

for some $\theta_j \in (0, 1)$, and this differs from $h_j(D_j f)(x)$ by less than $|h_j| \varepsilon/n$, using (41). By (42), it follows that

$$\left| f(x+h) - f(x) - \sum_{j=1}^n h_j (D_j f)(x) \right| \leq \sum_{j=1}^n |h_j| \varepsilon \leq |h| \varepsilon$$

for all h such that $|h| < r$.

This says that f is differentiable at x and that $f'(x)$ is the linear function which assigns the number $\sum h_j (D_j f)(x)$ to the vector $h = \sum h_j e_j$. The matrix $[f'(x)]$ consists of the row $(D_1 f)(x), \dots, (D_n f)(x)$; and since $D_1 f, \dots, D_n f$ are continuous functions on E , the concluding remarks of Sec. 9.9 show that $f \in \mathcal{C}^1(E)$.

THE CONTRACTION PRINCIPLE

We now interrupt our discussion of differentiation to insert a fixed point theorem that is valid in arbitrary complete metric spaces. It will be used in the proof of the inverse function theorem.

9.22 Definition Let X be a metric space, with metric d . If φ maps X into X and if there is a number $c < 1$ such that

$$(43) \quad d(\varphi(x), \varphi(y)) \leq c d(x, y)$$

for all $x, y \in X$, then φ is said to be a contraction of X into X .

9.23 Theorem If X is a complete metric space, and if φ is a contraction of X into X , then there exists one and only one $x \in X$ such that $\varphi(x) = x$.

In other words, φ has a unique fixed point. The uniqueness is a triviality. for if $\varphi(x) = x$ and $\varphi(y) = y$, then (43) gives $d(x, y) \leq c d(x, y)$, which can only happen when $d(x, y) = 0$.

The existence of a fixed point of φ is the essential part of the theorem. The proof actually furnishes a constructive method for locating the fixed point.

Proof Pick $x_0 \in X$ arbitrarily, and define $\{x_n\}$ recursively, by setting

$$(44) \quad x_{n+1} = \varphi(x_n) \quad (n = 0, 1, 2, \dots)$$

Choose $c < 1$ so that (43) holds. For $n \geq 1$ we then have

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq c d(x_n, x_{n-1}).$$

Hence induction gives

$$(45) \quad d(x_{n+1}, x_n) \leq c^n d(x_1, x_0) \quad (n = 0, 1, 2, \dots)$$

If $n < m$, it follows that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n+1}^m d(x_i, x_{i-1}) \\ &\leq (c^n + c^{n+1} + \dots + c^{m-1}) d(x_1, x_0) \\ &\leq [(1-c)^{-1} - c^m] d(x_1, x_0) \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\lim x_n = x$ for some $x \in X$.

Since φ is a contraction, φ is continuous (in fact, uniformly continuous) on X . Hence

$$\varphi(x) = \lim_{n \rightarrow \infty} \varphi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

THE INVERSE FUNCTION THEOREM

The inverse function theorem states, roughly speaking, that a continuously differentiable mapping f is invertible in a neighborhood of any point x at which the linear transformation $f'(x)$ is invertible:

9.24 Theorem Suppose f is a \mathcal{C}^1 -mapping of an open set $E \subset R^n$ into R^n , $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$. Then

- (a) there exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one-to-one on U , and $f(U) = V$;
- (b) if g is the inverse of f [which exists, by (a)], defined in V by

$$g(f(x)) = x \quad (x \in U),$$

then $g \in \mathcal{C}^1(V)$.

Writing the equation $y = f(x)$ in component form, we arrive at the following interpretation of the conclusion of the theorem: The system of n equations

$$y_i = f_i(x_1, \dots, x_n) \quad (1 \leq i \leq n)$$

can be solved for x_1, \dots, x_n in terms of y_1, \dots, y_n , if we restrict x and y to small enough neighborhoods of a and b ; the solutions are unique and continuously differentiable.

Proof

(a) Put $f'(a) = A$, and choose λ so that

$$(46) \quad 2\lambda \|A^{-1}\| = 1.$$