IEOR 4701: Stochastic Models in FE

Professor Whitt, Summer, 2007

SOLUTIONS to Post Office Questions

The Lack-Of-Memory Property

A nonnegative random variable X is said to have the *lack of memory* (LOM) *property* of

$$P(X > x + y|X > y) = P(X > x) \quad \text{for all} \quad x > 0, y > 0 .$$

Theorem 0.1 *A nonnegative random variable has the LOM property if and only if it has an exponential distribution.*

Trip to the Post Office

Five students from IEOR 4701 – Nicolas Abadie-Vennin, Pierre-Dimitri Gore-Coty, Rohit Saraf, Saraswati Rachupalli, and Thayne Batty – simultaneously enter an empty post office, where there are three clerks ready to serve them. Nicolas, Pierre-Dimitri and Rohit begin to receive service immediately, while Saraswati and Thayne wait in a single line, ready to be served by the first free clerk, with Saraswati at the head of the line (to be served first when a server becomes free). Suppose that the service times of the three clerks (for all customers) are independent exponential random variables, each with mean 2 minutes.

(a) What is the expected time (from the moment the students enter the post office) until Rohit completes service?

Let R be the service time of Rohit. $E[R] = 2$, given above directly

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(b) What is the probability that Rohit is still in service after 6 minutes?

Let R be the service time of Rohit. $P(R > t) = e^{-\lambda t}$, so $P(R > 6) = e^{-\lambda 6} = e^{-3}$

(c) What is the *conditional* probability that Rohit is still in service after 10 minutes, given that Rohit has not yet been served after 4 minutes?

$P(R > 10|R > 4) = P(R > 6) = e^{-\lambda 6} = e^{-3}$

(d) What is the *conditional* probability that Rohit is still in service after 10 minutes, given that Nicolas has not yet been served after 4 minutes?

$P(R > 10|N > 4) = P(R > 10) = e^{-\lambda 10} = e^{-5}$

(e) What is the probability that Rohit is the first to complete service?

1/3, a special case of rate of one divided by sum of rates

(f) What is the expected time (from the moment the students enter the post office) until the first student completes service?

Minimum of independent exponentials is again exponential with a rate equal to the sum of the rates, so the mean is 2/3

(g) What is the variance of the time (from the moment the students enter the post office) until the first student completes service?

Minimum of independent exponentials is again exponential with a rate equal to the sum of the rates, so the mean is 2/3, but then the variance of an exponential is the square of the mean, so the variance is 4/9

(h) What is the expected time (from the moment the students enter the post office) until Saraswati completes service?

Saraswati must wait until one of the first three finish service. We use the fact that a minimum of independent exponential random variables is again exponentially distributed with a rate equal to the sum of the component rates. Here the minimum has a rate equal to three times an individual service rate. Since the mean is the reciprocal of the rate, the mean time for the first service completion is 2/3 minute. We must add to that Saraswati's own expected service time. Hence, the expected time until Saraswati finishes service is $(2/3) + 2 = (8/3) =$ 2.33 minutes.

(i) What is the expected time (again since entering the post office) until *all* five students finish service?

All three clerks are working for the first three service completions; for the fourth service completion two clerks are working; for the fifth (last) service completion one clerk is working. Hence the overall expected time is: $(2/3) + (2/3) + (2/3) + (2/2) + (2/1) = 5$ minutes. We use the lack of memory property; at each service completion, the remaining service time of each service not completed is exponential, just as if the service began at that point.

(j) What is the variance of the time until *all* five students finish service?

The total time, for which the expectation is given in part (b) above, is the sum of five independent exponential random variables, with the specified means. The sum of exponential random variables is not exponentially distributed. The variance of the sum is the sum of the variances, however, since the random variables are independent. The variance of an exponential random variable is the square of its mean. Hence the desired variance is $(2/3)^2 + (2/3)^2 + (2/3)^2 + (2/2)^2 + (2/1)^2 = 19/3 = 6.33$.

(k) What is the probability that Saraswati is the *third* student to finish service?

2/9. Of course, Saraswati cannot be the first to finish. Since she is one of three in service when she starts service (after the first service completion), Saraswati is not the second to finish with probability 2/3. Conditional on not being the second to finish, Saraswati is the third to finish with probability 1/3, since she is then one of three in service. Hence the probability is $(2/3 \times (1/3) = (2/9)$.

(l) Suppose that you wanted to calculate the probability that the time required for all five students to complete service will exceed 10 minutes. What computational tool makes that calculation easy to perform? Briefly explain why.

The time required for all five students to complete service is the sum of five independent exponential random variables, but with different means. We have given the means above. The sum of the first three with identical means has a gamma (or Erlang) distribution, as indicated on p. 37. However, the sum of all five random times has a complicated distribution. The desired probability can be expressed as a multidimensional convolution integral. That multidimensional convolution integral could be computed by MATLAB. It would be made easier by reducing it to a three-dimensional integral, exploiting the gamma distribution for the first three service completions. For example the density of $X + Y$, when X and Y are independent nonnegative random variables with densities f_X and f_Y is

$$f_{X+Y}(x) = \int_0^x f_X(y) f_Y(x-y)\, dy \ .$$

You just iterate those integrals to compute the density of sums of more independent random variables. The computation gets harder, though, as the number of integrals increases.

The desired computation is easy to perform, however, by numerically inverting the Laplace transform; see the papers listed on the course computational tools web page.

To give a quick brief explanation. Let X be a random variable and let f be the pdf (density) of X. The Laplace transform $\hat{f}$ is defined as

$$\hat{f}(s) \equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \ .$$

From elementary Laplace transform theory, the Laplace transform of the complementary cumulative distribution function (ccdf) $F^c(x) \equiv 1 - F(x)$, where the cdf is $F(x) \equiv \int_0^x f(t)\, dt$, is

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s \ .$$

To be more concrete, let X be an exponentially distributed random variable with mean $1/\lambda$ (and thus rate λ). In this context, $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$. The Laplace transform here is

$$\begin{aligned}
\hat{f}(s) &\equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \\
&= \int_0^\infty e^{-sx} \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{-(s+\lambda)x}\, dx \\
&= \lambda/(\lambda + s)\ .
\end{aligned}$$

Directly, the Laplace transform of the ccdf in this exponential case is

$$\begin{aligned}
\hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\
&= \int_0^\infty e^{-sx} e^{-\lambda x}\, dx \\
&= \int_0^\infty e^{-(s+\lambda)x}\, dx \\
&= 1/(\lambda + s)\ .
\end{aligned}$$

In this exponential case we can easily verify that

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s\ .$$

But why are these transforms so useful? They are useful here, as in many other cases, because the transform of $X_1 + \cdots + X_n$, say $\hat{f}$, where $X_1, \ldots, X_n$ are independent random variables such that X_i has Laplace transform $\hat{f}_i$, is

$$\hat{f}(s) = \hat{f}_1(s) \times \hat{f}_2(s) \times \cdots \times \hat{f}_n(s)\ .$$

Thus, we can easily compute the Laplace transform of the ccdf of the time required for all students to complete service. In particular, the Laplace transform is

$$\begin{aligned}
\hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\
&= (1 - \hat{f}(s))/s \\
&= (1 - [\hat{f}_1(s) \times \cdots \times \hat{f}_5(s)])/s \\
&= (1 - [(\lambda_1/(\lambda_1 + s)) \times \cdots \times (\lambda_5/(\lambda_5 + s))])/s \\
&= (1 - [((3/2)/(3/2 + s))^3 \times (1/(1 + s)) \times ((1/2)/(1/2 + s))])/s\ .
\end{aligned}$$

The final expression is not so pleasing for humans to look at, but the computer is happy.

We can thus calculate the desired ccdf value $F^c(10)$ by numerically inverting that Laplace transform $\hat{F^c}(s)$.
