IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

Lecture notes on arbitrage pricing, August 6

Odds

Suppose that you go to a horse race and buy a 2 dollar ticket on Crazylegs to win. Suppose that the odds against Crazylegs winning are 4 to 1, often written $4-1$ or $4:1$. Of course, you lose your spent 2 dollars if Crazylegs does not win. If Crazylegs wins, then the 4 to 1 odds mean that you can bring your ticket to the ticket window and get your own money back plus 4 dollars for each dollar you bet. Thus, you would get 10 dollars for your winning 2-dollar ticket. You would have made 8 dollars profit.

(a) What probability of Crazylegs winning would make that a fair bet?

Fair odds would be associated with a probability

$$p = \frac{1}{1+o} = \frac{1}{1+4} = \frac{1}{5}$$

because the expected value of your ticket is

$$10p - 2 \ ,$$

which equals 0 if and only if $p = 1/5$.

(b) Suppose that three horses are racing in the upcoming Belmont Stakes horserace and the posted odds are

horse	odds
Seabiscuit	1-1
War Admiral	3-1
Rosemont	5-1

Suppose that you think the chances of the three horses winning are about equal. Suppose that you have a good credit line, so that you can borrow as much money as you want at the (outrageous) rate of 5% per week. How would you bet?

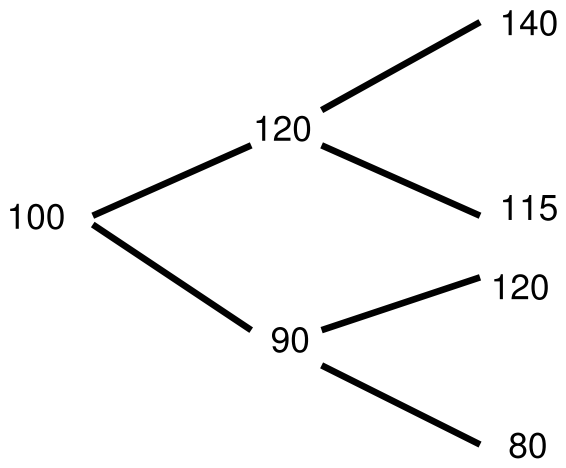
A Two-Period Binary Tree

Consider a stock that is initially priced at 100 dollars per share. Let the stock price evolve randomly over time according to the two-period binary tree shown in Figure .

We assume that, after one time period, the stock will either go up to 120 or drop to 90; these are the only two possibilities. If the stock does go up to 120 at then end of the first period, then we assume that the stock will either go up further to 140 or drop to 115 at the end of the second period. On the other hand, if instead the stock drops to 90 at the end of the first period, then we assume that the stock will either go up to 120 or drop further to 80 at the end of the second period. As above, we assume that all prices are in present value dollars, so that we can ignore interest.

Suppose now that you are offered a *European call option* with *strike price* 100 dollars per share and *expiry* 2. That is, you are offered the option to purchase the stock at the end of the

A Two-Period Binary Tree



second period for the initial price 100. That is, the option gives you the opportunity, but not the obligation, to buy shares of the stock at the end of period 2 for 100 dollars per share.

(a) What is the appropriate (unique arbitrage-free) price for the option at time 0?

(b) Find a hedging strategy that allows us to replicate the option through buying and selling the stock at times 0 and 1.

ANSWERS

(a) We can apply the Arbitrage Theorem - Theorem 10.1 - on page 635 to conclude that it suffices to find risk-neutral prices for each of the three random events in the tree. These probabilities are shown in parentheses on the arcs of the tree in Figure below. The three random events are then understood to be independent Bernoulli experiments.

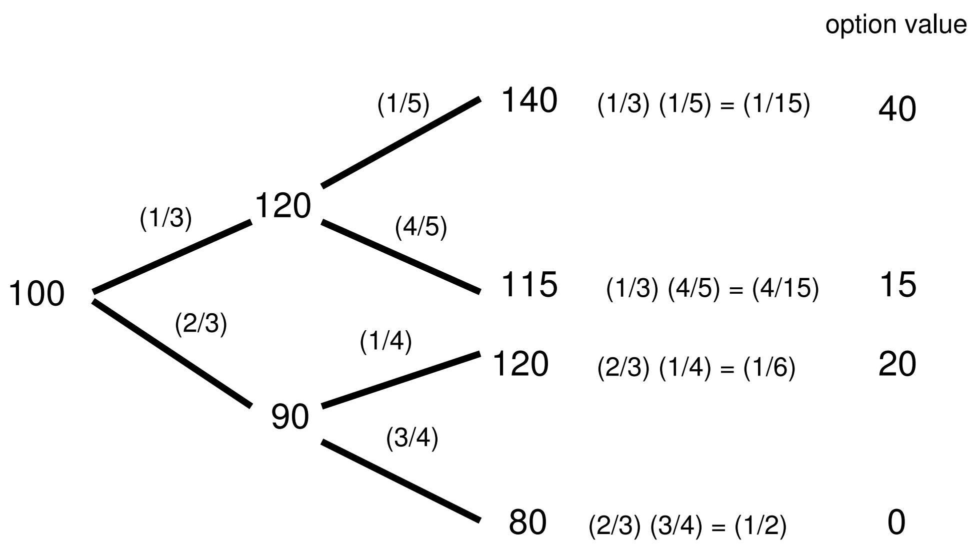
We use the independence to calculate the probabilities of each of the four possible outcomes at stage 2. We multiply the probabilities on the connecting arcs to get the final probabilities for each outcome. We can then calculate the expected value of the option. Since the call option gives us the opportunity to purchase the stock at time 2 for 100 dollars per share, we will only elect to do so if the stock price exceeds 100 at that time; i.e., we will do so in all cases except the case in which it drops twice to 80. The expected value of the stock option using the risk neutral probabilities is

$$40 \times (1/15) + 15 \times (4/15) + 20 \times (1/6) + 0 \times (1/2) = (100/15) + (20/6) = 10$$

So the fair (arbitrage-free) price of the option is 10 dollars per share.

(b) We now develop a hedging strategy. We can do this in more than one way. We give two methods. The second method employs method in the Lecture Notes on the Binomial Lattice Model.

A Two-Period Binary Tree with risk-neutral probabilities



Method 1.
Let X_i denote the stock price at the end of period i. Consider 4 possible actions:

A_0 Put one dollar in the bank and end up with one dollar in all possible scenarios
A_1 Buy one share of stock at time 0 and sell it at time 1
A_2 Buy one share of the stock at time 1 and sell at time 2, if the stock goes up to 120 at time 1
A_3 Buy one share of the stock at time 1 and sell at time 2, if the stock goes down to 90 at time 1

Let z_i be the amount (number of shares) done of action i. We want to find values of z_i that replicate the option under every possible event. To do so, we construct the following table: Table 1.

X_1	X_2	A_0	A_1	A_2	A_3	option
120	140	1	20	20	0	40
120	115	1	20	-5	0	15
90	120	1	-10	0	30	20
90	80	1	-10	0	-10	0

Table 1: Creation of the hedging strategy to replicate the option.

Considering both times 1 and 2, we see that there are four possible outcomes. We need to replicate the option in each of these cases. That leads to four equations in four unknowns. each equation is of the form $A_0 z_0 + A_1 z_1 + A_2 z_2 + A_3 z_3 = v$. Here are the four equations:

$$z_0 + 20z_1 + 20z_2 + 0z_3 = 40$$

$$\begin{aligned} z_0 + 20z_1 - 5z_2 + 0z_3 &= 15 \\ z_0 - 10z_1 + 0z_2 + 30z_3 &= 20 \\ z_0 - 10z_1 + 0z_2 - 10z_3 &= 0 \end{aligned} \tag{1}$$

By subtracting the first two equations, we get $25z_2 = 25$ or $z_2 = 1$. By subtracting the last two equations, we get $40z_3 = 20$ or $z_3 = 1/2$. If we substitute into the first and third equations, then we get two equations in two unknowns:

$$\begin{aligned} z_0 + 20z_1 &= 20 \\ z_0 - 10z_1 &= 5 \end{aligned}$$

Subtracting these, we get $30z_1 = 15$ or $z_1 = 1/2$. Substituting, we get $z_0 = 10$. The variable z_0 is the price of the option. We get $z_0 = 10$, which is consistent with the expected value under the risk-neutral probabilities. We replicate the option itself with $(z_0, z_1, z_2, z_3) = (10, 1/2, 1, 1/2)$.

Method 2.

Being more systematic, we also can apply the single-period two-alternative analysis in Section 3.2 of the Binomial Lattice Model Lecture Notes, starting at the end and working back. We find a portfolio (α, β) at each decision point, where α is the number of shares of the stock and β is the amount of the risk-free asset. We use formulas (12) and (13) in those lecture notes. First, after one period, when the stock price is 120, we get $(\alpha, \beta) = (1, -100)$; the option is worth 20 at this stage. Second, after one period, when the stock price is 90, we get $(\alpha, \beta) = (1/2, -40)$; the option is worth 5 at this stage. Third, at the very beginning, when the stock price is 100, we have another two-alternative tree with option values 20 for up and 5 for down. At this point, we get $(\alpha, \beta) = (1/2, -40)$; the option at the beginning is worth 10. We have assumed that the interest rate here is $r = 1$. It is easy to include positive interest rate.