference between a 2-year lockup and a 1-year lockup is 1.2676%. Figure 4 in §4 shows the lockup premium function for three values of δ : 0.00, 0.03 and 0.06.

6. Sensitivity Analysis for the DTMC model

The mathematical models developed here are useful to estimate how the lockup premium depends on the different variables. We describe highlights of such analyses here and present more details in the appendix.

Our results here are related to the standard *base case* with $\gamma_G = \gamma_S = \gamma = 0.5$, $\sigma = 0.1$, $Y_S = -0.15$, $Y_D = -0.20$ and $\delta = 0.03$, as in the second row of Table 1.

Figure 9 shows the lockup premium for three values of γ : 0.4, 0.5 and 0.6 while Figure 10 shows the lockup premium for three values of σ : 0.09, 0.10 and 0.11. In both cases, these changes produce minor changes in Y_G and the other model parameters; see the Appendix.

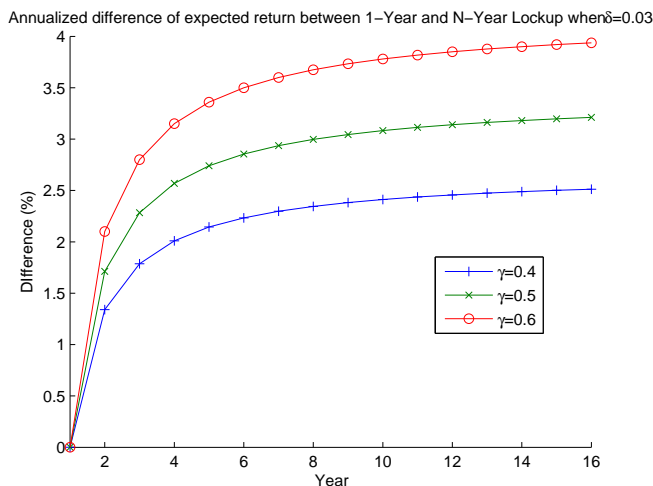


Figure 9: The lockup premium for the DTMC model in the base case with three values of γ : 0.4, 0.5 and 0.6.

We next consider how the DTMC model parameters p , q and r depend on the other driving variables. To supplement Figure 7 and the commentary in §5.3, Figure 11 shows how these parameters p , q and r depend on γ (assuming $\gamma_G = \gamma_S = \gamma$) and each of the return values Y_G , Y_S and Y_D , taken one at a time. We see that the model becomes unstable if γ gets very large, but there is nice near-linear behavior for values of $\gamma \leq 0.5$. We also see that the parameters p , q and r are considerably more sensitive to Y_G than the other two returns Y_S and Y_D .

Lastly, we consider how the DTMC lockup premium for a fixed lockup period depends on three variables δ , γ , and σ . Figure 12 shows how the three-year lockup premium depends on two of the three variables while fixing the remaining variable. We see that the three-year lockup premium is reasonably well approximated by a linear function of γ and σ , respectively; there is concavity in γ but convexity in σ . Also, the three-year lockup premium is relatively

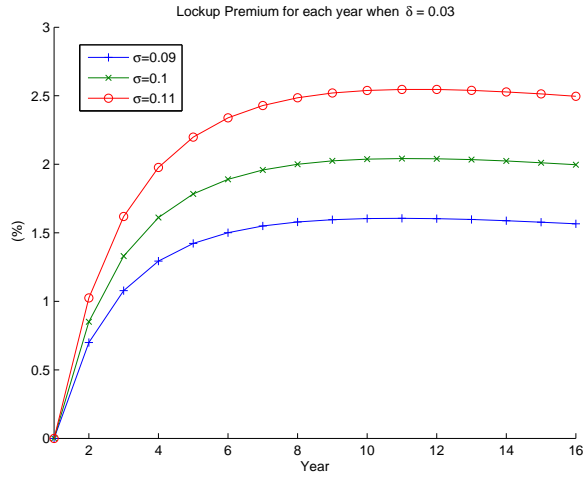


Figure 10: The lockup premium for the DTMC model in the base case with three values of σ : 0.09, 0.10 and 0.11.

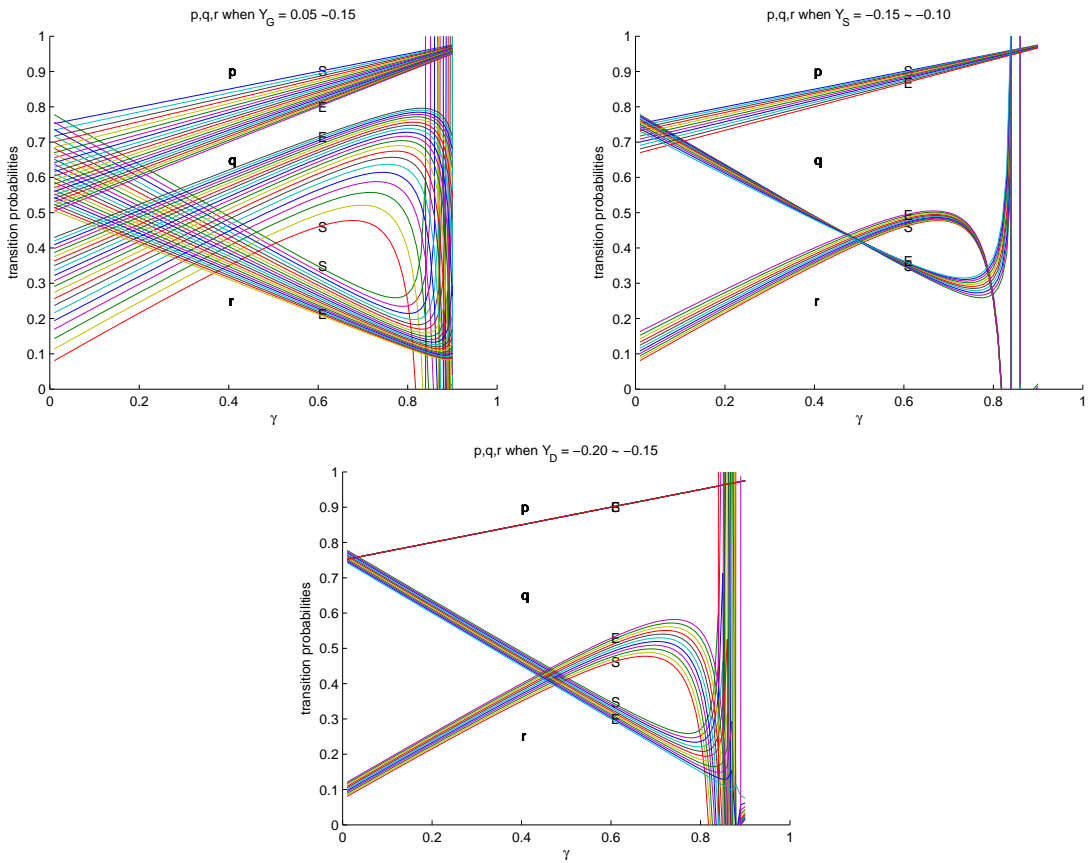
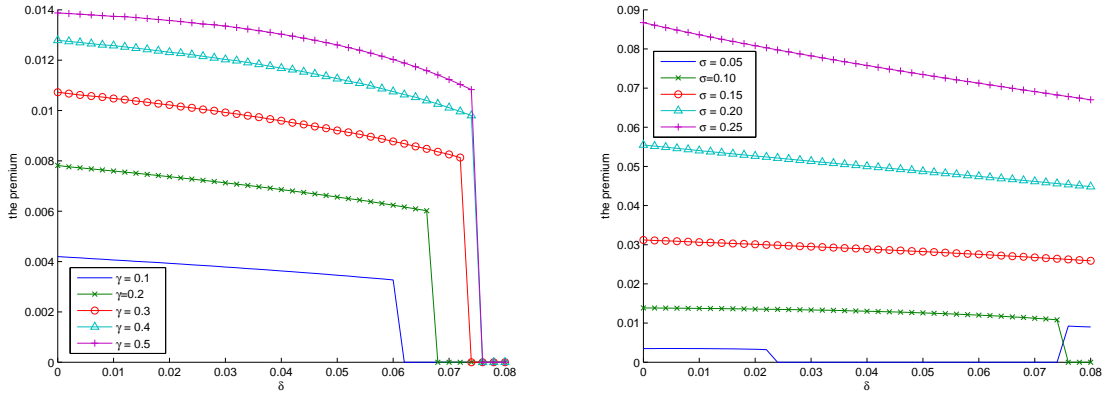
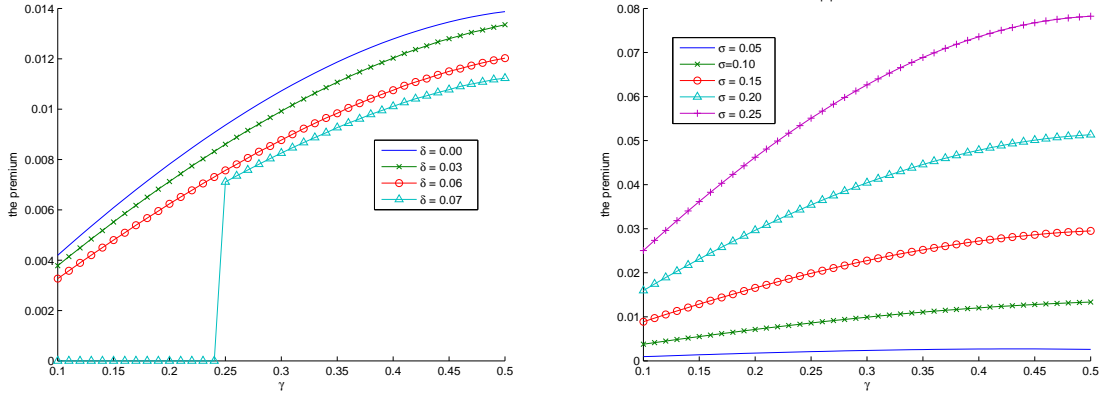


Figure 11: The parameters p , q and r as a function of γ in the base case for values of Y_G ranging from 0.5 (starting value, denoted by S) to 0.15 (ending value, denoted by E), Y_S ranging from -0.15 to -0.10 , and Y_D ranging from -0.20 to -0.15

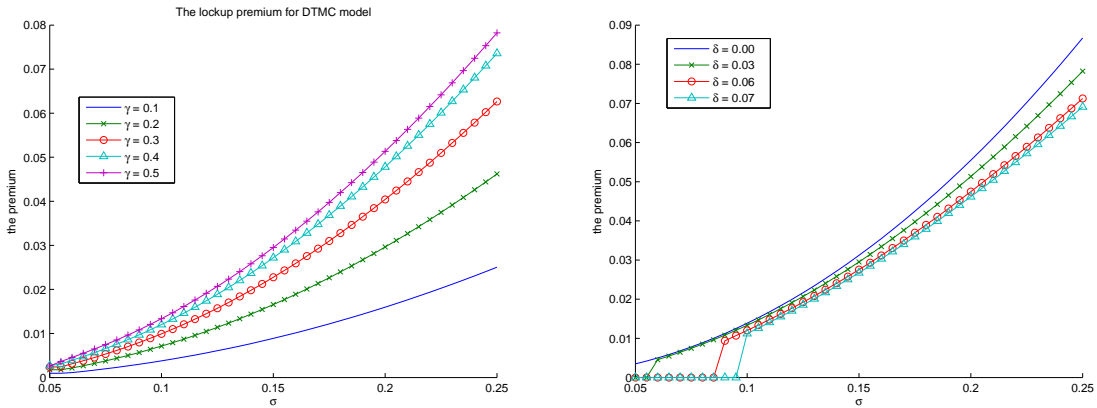
insensitive to δ . However, we note that the CTMC model predicts greater impact of δ on the three-year lockup premium, as illustrated in Figure 3.



For $\delta = 0.00$ to 0.08 , (i) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\sigma = 0.1$ (ii) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\gamma = 0.5$



For $\gamma = 0.1$ to 0.5 , (iii) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\sigma = 0.1$ (iv) $\sigma = 0.05, 0.1, 0.15, 0.2, 0.25$ with $\delta = 0.03$



For $\sigma = 0.05$ to 0.25 , (v) $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ with $\delta = 0.03$ (vi) $\delta = 0.00, 0.03, 0.06, 0.07$ with $\gamma = 0.5$

Figure 12: The three-year lockup premium for the DTMC model with $Y_S = -0.15, Y_D = -0.20$. The lockup premium is set to 0 if q or r becomes negative.

7. The CTMC Model

In the hope of providing better predictions, we now propose a more sophisticated model, which uses a CTMC - specifically a continuous-time birth-and-death (BD) process - and a DTMC; see Chapter 6 of Ross (2003) for background. Now the fund changes state in continuous time, but the investment updates take place in discrete time. We can study the model behavior as a function of the time T allowed for updates, if we wish, but we let $T = 1$ in our numerical examples.

With the DTMC model, when a fund starts in a good state, at least two years are required for the fund to become dead. In contrast, with the CTMC model, a fund can become dead at any time. There is a cost, however: For the CTMC model, we are unable to fit the model parameters simply by solving three equations in three unknowns as we had for the DTMC in §5.3. Instead, we do the model fitting numerically. However, the parameter fitting for the CTMC model is not substantially harder than for the DTMC model, when we consider the iteration needed to find Y_G for given σ with the DTMC, discussed in §5.4.

In our proposed CTMC model we replace the three-state absorbing DTMC in (5.1) by a two-state absorbing BD process. The states now are G and S ; we do not directly use the state D here, but we will be able to account for it. As usual, we specify the BD process by specifying its infinitesimal transition matrix Q . That means we specify the birth and death rates. Let μ_G be the death rate in G , the rate of transition down to state S from state G . Let λ_S be the birth rate in state S , the rate of transition up to state G from state S . Let μ_S be the death rate in state S , the rate of transition down to state D from state S . We may leave state S to go to state D , but we get absorbed in D . We do not need to include the state D in our transition rate matrix. Here is the infinitesimal transition matrix, with the parameters above:

$$Q = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} -\mu_G & \mu_G \\ \lambda_S & -(\lambda_S + \mu_S) \end{pmatrix}. \quad (7.1)$$

7.1. The Transition Matrix

We now want to derive the time-dependent transition probability matrix $P(t)$ for this BD process. It is well-known that $P(t)$ is the solution to the matrix ordinary differential equation

$$P(t)' = P(t)Q, \quad P(0) = I, \quad (7.2)$$

where I is the identity matrix, so that $P(t)$ is the matrix exponential $P(t) = e^{tQ}$. If we diagonalize Q so that $Q = UDU^{-1}$, where D is a diagonal matrix and $UU^{-1} = I$, then we

can write $P(t) = Ue^{tD}U^{-1}$; see §4.8 and the appendix of Karlin and Taylor (1975). Since D is a diagonal matrix, the i^{th} diagonal element of e^{tD} is related to the corresponding diagonal element of D , i.e., $(e^{tD})_{i,i} = e^{D_{i,i}t}$ for $t > 0$. Let $\Lambda(t)$ be a diagonal matrix of the form

$$\Lambda(t) = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} e^{\eta_G t} & 0 \\ 0 & e^{\eta_S t} \end{pmatrix}, \quad (7.3)$$

with the two parameters η_G and η_S being the eigenvalues of the matrix Q , while the columns of U are the associated right eigenvectors. The resulting formula for $P(t)$ is

$$P(t) = U\Lambda(t)U^{-1}. \quad (7.4)$$

The characterization (7.4) implies that $P_{i,j}(t) = A_{i,j}e^{\eta_1 t} + B_{i,j}e^{\eta_2 t}$ for $t \geq 0$ and all state pairs (i, j) , where η_1 and η_2 are the eigenvalues of Q and $A_{i,j}$ and $B_{i,j}$ are appropriate constants. Since $P(0) = I$, we necessarily have $A_{i,i} + B_{i,i} = 1$ for $i = 1, 2$ and $A_{i,j} + B_{i,j} = 0$ for $i \neq j$. If $0 > \eta_1 > \eta_2$, then asymptotically $P_{i,j}(t) \sim A_{i,j}e^{-\eta_1 t}$ as $t \rightarrow \infty$, which means that the ratio approaches 1. As a consequence, necessarily $A_{i,j} > 0$ for all state pairs (i, j) ; $B_{i,j} = -A_{i,j}$ for $i \neq j$.

As usual, we find the eigenvalues of Q by finding the determinant of $\eta I - Q$. The characteristic polynomial as a function of the variable η is the quadratic equation

$$(\eta + \lambda_S + \mu_S)(\eta + \mu_G) - \lambda_S \mu_G = 0, \quad (7.5)$$

which has two strictly negative roots, as required for the formula in (7.3) to yield bonafide probabilities. In particular, solving the quadratic equation, we obtain

$$\eta = \frac{-(\lambda_S + \mu_S + \mu_G) \pm \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S \mu_G}}{2}. \quad (7.6)$$

Since the term inside the square root can be rewritten as $(\mu_G - \mu_S)^2 + \lambda_S^2 + 2\mu_G \lambda_S + 2\lambda_S \mu_S$, it is nonnegative. The first term clearly dominates the square root in absolute value. So we indeed have two negative roots.

Now we find eigenvectors corresponding to the eigenvalues in (7.6). Given eigenvalues, the eigenvectors form the null space of $(Q - \eta I)$, i.e., a matrix U such that $(Q - \eta I)U = 0$. We arrange eigenvalues η_G, η_S as η matrix:

$$\eta = \begin{pmatrix} \eta_G \\ \eta_S \end{pmatrix} = \begin{pmatrix} \frac{-(\lambda_S + \mu_S + \mu_G) - \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S \mu_G}}{2\lambda_S} \\ \frac{-(\lambda_S + \mu_S + \mu_G) + \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S \mu_G}}{2\lambda_S} \end{pmatrix}. \quad (7.7)$$

Such an eigenvectors matrix U , where the columns of U are eigenvectors of Q , can be found by algebraic manipulation or by symbolic calculation package like Mathematica. One such

eigenvalue matrix is

$$U = \begin{pmatrix} \frac{-(\lambda_S + \mu_S + \mu_G) - \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}}{2\lambda_S} & \frac{-(\lambda_S + \mu_S + \mu_G) + \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}}{2\lambda_S} \\ 1 & 1 \end{pmatrix}. \quad (7.8)$$

Its inverse matrix is then

$$U^{-1} = \begin{pmatrix} -\frac{\lambda_S}{\sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}} & \frac{\lambda_S - \mu_G + \mu_S + \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}}{2\sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}} \\ \frac{\lambda_S}{\sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}} & \frac{-\lambda_S + \mu_G - \mu_S + \sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}}{2\sqrt{(\lambda_S + \mu_S + \mu_G)^2 - 4\mu_S\mu_G}} \end{pmatrix}. \quad (7.9)$$

Thus, we now have derived the components of $P(t)$ in (7.4). We have derived $P(t)$ as a nonlinear function of μ_G , λ_S and μ_S from (7.7)-(7.9).

7.2. The Associated Ergodic DTMC

We use the time-dependent transition matrix $P(t)$ in the role of (5.1). We now specify an updating time interval of length T . We then replace a dead fund by a good fund at time T . So we make an ergodic two-state DTMC with transition matrix

$$P = \begin{matrix} G \\ S \end{matrix} \begin{pmatrix} 1 - P_{G,S}(T) & P_{G,S}(T) \\ 1 - P_{S,S}(T) & P_{S,S}(T) \end{pmatrix}. \quad (7.10)$$

We construct P in (7.10) by letting $P_{G,S} = P_{G,S}(T)$ and $P_{S,S} = P_{S,S}(T)$ and then making the DTMC ergodic by letting the row sums be 1. In other words, we insert an instantaneous transition from state D to G at time T , which is the time of a single transition in the DTMC.

Paralleling (5.3), this two-state DTMC has steady-state probability vector π , where

$$\pi \equiv (\pi_G, \pi_S) = \left(\frac{1 - P_{S,S}(T)}{1 - P_{S,S}(T) + P_{G,S}(T)}, \frac{P_{G,S}(T)}{1 - P_{S,S}(T) + P_{G,S}(T)} \right). \quad (7.11)$$

7.3. Parameter Fitting in the CTMC Model

We now proceed toward parameter fitting for this new model. Paralleling (5.4), we have the time-dependent death rate being

$$\delta \equiv \delta(T) = \pi_G\{1 - P_{G,G}(T) - P_{G,S}(T)\} + \pi_S\{1 - P_{S,G}(T) - P_{S,S}(T)\}. \quad (7.12)$$

Just as for the DTMC, we can derive the survival probability from the CTMC model, which is closely related to the death rate. At time t , the survival probability of a fund is defined as $S(t) = P_{G,G}(t) + P_{G,S}(t)$ for $t \geq 0$. Figure 13 displays the survival probabilities for the CTMC model with $\delta = 0.03, 0.06$ and 0.09 . The survival probability for $\delta = 0.09$ is possible only in the CTMC model, since for the DTMC model, r becomes negative when $\delta \approx 0.07$. As we see from Figure 13, median fund life is less than 10 year for $\delta = 0.09$. Since this median hedge

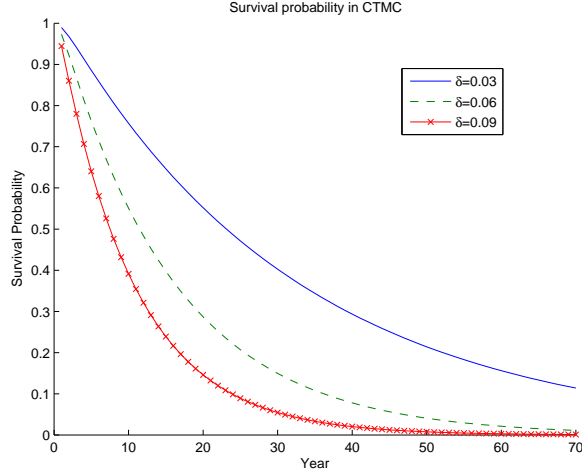


Figure 13: Survival probability curves for the CTMC model with $\delta = 0.03, 0.06,$ and 0.09 . The parameters values $p, q, r, Y_G, Y_S,$ and Y_D are from table 1.

fund life is within the range of Rouah (2006) and Park (2006), it should be worth considering $\delta = 0.09$ with the CTMC model.

In addition to equation (7.12), we will need analogs of equations (5.5) and (5.6). These will involve transitions in the DTMC over the time interval of length T . In particular, we obtain the new equations

$$\gamma_G \cdot Y_G = P_{G,G}(T) \cdot Y_G + P_{G,S}(T) \cdot Y_S + \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D \quad (7.13)$$

and

$$\gamma_S \cdot Y_S = P_{S,G}(T) \cdot Y_G + P_{S,S}(T) \cdot Y_S + \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D . \quad (7.14)$$

This is just as for the DTMC model before, except that we have to add the term for a D state when the fund starts in the G state at the beginning of the year. Since we are thinking of yearly updates, we let $T = 1$.

We now want to do the model fitting. We want to determine the three parameters μ_G, λ_S and μ_S , exploiting the three equations (7.12), (7.13) and (7.14), but we have been unable to obtain explicit solutions for the desired parameters as we did in §5.3. So we use an iterative algorithm.

We start with a candidate initial parameter triple $(\mu_G, \lambda_S, \mu_S)$. Given that parameter triple and the specified time T , we calculate the transition probabilities $P_{G,G}(T), P_{G,S}(T), P_{S,G}(T),$ and $P_{S,S}(T)$ in (7.4)–(7.6) by calculating the eigenvalues and eigenvectors of the infinitesimal matrix Q in (7.1). Afterwards we calculate the steady-state probability vector $\pi \equiv (\pi_G, \pi_S)$ in (7.11) of the two-state DTMC in (7.10). We then calculate the right-hand sides of the three

equations (7.12)–(7.14). Our goal is to have three bonafide equations, where the two sides of the equations are equal, but in the iteration we will not achieve that. Based on the errors we see, we update the parameter triple $(\mu_G, \lambda_S, \mu_S)$ and repeat until the errors in the three equations (7.12)–(7.14) are negligible.

Since we are confronted with a three-dimensional iteration, we do not want to proceed in a haphazard way. Hence, we apply nonlinear programming to do this iteration. The idea is to find parameter triple $(\lambda_S, \mu_S, \mu_G)$ minimizing errors between the right-hand and left-hand sides of equations (7.12), (7.13) and (7.14). To formulate a minimization problem, we define three error functions ϵ_1 , ϵ_2 and ϵ_3 as a function of parameter triple $(\lambda_S, \mu_S, \mu_G)$ as follows:

$$\begin{aligned}\epsilon_1 &\equiv \epsilon_1(\lambda_S, \mu_S, \mu_G) = \delta(T) - \pi_G\{1 - P_{G,G}(T) - P_{G,S}(T)\} - \pi_S\{1 - P_{S,G}(T) - P_{S,S}(T)\}, \\ \epsilon_2 &\equiv \epsilon_2(\lambda_S, \mu_S, \mu_G) = \gamma_G \cdot Y_G - P_{G,G}(T) \cdot Y_G - P_{G,S}(T) \cdot Y_S - \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D, \\ \epsilon_3 &\equiv \epsilon_3(\lambda_S, \mu_S, \mu_G) = \gamma_S \cdot Y_S - P_{S,G}(T) \cdot Y_G - P_{S,S}(T) \cdot Y_S - \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D.\end{aligned}\tag{7.15}$$

Our objective, then, is to find λ_S , μ_S and μ_G such that $\epsilon_1(\lambda_S, \mu_S, \mu_G) = \epsilon_2(\lambda_S, \mu_S, \mu_G) = \epsilon_3(\lambda_S, \mu_S, \mu_G) = 0$. To obtain values of ϵ_1, ϵ_2 , and ϵ_3 for a given parameter triple of λ_S , μ_S and μ_G , we have to calculate $P_{G,G}(T), P_{G,S}(T), P_{S,G}(T)$, which are elements of $P(t)$ matrix in (7.4). As indicated above, this involves finding eigenvalues and eigenvectors of Q matrix in (7.1). From (7.6), we derived eigenvalues as a function of λ_S , μ_S and μ_G . Given the eigenvalues, the eigenvectors can be calculated as in (7.8), but also in other ways. Since Q is only a 2×2 matrix, calculation of the eigenvectors for given eigenvalues can be done easily. One way is to use the Schur decomposition algorithm, as in Anderson et al. (1999), which is implemented in MATLAB as the *eig* function. Then $\Lambda(t)$ can be calculated easily from (7.3), so we can easily compute the U and Λ matrices numerically. The final step is to compute $P_{G,G}(T)$, $P_{G,S}(T)$ and $P_{S,G}(T)$ from $P(t) = U\Lambda(t)U^{-1}$.

We can obtain the desired parameter triple $(\lambda_S, \mu_S, \mu_G)$ by solving the following constrained minimization problem:

$$\begin{aligned}&\min_{\lambda_S, \mu_S, \mu_G} \max\{|\epsilon_1|, |\epsilon_2|, |\epsilon_3|\} \\ &\text{such that} \\ \epsilon_1 &= \delta(T) - \pi_G(1 - P_{G,G}(T) - P_{G,S}(T)) - \pi_S\{1 - P_{S,G}(T) - P_{S,S}(T)\}, \\ \epsilon_2 &= \gamma_G \cdot Y_G - P_{G,G}(T) \cdot Y_G - P_{G,S}(T) \cdot Y_S - \{1 - P_{G,G}(T) - P_{G,S}(T)\}Y_D, \\ \epsilon_3 &= \gamma_S \cdot Y_S - P_{S,G}(T) \cdot Y_G - P_{S,S}(T) \cdot Y_S - \{1 - P_{S,G}(T) - P_{S,S}(T)\}Y_D \\ \lambda_S, \mu_S, \mu_G &\geq 0\end{aligned}\tag{7.16}$$

We regard γ_G , γ_S , Y_G , Y_S and Y_D as given constants, and we regard the BD rates λ_S , μ_S and μ_G as the variables. Since the transition probabilities $P_{G,G}(T)$, $P_{G,S}(T)$ and $P_{S,G}(T)$ are functions of the BD rates λ_S , μ_S and μ_G through the eigenvalue and eigenvector calculation, we must regard (7.16) as a nonlinear programming (NLP) problem, for which it is natural to apply an iterative procedure. However, since we only have three variables, we are able to solve the NLP (7.16) easily. One effective way is to use Sequential Quadratic Programming (SQP), as in Schittkowski (1985). With SQP, at each iteration, an approximation is made of the Hessian of the Lagrangian function using a quasi-Newton updating method. That is then used to generate a QP subproblem whose solution is used to form a search direction for a line search procedure. This algorithm is implemented in MATLAB via the functions *fminsearch* and *fmincon*. Both functions solve (7.16) within seconds.

In addition to fitting λ_S , μ_S and μ_G , we want to calibrate σ^2 . To do so, we need to adjust the definition of σ^2 for the CTMC model. Suppose that π_G and π_S are the stationary probabilities for the transition matrix in (7.10). We let π'_D be the stationary probability that the fund dies at the end of 1 year when it starts alive before. This is equal to death rate δ in our definition:

$$\pi'_D = \delta = \pi_G\{1 - P_{G,G}(1) - P_{G,S}(1)\} + \pi_S\{1 - P_{S,G}(1) - P_{S,S}(1)\} \quad (7.17)$$

We then also define π'_G and π'_S accordingly, using (7.11):

$$\begin{aligned} \pi'_G &= \pi_G \cdot P_{G,G}(1) + \pi_S \cdot P_{S,G}(1) \\ \pi'_S &= \pi_G \cdot P_{G,S}(1) + \pi_S \cdot P_{S,S}(1) \end{aligned} \quad (7.18)$$

where (π_G, π_S) is defined in (7.11). Finally, the variance satisfies

$$\sigma^2 = \pi'_G \cdot Y_G^2 + \pi'_S \cdot Y_S^2 + \pi'_D \cdot Y_D^2, \quad (7.19)$$

where π' is defined in (7.17) and (7.18). It turns out that we can easily achieve any desired σ , such as $\sigma \approx 0.1$, by iterating Y_G . Given (7.19), this iteration step is essentially the same as for the DTMC in §5.4.

Below are parameter values obtained using the NLP in (7.16) and iterating Y_G values. In the following table, ϵ records the maximum absolute value of errors in equations (7.12), (7.13) and (7.14). As before, we let $T = 1$.

Unlike the DTMC model, where the parameter r becomes negative if δ exceeds 0.07 for the base-case parameter values, for the CTMC we can fit the model to δ up to around 0.13. When $\delta \approx 0.13$, we observe that the CTMC lockup premium becomes nearly 0.

Table 2: Parameter value sets for the CTMC model with $\gamma_G = \gamma_S = 0.5$

δ	μ_G	λ_S	μ_S	Y_G	Y_S	Y_D	Calculated σ	ϵ
0	0.2410	0.4791	0.0000	0.0670	-0.15	-0.20	0.1003	2.3033×10^{-6}
0.03	0.2204	0.5531	0.1240	0.0690	-0.15	-0.20	0.1005	3.9441×10^{-8}
0.06	0.2264	0.7017	0.3488	0.0700	-0.15	-0.20	0.1001	1.2849×10^{-6}
0.07	0.2286	0.7916	0.4741	0.0701	-0.15	-0.20	0.0997	1.4160×10^{-6}
0.09	0.2381	1.1063	0.8806	0.0710	-0.15	-0.20	0.0997	5.9465×10^{-7}

7.4. The Lockup Premium Calculation

Once we have fit all the parameters, we can calculate the lockup premium. The procedure is essentially the same as in §5.6. For a 1-year Lockup, the fund's annual return is

$$R_1 = P_{G,G}(1) \cdot Y_G + P_{G,S}(1) \cdot Y_S + \{1 - P_{G,G}(1) - P_{G,S}(1)\}Y_D = \gamma_G \cdot Y_G \quad (7.20)$$

The fund's expected return after the i^{th} year if the fund is under lockup is

$$\begin{aligned} R_i &= P_{G,G}(i) \cdot Y_G + P_{G,S}(i) \cdot Y_S \\ &+ [P_{G,G}(i-1) \cdot \{1 - P_{G,G}(1) - P_{G,S}(1)\} + P_{G,S}(i-1) \cdot \{1 - P_{S,G}(1) - P_{S,S}(1)\}] \cdot Y_D \\ &+ \{1 - P_{G,G}(i-1) - P_{G,S}(i-1)\} \cdot \gamma_G \cdot Y_G. \end{aligned} \quad (7.21)$$

Just as for the DTMC in §5.6, the cumulative difference of expected returns between a 1-year lockup and an n -year lockup is

$$C_n = \sum_{i=1}^n (R_1 - R_i) = nR_1 - \sum_{i=1}^n R_i \quad (7.22)$$

The lockup premium is then the average difference $A_n = C_n/n$. Figure 3 in §1 shows the lockup premium functions for four different values of δ , ranging from 0.00 to 0.09. The remaining parameter values are as specified in Table 2.

7.5. Premium Comparison between CTMC and DTMC

Having created both the DTMC and CTMC models, it is interesting to see how they compare. Since funds die more quickly in the CTMC model, we expect that the lockup premium for the CTMC to be lower than for the DTMC model, and that is what we see. There is no difference at all for $\delta = 0$, but as δ increases the difference between the estimated lockup premiums increases. We provide several plots and tables in the Appendix. Here we illustrate by showing the larger differences for $\delta = 0.07$ in Table 3 below.

Table 3: Lockup premium comparisons when $\delta = 0.07$

Lockup Period	2	3	4	5	6
DTMC	0.8822	1.1288	1.2215	1.2546	1.2593
CTMC	0.6428	0.9296	1.0642	1.1270	1.1528
(DTMC-CTMC)/DTMC (%)	27.1320	17.6480	12.8814	10.1705	8.4604

8. Conclusion

We have defined the hedge fund lockup premium as the average difference (per year) between the annual returns from investments in hedge funds, where one has a nominal one-year lockup and the other has an extended n -year lockup. We have developed DTMC and CTMC models to estimate the hedge-fund lockup premium as a function of the length n of the extended lockup period. To account for immediate redemption of investment when a hedge fund fails, we include a death state in the model. The lockup premium represents the cost of not being able to switch from sick funds to good funds while under the lockup condition. The effect of the lockup is mitigated by the death rate, and so is more difficult to analyze.

We have shown how the Markov chain models can be fit to basic hedge-fund performance measures, notably, the persistence of returns, γ (also allowing different γ_G and γ_S), the standard deviation of returns, σ , and the hedge-fund death rate δ . We then have applied the models to estimate how the lockup premium depends on these important performance measures. The models quantify how the lockup premium increases as a function of the persistence factor γ and the standard deviation σ , but decreases as a function of the death rate δ .

We examined the literature to see what researchers have concluded about hedge-fund performance persistence and the other hedge-fund performance measures, but we have found varying conclusions. We also performed our own statistical analysis using the TASS hedge fund data to estimate these hedge fund performance measures. We found strong evidence of persistence, but the specific persistence values cannot be predicted with great confidence, as is evident from the scatter plots in Figure 2. Thus we think we have been more successful showing how the lockup premium depends on the hedge-fund performance measures than in determining these performance measures themselves.

In §4 we provided a simple analysis without Markov chains to quantify the lockup premium in the case of no death. That analysis yields the explicit no-death lockup premium formulas in (4.4) and (4.7). In that case the lockup premium tends to be proportional to σ . In all

cases, the lockup premium is a concave function of n , initially increasing and then eventually decreasing for $\delta > 0$ because the fund will eventually die, so that the lockup will eventually provide no extra penalty. The simple approximation with $\delta = 0$ yields an upper bound.

The model fitting requires solving equations. For the DTMC, we were able to give explicit formulas for the three DTMC parameters p , q and r as a function of Y_G , Y_S , Y_D , γ_G and γ_S , but in order to calibrate the standard deviation of returns, σ , we needed to use an iterative method. For the CTMC we used a more involved iterative method based on nonlinear programming. For both models, we developed efficient algorithms for doing the model fitting.

We conclude that all three performance measures - δ , γ and σ - can have a significant impact on the lockup premium, but we predict that the effect will be negligible if either γ or σ is small. We estimated these key hedge-fund performance measures from the TASS data, but further work needs to be done to obtain reliable estimates.

The CTMC model is more realistic because the DTMC model requires two years for a transition from G to D . The CTMC model allows a wider range of δ - up to 0.13 instead of only up to 0.07 for the DTMC model - for the base case of parameters. Figure 3 shows for the CTMC model that the lockup premium for $\delta = 0.09$ is about half what it would be with $\delta = 0.00$.

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