IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

Lecture notes on the Binomial Lattice Model, August 13

1 Introduction

1.1 Bernoulli Trials

A sequence of *Bernoulli trials* is a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$, where

$$P(X_n = 1) = p = 1 - P(X_n = 0), \quad n \geq 1.$$

We use Bernoulli trials to model the outcomes of successive coin tosses; we may say that $X_n = 1$ if we get heads on the n^{th} toss. Then the partial sum $S_n \equiv X_1 + \cdots + X_n$ is the number of heads on the first n tosses. The random variables S_n has a *binomial distribution* with parameters n and p:

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n \ ;$$

see Sections 2.2.1 and 2.2.2 of Ross, on pages 28-29. These are bonafide probabilities because they are positive and

$$\sum_{k=0}^{n} P(S_n = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 \ .$$

To verify this last claim, we can apply the *binomial theorem*, giving the value of $(x+y)^n$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \ .$$

For the binomial distribution, the sum of the probabilities is thus a representation of $(p + (1 - p))^n = 1^n = 1$.

1.2 The Binomial Lattice Model

The *binomial lattice model* is a modification of a sequence of Bernoulli trials, created to model stock prices. We again consider a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$, each assuming only two values. We make two changes: First, we allow the two possible values of X_n to be general: Instead of 1 and 0, the possible values of X_n are u and d for *up* and *down*. We now have

$$P(X_n = u) = p = 1 - P(X_n = d), \quad n \geq 1.$$

Second, we consider a multiplicative model instead of an additive model. We let the initial value be $S_0 = S$ and let

$$S_n = S_0 \times X_1 \times X_2 \times \cdots \times X_n, \quad n \geq 1 \ .$$

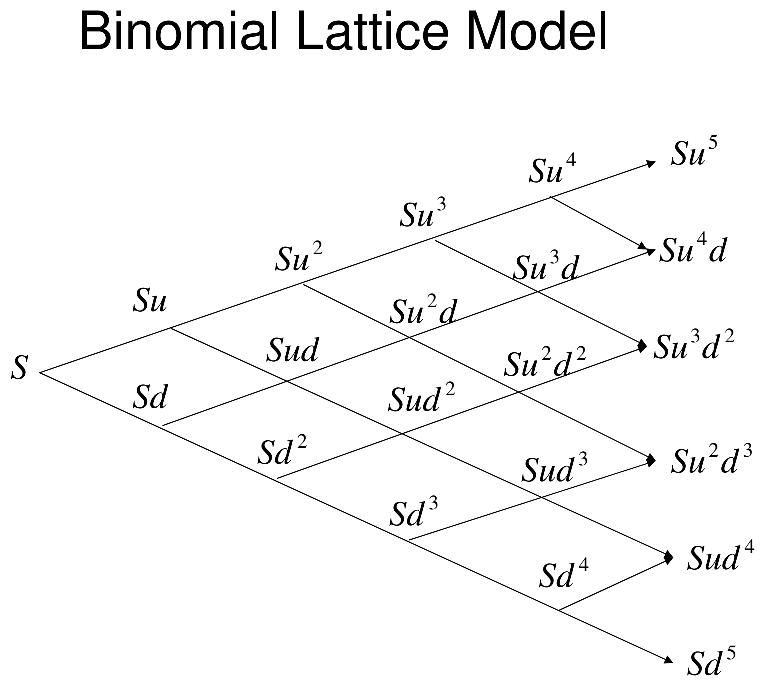

Figure 1: The general tree for the binomial lattice model with $n = 5$.

Pictorially, the possible values of S_k for $0 \leq k \leq n = 5$ with a binomial lattice model are depicted in Figure 1.2. This is often called a binomial tree.

In the displayed tree there are $n = 5$ stages (beyond the initial stage 0). The random variables S_n is intended to represent the stock price at stage n. The random variable X_n is then the random multiple taking S_{n-1} into S_n:

$$S_n = S_{n-1} X_n, \quad n \geq 1 \ .$$

Since the random multipliers are assumed to be i.i.d., the stochastic process $\{S_n : n \geq 0\}$ is a Markov chain: the probability of future states conditional on the past and present depends only on the present state. The possible values of S_n are still determined by binomial(n, p) probabilities

$$P\left(S_n = S u^k d^{n-k}\right) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n \ . \tag{1}$$

2 An Approximation to Geometric Brownian Motion

The binomial lattice model is often introduced as a discrete approximation to geometric Brownian motion (GBM), which in turn is a commonly used continuous-time stochastic process to model security prices. After taking logarithms, this discrete approximation corresponds to the the relatively familiar approximation of Brownian motion (BM) by a simple random walk, as described at the beginning of Chapter 10 of Ross, but here we have a non-zero drift. This non-zero drift makes the analysis somewhat more complicated, but otherwise the story is just as in Section 10.1 of Ross.

In fact, it is natural to think of the more elementary random walk as the basic object. When we want to understand what is Brownian motion, we think of Brownian motion as the limit of a sequence of the random walks as the time periods get short. This can be formalized as a proper limit, called a *stochastic-process limit*: a sequence of appropriately defined random walks converges to Brownian motion. From a practical perspective, to understand BM, we think of it as a random walk. Similarly, GBM can be expressed as the limit of a sequence of BLM's. To understand GBM, we think of it as a BLM. We can perform calculations by either using GBM or an associated BLM. In this section we will show how to relate them.

2.1 Brownian motion

We first give some minimal background on Brownian motion; see Chapter 10 of Ross. If $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion (with 0 drift coefficient and unit (1) variance coefficient), we can get an associated (μ, σ^2) Brownian motion $\{X(t) : t \geq 0\}$, with drift coefficient μ and variance coefficient σ^2 (i.e., $E[X(t)] = \mu t$ and $Var(X(t)) = \sigma^2 t$ for all $t \geq 0$) by setting

$$X(t) \equiv \mu t + \sigma B(t), \quad t \geq 0 \ .$$

Clearly, $X(t)$ is distributed as $N(\mu t, \sigma^2 t)$, where $N(a, b)$ denotes a normal random variable with mean a and variance b.

2.2 Geometric Brownian Motion

Then geometric Brownian motion (GBM) with parameters μ and σ, and with initial value $Y(0)$ regarded as constant, is defined by

$$Y(t) \equiv Y(0)e^{X(t)} = Y(0)e^{\mu t + \sigma B(t)}, \quad t \geq 0 \ .$$

Dividing by the constant $Y(0)$ and taking logarithms, we get

$$\ln\left(Y(t)/Y(0)\right) = X(t) = \mu t + \sigma B(t), \quad t \geq 0 \ .$$

Notice that the parameters μ and σ^2 are the mean and variance, respectively, of $\ln\left(Y(t)/Y(0)\right)$ at $t = 1$. They are not the mean and variance of $Y(t)$ or $Y(t)/Y(0)$ at $t = 1$. This construction gives $Y(t)$ a *lognormal distribution* with mean

$$E[Y(t)/Y(0)] = E[e^{X(t)}] = Y(0)E[e^{\mu t + \sigma B(t)}] = Y(0)e^{\mu t + \sigma^2 t/2} \ ,$$

second moment

$$E[(Y(t)/Y(0))^2] = E[(e^{X(t)})^2] = E[(e^{2X(t)}] = e^{2\mu t + 2\sigma^2 t} \ ,$$

and variance

$$Var((Y(t)/Y(0)) = E[(Y(t)/Y(0))^2] - (E[Y(t)/Y(0)])^2 = e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1)$$

These calculations are based on simple properties of the exponential function and the form of the moment generating function of a normal random variable; see Example 2.42 on page 67.

2.3 Approximating GBM by a BLM

To approximate GBM by a binomial lattice model (BLM), we first simplify by letting $d = 1/u$ in our binomial lattice model. This choice would be obvious if $\mu = 0$, because then $\ln(Y(t)/Y(0)$ would have a normal distribution with mean 0. It would then be natural to have $\ln(d) = -\ln(u)$, which is equivalent to $d = 1/u$. We do this more generally because this is the standard approach.

That leaves two parameters to specify: u and p. To make the approximation, we have to decide how we are going to break up time. It is common to think of time t expressed in years, so that $t = 1$ corresponds to one year. Suppose that we do indeed measure time in years and let Δ be the time period in the discrete model. If time in the discrete model represents weeks, then $\Delta = 1/52 \approx 0.0192$; if time in the discrete model represents months, then $\Delta = 1/12 \approx 0.0833$; time in the discrete model represents days, then $\Delta = 1/365 \approx 0.002740$ (ignoring the leapyear possibility).

Given time measured in years and the parameters μ and σ measured in that time scale, and having chosen the length of a time period Δ in the approximating BLM ($\Delta = 1/n$ if $n\Delta = 1$), we can make the approximation by matching the first two moments of the distribution of $\ln(S_n/S_0)$ to $\ln(Y(n\Delta)/Y(0))$. Doing so gives

$$E[\ln(S_n/S_0)] = nE[\ln(S_1/S_0)] = n\mu\Delta$$

and

$$Var(\ln(S_n/S_0)) = nVar(\ln(S_1/S_0)) = n\sigma^2\Delta$$

Since both the left and right sides of both equations are proportional to n, we can divide through by n, which is tantamount to considering the case of $n = 1$:

$$E[\ln(S_1/S_0)] = \mu\Delta$$

and

$$Var(\ln(S_1/S_0)) = \sigma^2\Delta$$

However, letting $U = \ln(u)$, we see that these two equations are equivalent to

$$pU + (1-p)(-U) = (2p-1)U = \mu\Delta \tag{2}$$

and, working with the second moment instead of the variance,

$$pU^2 + (1-p)(-U)^2 = U^2 = \sigma^2\Delta + (\mu\Delta)^2 \ . \tag{3}$$

This gives us two equations in the two unknowns p and U. They can be combined to produce a single quadratic equation in one unknown, but a simplification follows from the observation that Δ is supposed to be small, so that Δ^2 should be much smaller than Δ. Assuming that μ is of the same order as σ, we then can deduce that $(\mu\Delta)^2$ should be small compared to $\sigma^2\Delta$. Assuming that indeed $(\mu\Delta)^2$ is suitably small compared to $\sigma^2\Delta$, we can omit the term $(\mu\Delta)^2$ with little loss in accuracy, and that is what we do.

Hence we replace the last equation (3) by

$$U^2 \approx \sigma^2\Delta \ , \tag{4}$$

which immediately yields

$$U = \sigma\sqrt{\Delta} \quad \text{and} \quad u = e^U = e^{\sigma\sqrt{\Delta}} \ . \tag{5}$$

Since $d = 1/u$, we also have

$$d = e^{-U} = e^{-\sigma\sqrt{\Delta}} \ . \tag{6}$$

Finally, plugging (5) into (2), we obtain

$$p = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta} \ . \tag{7}$$

Notice that (2) for the mean is satisfied exactly after this simplifying approximation, so that we have introduced no error in the mean, but there is a small error in the second moment and the variance; (3) is not satisfied exactly when we go to (4).

Summary. Given a specified time scale such as years ($t = 1$ means one year) in which the parameters μ and σ^2 are specified, let each discrete time period be of length Δ (assumed to be relatively short, such as a month, week or day in the yearly time scale), we let $S_0 = Y(0) = S$ be the initial price of the stock and then let the parameters u, d and p in the approximating binomial lattice model be given by (5), (6), and (7), i.e.,

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} \\ d &= 1/u = e^{-\sigma\sqrt{\Delta}} \\ p &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta} \ . \end{aligned} \tag{8}$$

Example 2.1 Consider a stock with a yearly time scale, where the parameters μ and σ are given by with $\mu = 0.15$ and $\sigma = 0.30$. (That is a relatively volatile stock; a more common parameter value would be $\sigma = 0.15$.) Assume that the initial stock price is $S_0 = 100$. Make a binomial lattice model with weekly time periods, assuming that the stock prices follow a GBM in continuous time.

Answer: With the time specifications, $\Delta = 1/52 \approx 0.01923$. From (8), we get

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} = e^{0.30/\sqrt{52}} \approx 1.04248 \\ d &= 1/u \approx 0.95925 \\ p &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta} = \frac{1}{2} + \frac{0.15}{20.30\sqrt{52}} = 0.534669 \ . \end{aligned} \tag{9}$$

Given that the initial stock price is $S_0 = 100$, we get the tree in Figure 2.1. We do not show the probabilities in this tree. They are as in (1) with $p = 0.534669$.

3 Option Pricing with the Binomial Lattice Model

We now consider option pricing with the BLM. The setting is a BLM model as defined above, allowing general up and down values u and d. In particular, we now do not assume that $d = 1/u$, but we will make that assumption when we obtain the BLM as an approximation of GBM. Thus, we assume that there is a stock with initial price S evolving according to a BLM, with states and prices as shown in Figure 1.2 and probabilities as given in (1). However, it turns out that we will not use the initial specified probability p. (That will be explained below!)

In addition we add a risk-free asset (money) with fixed interest rate r, $0 < r < 1$, satisfying $u > 1 + r > d$. We assume that you earn interest at rate r per period when you put money in the bank; i.e., 1 dollar put in the bank at the beginning of any period returns $1 + r$ dollars

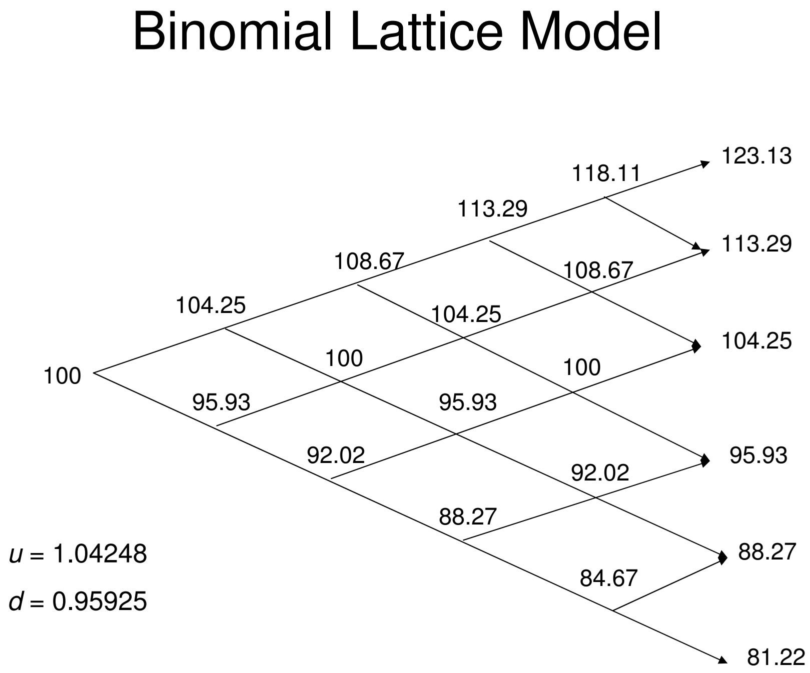

Figure 2: The binomial lattice model approximating the specified geometric Brownian motion in Example 2.1. The time periods are weeks in a yearly time scale. Time periods 0 through 5 are shown.

at the end of that period (which coincides with the beginning of the next period). We assume that the investor can buy or sell stock each period, and can put money in the bank or take it out at the common interest rate r.

In this context (with the stock and risk-free asset as described above), we consider a European call option giving the opportunity to buy shares of a specific stock at the expiration time $T = n$ (after period n) for (the strike price) K per share. Let S_n be the value of the stock at the end of period n. At that time, the the option is worth $(S_n - K)^+$, where $(x)^+ \equiv \max\{x, 0\}$.

We assume the investor can either buy or sell shares of this option. We want to derive a fair price for this option. In this setting we will see that there exists one and only one price for the option that is arbitrage free; i.e., there is one required price that must prevail to prevent the investor from making an arbitrarily large profit without any risk.

3.1 Replication

The key idea is that the option can be replicated by a strategy of buying or selling the stock and borrowing or lending the risk-free asset. This replication forces the payoffs for the option to be identical to the investment strategy for the stock and risk-free asset for every possible random outcome of the stock. Hence there is only one permissible price for the option (under these model assumptions). If that price does not prevail, then an arbitrage opportunity necessarily exists: We have two investment opportunities with identical outcomes; if the prices are different, then we buy a massive amount of the cheaper one and sell the same massive amount of the more expensive one, and consequently make massive profit, guaranteed, with no risk. It is significant that this argument does not depend on the transition probabilities in the BLM. As stated above, for this reason, we do not use the probability p at all.

In fact, the replication scheme is usually regarded as being even more important than the arbitrage-free price of the option, which the replication scheme allows us to derive. In general, **we want to know both the arbitrage-free price of the option and the replication scheme for matching all option payoffs by an investment strategy with the stock and the risk-free asset.**

This replication approach applies to arbitrary derivatives of the stock; it is not limited to European call options. It suffices to specify the payoffs of the derivative in every possible state of the stock at the end of period n. It is important that this derivative have its payoffs depend on the stock prices; the derivative value is a direct function of the stock prices. That is what the term *derivative* means in this financial setting. (This is a totally different meaning from derivative in calculus.)

3.2 A Single Period Suffices

The BLM looks complicated with the expanding tree of states as the number of periods increases. The analysis simplifies greatly because it suffices to consider only a single time period; i.e., it suffices to consider a one-period tree with two possible outcomes. The more general model can be treated by iteratively applying established results for the simple one-period tree.

With one period, we have the stock starting at price S, which will either go up to Su or down to Sd. There is interest at the fixed rate r, where $u > 1 + r > d$. That extra inequality is to avoid trivial uninteresting cases: If $1 + r > u$, then you would never invest in the stock; if $d > 1 + r$, then you would never put money in the bank.

To treat general derivatives in this one-period setting, we let C_u be the payoff of the derivative if the stock goes up, and we let C_d be the payoff if it goes down. It is natural to have $C_u > C_d$ as well as $u > 1 + r > d$, but that is not required.

We will now construct a special portfolio (α, β) of the stock and the risk-free asset in order to replicate the option in both cases (if it goes up or if it goes down). (We let α and β be arbitrary real numbers, positive or negative; e.g., we can buy or sell fractional shares of the stock.) The portfolio (α, β) means that we buy α shares of the stock and put β dollars in the bank. If α is negative, we sell the stock; if β is negative, then we borrow from the bank (at the same interest rate r). The initial value of this portfolio is $\alpha S + \beta$.

To achieve replication, the value of the portfolio at the end of our single period must match the value of the derivative at the same time, if the stock goes up or if it goes down. That leads to two equations:

$$\alpha u S + \beta(1 + r) = C_u \quad \text{if the stock price goes up} \tag{10}$$

and

$$\alpha d S + \beta(1 + r) = C_d \quad \text{if the stock price goes down} \tag{11}$$

Those two equations constitute two equations in the two unknowns α and β. We can solve for these variables, obtaining the **replicating portfolio**:

$$\alpha = \frac{C_u - C_d}{S(u - d)} \tag{12}$$

and

$$\beta = \frac{uC_d - dC_u}{(1 + r)(u - d)} \ . \tag{13}$$

In summary, we replicate the derivative payoffs exactly if we initially (at time 0) buy α shares of the stock and put β dollars in the bank, where these specific values are given above in (12) and (13).

Since the portfolio replicates the derivative, the values of these two must be identical. Hence the **initial value (unique arbitrage-free price) of the derivative** is C_0, where

$$C_0 = \alpha S + \beta = \frac{C_u - C_d}{u - d} + \frac{uC_d - dC_u}{(1+r)(u-d)} \ . \tag{14}$$

However, this expression for C_0 in (14) can be simplified to yield

$$C_0 = \frac{1}{1+r} \left(p^* C_u + (1 - p^*) C_d\right) \ , \tag{15}$$

where

$$p^* = \frac{1 + r - d}{u - d} \quad \text{and} \quad 1 - p^* = \frac{u - (1+r)}{u - d} \ . \tag{16}$$

Since we have assumed that $u > 1 + r > d$, we see that p^* is a probability, called the **risk-neutral probability**. Notice that this risk-neutral probability p^* in (16) need not agree with any a priori probability p specified for the BLM. If we started with a probability p, then we would perform a **change of measure** to change to the risk-neutral probability distribution based on p^*.

Let C_n be the value of the derivative at time n. We know that C_1 is a random variable, equal to C_u if the stock goes up, and equal to C_d if the stock goes down. The risk-neutral probability makes C_0 equal to the expected present value (at time 0) of C_1 with respect to the probability p^*. That is, formula (15) can be restated in terms of an expected value with respect to the risk-neutral probability distribution as follows:

$$C_0 = \frac{1}{1+r} E_*[C_1] \ , \tag{17}$$

where E_* denotes the expectation with respect to the Bernoulli distribution in (16) with probability p^*. In fact, the same is true for the stock itself. Let S_n be the value of the stock at time n. Then $S_0 = S$ and S_1 is a random variable, either equal to uS if the stock goes up or dS if the stock goes down. Then

$$\begin{aligned}
\frac{1}{1+r} E_*[S_1] &= \frac{1}{1+r} \left(p^* S u + (1 - p^*) S d\right) \\
&= \frac{1}{1+r} \left(\frac{1 + r - d}{u - d} S u + \frac{u - (1+r)}{u - d} S d\right) \\
&= \frac{1}{1+r} \left(\frac{(1+r)(u-d)}{u - d} S\right) = S \ .
\end{aligned} \tag{18}$$

This is a very important conclusion: **We can find the arbitrage-free price of the derivative by first finding the risk-neutral probability for the stock.** We then can compute the expected value to get the option price at time 0. Indeed, it is common to start by finding the risk-neutral probability for the stock. The Arbitrage Theorem supports this step more generally. It states, for a more general setting, that there exists an arbitrage-free price for the derivative if and only if there exists associated risk-neutral probabilities. What is special about the BLM, however, is that there can be at most one risk-neutral probability. To have a simple clean story, we rely heavily on having only two possible outcomes.

3.3 Extending to More Periods

Now suppose that there are n periods instead of only one. We are given the values of the derivative for all possible outcomes at period n. Let S_n be the price of the stock in period n; let C_n be the arbitrage-free value of the derivative in period n. We are given the values of the derivative for all possible outcomes at period n, but it remains to determine appropriate (arbitrage-free) values C_k of the derivative, as a function of the state then, for $k < n$. We can use the previous single-period analysis to determine the arbitrage-free prices of the derivative at the previous period $n-1$, using the logic just presented. The situation for the stock is the same in period $n-1$, except that the initial price at that time will be different. But starting from any stock price S_{n-1}, it will either go up to uS_{n-1} or go down to dS_{n-1}. The single-period analysis applies again, yielding

$$C_{n-1} = \frac{1}{1+r} E_*[C_n] \ , \tag{19}$$

where, as before, E_* denotes the expectation with respect to the Bernoulli distribution in (16) with probability p^* in (16). Notice from (16) that these risk-neutral probabilities are independent of the period n and the history up to the state at that period.

We can thus calculate all the arbitrage-free derivative prices at stage $n-1$, using the one-period analysis above. We then can proceed recursively in this way to stage $n-2$ and so on back to stage 1 and then stage 0. When we proceed in this way, we derive the replication strategy for every conceivable stock price S_k at time k for all k, $0 \leq k \leq n-1$.

Given the derivative payoffs at time n and the risk-neutral probability p^*, we can easily compute the expectation in (19). We can then recursively perform similar calculations until we arrive at the arbitrage-free price of the derivative at time 0, C_0, which was to be determined. We could also write

$$C_0 = \left(\frac{1}{1+r}\right)^n E_*\left[C_n\right] \ , \tag{20}$$

where E_* is the expectation with respect to a binomial probability distribution with parameters n and p^*.

For the special case in which the derivative is actually a call option having payoffs $C_n = (S_n - K)^+$, we have

$$\begin{aligned}
C_0 &= \left(\frac{1}{1+r}\right)^n E_*\left[C_n\right] \\
&= \left(\frac{1}{1+r}\right)^n E_*\left[(S_n - K)^+\right] \\
&= \left(\frac{1}{1+r}\right)^n \sum_{k=0}^{n} (u^k d^{n-k} - K)^+ P(S_n = u^k d^{n-k}) \\
&= \left(\frac{1}{1+r}\right)^n \sum_{k=0}^{n} (u^k d^{n-k} - K)^+ \binom{n}{k} (p^*)^k (1-p^*)^{n-k} \ .
\end{aligned} \tag{21}$$

However, we usually want to compute all possible option values C_k depending on all possible stock prices S_k, because that yields the replication strategy (for all time periods) as a byproduct.

Example 3.1 Now return to Example 2.1, where we developed a BLM approximation for a GBM with parameters $\mu = 0.15$ and $\sigma = 0.30$ with a yearly time scale. As before, assume that the initial stock price is $S_0 = 100$. Figure 2.1 displays a binomial lattice model with weekly time periods. Now suppose that the interest rate is 8% compounded weekly. Find the arbitrage-free price of an option to buy the stock at $K = 98$ dollars per share after 5 weeks.

Answer: We previously derived the parameters of the BLM. With the time specifications, $\Delta = 1/52 \approx 0.01923$. From (8), we got

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} = e^{0.30/\sqrt{52}} \approx 1.04248 \\ d &= 1/u \approx 0.95925 \\ p &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta} = \frac{1}{2} + \frac{0.15}{2(0.30)\sqrt{52}} = 0.534669 \ . \end{aligned} \tag{22}$$

Given that the initial stock price is $S_0 = 100$, we get the tree in Figure 2.1, just as before. However, instead of the probability p just computed, we use the risk-neutral probability p^*. To compute it, we need to determine r. We have assumed that the interest is compounded weekly, which is consistent with our time scale. The interest rate per week is then

$$r = 0.08/52 = 0.001538 \ ,$$

which yields an annual interest of $(1 + 0.001538)^{52} = 1.083$, corresponding to 8.3%, which exceeds the originally specified 8% because of the compounding. (To check that this is approximately correct, note that $e^{0.08} = 1.083$.) Over individual weeks, the interest rate does not play a big role. The risk-neutral probability then is

$$p^* = \frac{1 - r - d}{u - d} = \frac{1 + 0.001538 - 0.9593}{1.0425 - 0.9593} = \frac{0.0422}{0.0832} = 0.5072$$

Now we proceed to develop the values C_k of the option in period k depending on the state of the stock. To do so, we first put the values of the option in at the final period $n = 5$. The values there are $(S_n - K)^+ = (S_5 - 98)^+$. We then calculate the values of the option in preceding periods using equation (19). The option values are displayed in Figure 3.1. For $n = 5$, the option values depend on the stock prices that may occur at that period; i.e., we have just computed $(S_n - K)^+ = (S_5 - 98)^+$ in each case, where S_5 is given in Figure 2.1. We then calculate the arbitrage-free option prices in previous periods by computing the discounted expected present value, as in (19). The values are displayed in Figure 3.1. We see that the arbitrage-free option price initially is $C_0 = 5.28$ dollars per share.

Figure 3.1 shows all the option values over time, but it does not show the replicating strategy, but it is easy to construct the replicating strategy. It is given by equations (12) and (13). To determine the strategy at any point in the tree, let C_u and C_d be the option values at the next stage if the stock goes up or down, respectively. To illustrate, suppose that we are at state Su^3 after the stock price has gone up three times. From Figures 2.1 and 3.1, we see that the stock price in this state is 113.29, while the option value is 15.57. From that node in the tree, $C_u = 20.25$ and $C_d = 10.82$. We use these to calculate the replicating strategy in this state. We get

$$\alpha = \frac{C_u - C_d}{S(u - d)} = \frac{20.25 - 10.82}{113.29(1.04248 - 0.95925)} = \frac{9.43}{9.43} = 1.00 \tag{23}$$

and

$$\beta = \frac{uC_d - dC_u}{(1 + r)(u - d)} = \frac{1.04248(10.82) - 0.95925(20.25)}{(1.001538)(0.0832)} = -97.75 \ . \tag{24}$$

Note that $1.00 \times 113.29 - 97.75 \times 1 = 113.29 - 97.75 = 15.54$. This is close to the directly computed value of the option of 15.57.

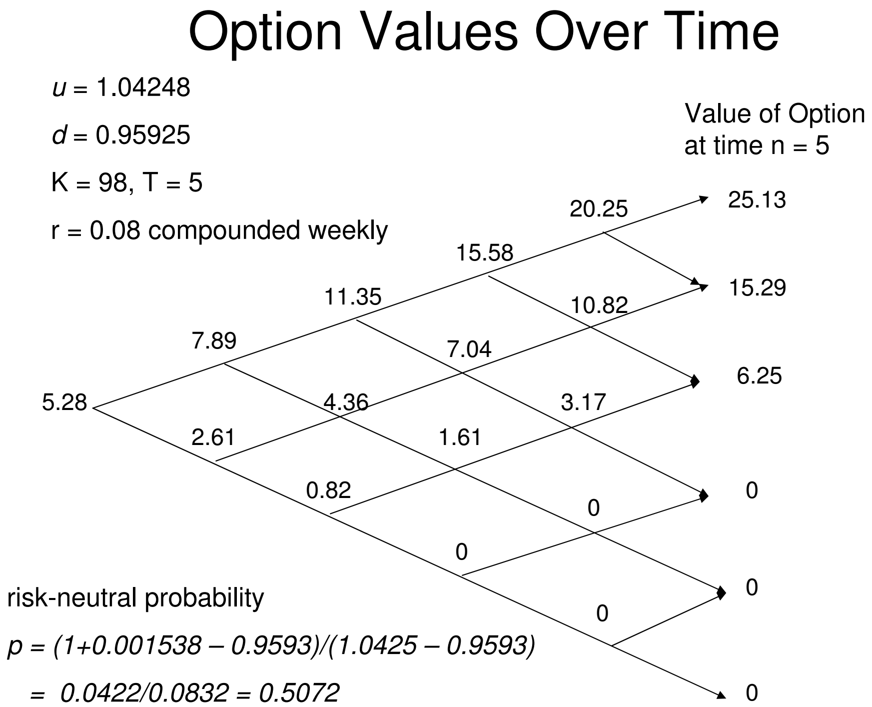

Figure 3: The option prices over time determined by the binomial lattice model for Example 3.1.

3.4 Testing Your Understanding

Example 3.2 In Example 2.1 we developed a BLM approximation for a GBM with parameters $\mu = 0.15$ and $\sigma = 0.30$ with a yearly time scale, and used it to price an option. As before, assume that the initial stock price is $S_0 = 100$. Figure 2.1 displays a binomial lattice model with weekly time periods. We now make some changes. Now suppose that the interest rate is 26% compounded weekly. Find the arbitrage-free price of an option to buy the stock at $K = 98$ dollars per share after 3 weeks. Notice that the interest rate r and the expiration time T have been changed.

(a) What are the option values over time?

(a) What is the replicating strategy to use at time 2 after the stock has gone up twice, i.e., in state Su^2?

Answers: (a) The BLM is just as in Figures 1.2 and 2.1, but now the arbitrage-free option values over time change. Note that $r = 0.26/52 = 0.005$, so that the risk-neutral probabilty is $p^* = (1.005 - d)/(u - d) = 0.5493$. We can use this risk-neutral probability to compute the option values in previous periods. They now are as shown in Figure 3.2 instead of as in Figure 3.1.

(b) Figure 3.2 shows all the option values over time, but it does not show the replicating strategy, but it is again easy to construct the replicating strategy. It is given by equations (12) and (13). To determine the strategy at any point in the tree, let C_u and C_d be the option values at the next stage if the stock goes up or down, respectively. As asked, suppose that we

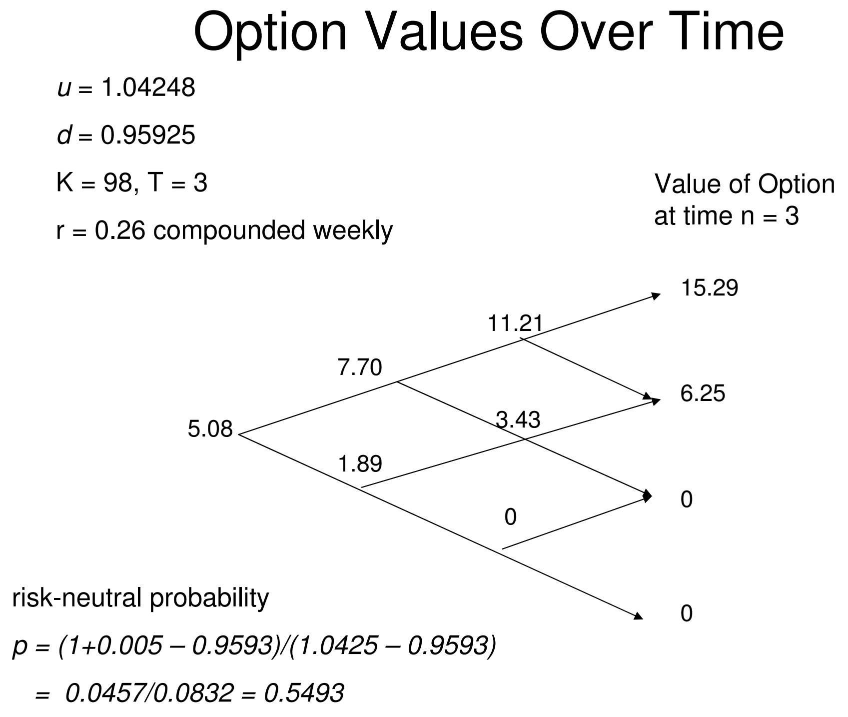

Figure 4: The option prices over time determined by the binomial lattice model for Example 3.2.

are at state Su^2 after the stock price has gone up two times. From Figures 2.1 and 3.1, we see that the stock price in this state is 108.67, while the option value is 11.21. From that node in the tree, $C_u = 15.29$ and $C_d = 6.25$. We use these to calculate the replicating strategy in this state. We get

$$\alpha = \frac{C_u - C_d}{S(u-d)} = \frac{15.29 - 6.25}{108.67(1.04248 - 0.95925)} = \frac{9.04}{9.04} = 1.00 \tag{25}$$

and

$$\beta = \frac{uC_d - dC_u}{(1+r)(u-d)} = \frac{1.04248(6.25) - 0.95925(15.29)}{(1.005)(0.0832)} = -97.49\ . \tag{26}$$

Note that $1.00 \times 108.67 - 97.49 \times 1 = 108.67 - 97.49 = 11.18$. This is close to the directly computed value of the option of 11.21. The small error seems to be due to rounding.

4 Martingales

The risk neutral probability that was key to having an arbitrage-free price of the derivative corresponds to having the discounted stock price process be a martingale under the risk-neutral probability distribution. The discounted stock price process is $\{(1+r)^{-n}S_n : n \geq 0\}$. The key martingale property is

$$\begin{aligned} E_*\left[(1+r)^{-(n+1)}S_{n+1}|(1+r)^{-k}S_k, 0 \leq k \leq n\right] &= E_*\left[(1+r)^{-(n+1)}S_{n+1}|S_k, 0 \leq k \leq n\right] \\ &= E_*\left[(1+r)^{-(n+1)}S_{n+1}|S_n\right] \end{aligned}$$

$$= \quad (1+r)^{-n} S_n \ , \tag{27}$$

where E_* denotes expectation with respect to the risk-neutral measure.

The Arbitrage Theorem says that there is no arbitrage opportunity if and only if there exists at least one underlying probability distribution under which the stochastic process of discounted stock prices is a martingale. In general, there may be many such underlying probability distributions. For BLM's, where there are only two possible outcomes at each stage, and for GBM, which is governed by only two parameters, there turns out to be a unique underlying probability distribution, so that we can find a unique arbitrage-free price for each derivative. That makes BLM's and GBM's very useful.

5 Black-Scholes

The direct continuous-time analog of the derivative pricing via a BLM is derivative pricing for GBM. That leads to the Black-Scholes or Black-Scholes-Merton theory.