IEOR 4701: Stochastic Models in Financial Engineering
Summer 2007, Professor Whitt
SOLUTIONS to Conditional Probability Problems

Chapter 1 of Ross. Here are key points:

1. Probability theory is a branch of *mathematics*, so it is important to pay attention to the *definitions* and *axioms*.

In particular, Chapter 1 emphasizes that a key idea overall is to remember and apply the definition of *conditional probability*:

$$P(A|B) \equiv \frac{P(AB)}{P(B)} \ ,$$

where $AB \equiv A \cap B$ denotes the *intersection* of the sets (events) A and B.

2. We are focusing on problem solving. For that purpose, a good general strategy is *divide and conquer*: break the problem into smaller pieces that are easier to analyze. Skipping steps can cause errors.

3. It is helpful to draw *pictures*.

The following are exercises at the end of Chapter 1 in Ross.

1.28 If the occurrence of event B makes event A more likely, does the occurrence of event A make event B more likely?

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The key thing here is to formulate the problem precisely. The question can be expressed as follows: If $P(A|B) \geq P(A)$ is it necessarily true that $P(B|A) \geq P(B)$. Once the question has been so formulated, it is easy to see that the answer is indeed "Yes." As a second step, apply the definition of conditional probability to write down what the conditional probabilities are. You are then left with trivial algebra. In particular, the condition $P(A|B) \geq P(A)$ is equivalent to $P(AB)/P(B) \geq P(A)$ or $P(AB) \geq P(A)P(B)$. On the other hand, the desired conclusion $P(B|A) \geq P(B)$ is equivalent to $P(AB)/P(A) \geq P(B)$ or $P(AB) \geq P(A)P(B)$.

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1.29 Suppose that $P(E) = 0.6$. What can you say about $P(E|F)$ when
(a) E and F are mutually exclusive?
(b) $E \subseteq F$?
(c) $F \subseteq E$?

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Use the definition of conditional probability: $PE|F) = P(EF)/P(F)$. Then observe what is $P(EF)$ because of what is EF. "Mutually exclusive" means that the intersection is empty. Hence $EF = \phi$ (empty set) in (a), $EF = E$ in (b) and $EF = F$ in (c). Hence the answers are: (a) 0, (b) $P(E)/P(F) = 0.6/P(F) \geq 0.6$, (c) $P(EF)/P(F) = P(F)/P(F) = 1.0$.

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1.18 (a) A family has two children. What is the probability that both are girls, given that at least one is a girl? (Assume that each child is equally likely to be a boy or a girl.)

(b) Does the answer change if we rephrase the question: What is the probability that both are girls, given that the older child is a girl?

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Start by formalizing. Put the problem into the mathematical framework. The sample space is the *set*

$$S = \{(g,g), (g,b), (b,g), (b,b)\},$$

where (b,g) means that the first (older) child is a boy and the second (younger) child is a girl. Then events are subsets of the sample space. Let B be the event that both are girls, i.e., $B = \{(g,g)\}$. The key idea here is that the events L and O are different, where L is the event that at least one child is a girl, while O is the event that the older child is a girl. the events L and O are:

$$L = \{(g,g), (g,b), (b,g)\};$$

and

$$O = \{(g,g), (g,b)\} \ .$$

Thus $P(B|L) = 1/3$, while $P(B|O) = 1/2$.

You need to recall some elementary set theory. We use braces to denote a set, as in S or E above. Note that x, $\{x\}$ and $\{\{x\}\}$ are different objects: x is an element of the set $\{x\}$, while $\{x\}$ is an element of the set $\{\{x\}\}$; x is not an element of the set $\{\{x\}\}$. A set containing the element x is not the same as the element x itself.

Formally, a *probability measure* assigns probabilities to subsets of the sample space; those subsets are called events.

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1.33 In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be in the class if sex and class are to be independent when a student is selected at random?

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Let n be the number of sophomore girls. That is the single unknown. Make an equation with one unknown and solve it. Let the definition of independence give you the equation:

$$P(FB) = P(F)P(B)$$

so that the equation becomes

$$\frac{4}{16+n} = \frac{10}{16+n} \times \frac{10}{16+n} \ ,$$

from which it is easy to see that the answer is $n = 9$. Any of the other events would yield the same answer; e.g., we could have used instead

$$P(SG) = P(S)P(G) \ .$$

Actually, it is more natural to work with random variables: Let X represent the sex of the student, with 1 indicating male and 0 indicating female. Let Y represent the class or year of the

student, with 1 indicating freshman and 0 indicating sophomore. Then we want to conclude that the random variables X and Y are independent. We need to know the definition. The random variables are independent if

$$P(X = i, Y = j) = P(X = i)P(Y = j) \quad \text{for all} \quad i \quad \text{and} \quad j$$

where it suffices to consider values 0 and 1 for i and j. This holds. For example,

$$P(X = 1, Y = 1) = P(BF) = P(B)P(F) = P(X = 1)P(Y = 1)$$

consistent with above.

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1.42 There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. Suppose that one of these three coins is selected at random and flipped. What is the conditional probability that the randomly selected coin was the two-headed coin, given that the outcome of the flipping showed heads?

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This is an application of Bayes Theorem, as discussed in Section 1.6. Draw a probability tree. From the root, draw three branches to show the possibilities of which coin is chosen; let the probability weight on each branch be 1/3. Then from each of these branches, make further branches showing the outcome of the coin toss, conditional on the selected coin. Use Bayes theorem to *reverse the conditional probability*: get $P(B|A)$ from $P(A|B)$. See examples in Section 1.6. In particular,

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\begin{aligned}
P(\text{Two}|\text{Head}) &= \frac{P(\text{Two Head})}{P(\text{Head})} \\
&= \frac{P(\text{Two})P(\text{Head}|\text{Two})}{P(\text{Two})P(\text{Head}|\text{Two}) + P(\text{Fair})P(\text{Head}|\text{Fair}) + P(\text{Bias})P(\text{Head}|\text{Bias})} \\
&= \frac{1/3 \times 1}{1/3 \times 1 + 1/3 \times 1/2 + 1/3 \times 3/4} \\
&= \frac{1/3}{1/3 + 1/6 + 1/4} = \frac{4/12}{4/12 + 2/12 + 3/12} = \frac{4}{9}
\end{aligned}
\tag{1}
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1.26 and 1.27 Four hands of 13 cards each are to be dealt from a deck of 52 playing cards. What is the probability that there is one ace in each of the four hands?

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We want to apply a previous exercise:

1.23 **Divide and Conquer with Conditional Probability**

We are asked to calculate an event E that can be represented as the intersection of four events. We thus elect to do the calculation in 4 steps: $E = E_1 E_2 E_3 E_4$. Here is the key fact:

$$P(E_1 E_2 E_3 E_4) = P(E_1) \times P(E_1 E_2|E_1) \times P(E_1 E_2 E_3|E_1 E_2) \times P(E_1 E_2 E_3 E_4|E_1 E_2 E_3) \ ,$$

because

$$P(E_1 E_2 E_3 E_4) = P(E_1)\frac{P(E_1 E_2)}{P(E_1)}\frac{P(E_1 E_2 E_3)}{P(E_1 E_2)}\frac{P(E_1 E_2 E_3 E_4)}{P(E_1 E_2 E_3)} \ .$$

due to the cancellation.

1.26 We apply the reasoning above with the four events:

$$E_1 = \{\text{the first hand has exactly one ace}\}$$

$$E_2 = \{\text{the second hand has exactly one ace}\}$$

$$E_3 = \{\text{the third hand has exactly one ace}\}$$

$$E_4 = \{\text{the fourth hand has exactly one ace}\}$$

By counting the number of hands with one ace, and dividing by the total number of hands, we get

$$P(E_1) = \frac{\frac{4!}{1!3!}\frac{48!}{12!36!}}{\frac{52!}{13!39!}} \ .$$

Continuing in that way, we get

$$P(E_1 E_2 | E_1) = \frac{\frac{3!}{1!2!}\frac{36!}{12!24!}}{\frac{39!}{13!26!}}$$

$$P(E_1 E_2 E_3 | E_1 E_2) = \frac{\frac{2!}{1!1!}\frac{24!}{12!12!}}{\frac{26!}{13!13!}}$$

$$P(E_1 E_2 E_3 E_4 | E_1 E_2 E_3) = \frac{\frac{1!}{1!0!}\frac{12!}{12!0!}}{\frac{13!}{13!0!}}$$

and so forth. When done, cancellation simplifies to the same answer as given in 1.27 below. In particular, we get

$$P(E_1 E_2 E_3 E_4) = \frac{4 \times 13 \times 37 \times 38 \times 39}{52 \times 51 \times 50 \times 49} \times \frac{3 \times 13 \times 26 \times 25}{39 \times 38 \times 37} \times \frac{2 \times 13 \times 13}{26 \times 25} \times 1 \ ,$$

so that

$$P(E_1 E_2 E_3 E_4) = \frac{39 \times 26 \times 13}{51 \times 50 \times 49} \ .$$

1.27 Apply Exercise 1.23 again. This is the easy way.

$$E_1 = \{\text{one hand contains the ace of spades}\}$$

$$E_2 = \{\text{the ace of spades and the ace of hearts are in different hands}\}$$

$$E_3 = \{\text{the ace of spades, ace of hearts and ace of diamonds are all in different hands}\}$$

$$E_4 = \{\text{all four aces are in different hands}\}$$

The answer is in the back. First, $P(E_1) = 1$. Next,

$$P(E_2|E_1) = \frac{39}{51}$$

since 12 cards are in the same pile as the ace of spades while 39 are not. And so forth:

$$P(E_1 E_2 E_3 E_4) = \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49} \; .$$

So we see that the problem is very easy with the right approach.

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