IEOR 4701: Stochastic Models in FE

Summer 2007, Professor Whitt

Class Lecture Notes: Wednesday, August 1.

Renewal Theory

1. **Renewal Reward Processes, Section 7.4**

The discussion was based on Example 7.12: A car-buying model.

The long-run average reward per time is equal to the expected reward per cycle divided by the expected length of a cycle. That is the conclusion of Proposition 7.3. Its proof draws on Proposition 7.1, which is the strong law of large numbers for renewal processes. The basic random variables are X_n and R_n. The nonnegative random variable X_n is the length of the n^{th} cycle. The random variable R_n is the reward earned during the n^{th} cycle. We allow the random variables X_n and R_n to be dependent, but we assume that the sequence $\{(X_n, R_n) : n \geq 1\}$ is a sequence of i.i.d. random vectors; e.g., (X_1, R_1) is independent of (X_2, R_2). The total reward earned up to time t is

$$R(t) \equiv \sum_{i=1}^{N(t)} R_i, \quad t \geq 0 ,$$

where $N(t)$ is the number of cycles or renewals up to time t, i.e.,

$$N(t) \equiv \max\{n : X_1 + \cdots + X_n \leq t\}, t \geq 0 .$$

The process $\{N(t) : t \geq 0\}$ is a renewal process with inter-renewal times X_n. The main result we apply is the strong law of large numbers:

$$\frac{R(t)}{t} \rightarrow \frac{E[R_1]}{E[X_1]} \quad \text{as} \quad t \rightarrow \infty .$$

The expected reward per cycle is $E[R_1]$; the expected length of a cycle is $E[X_1]$.

2. **The Elementary Renewal Theorem, Wald's equation and stopping times**

The elementary renewal theorem is given on top of page 425. It states that $E[N(t)]/t \rightarrow 1/EX$ as $t \rightarrow \infty$. It is proved in Exercises 7.13 and 7.14 at the end of the chapter. It uses Wald's equation, which in turn uses stopping times. See Professor Sigman's notes on stopping times, posted on our web page, and handed out in class. The elementary renewal theorem is used to relate to the DTMC result stating that the expected time to return to state i, starting from state i, is $1/\pi_i$, where P is the transition matrix for an irreducible DTMC with $\pi = \pi P$.