IEOR 4701: Stochastic Models in FE

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A Quick Introduction to Stochastic Calculus

1 Introduction

The purpose of these notes is to provide a quick introduction to stochastic calculus. We will first focus on the Ito integral, which is a stochastic integral. We will do that mostly by focusing hard on one example, in which we integrate Brownian motion with respect to Brownian motion. We will then briefly outline the way an Ito integral is defined. It will be apparent that the theory of stochastic integration draws heavily on the theory of martingales. A key concept is the notion of quadratic variation.

After defining the Ito integral, we shall introduce stochastic differential equations (SDE's) and state Ito's Lemma. Ito's lemma provides a way to construct new SDE's from given ones. It is the stochastic calculus counterpart of the chain rule in calculus. It can be understood by considering a Taylor series expansion and understanding how it should be modified in this stochastic setting. Drawing on quadratic variation, we replace the squared differential $(dB)^2$ by dt.

Finally, we will state the Black-Scholes partial differential equation for the arbitrage-free time-dependent and state-dependent price of a (financial) derivative of a stock, assuming that the stock is governed by geometric Brownian motion. Ito's lemma converts an SDE for the stock price into another SDE for the derivative of that stock price. An arbitrage-free argument produces the final Black-Scholes PDE.

2 A Revealing Example

We will discuss the special stochastic integral $\int B\,dB$, where $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion (BM), and its value

$$\int_0^t B(s)\,dB(s) = \frac{B(t)^2}{2} - \frac{t}{2}, \quad t \geq 0 \ . \tag{1}$$

We assume that B is the *same* BM in both places, as integrand and integrator.

2.1 The Chain Rule from Calculus

Suppose that u is a differentiable real-valued function of a real variable, which is the composition of two other functions: $u = f \circ g$, i.e., $u(t) = f(g(t))$, $t \geq 0$, where f and g are differentiable functions. The chain rule gives us the derivative of u in terms of the derivatives of the component functions f and g:

$$u'(t) \equiv \frac{du}{dt} = f'(g(t))g'(t), \quad t \geq 0 \ .$$

The chain rule leads to an associated formula for integrals:

$$\int_0^t b\,db \equiv \int_0^t b(s)b'(s)\,ds = \frac{b(t)^2}{2}\ , \tag{2}$$

provided that b is a differentiable function, because, we can apply the chain rule to the alleged value of the integral: Here $u = f \circ g$, where $f(x) \equiv x^2/2$ and $g = b$. Applying the chain rule with $u(t) = b(t)^2/2$, we get

$$\frac{du}{dt} = \frac{d(b(t)^2/2)}{dt} = b(t)b'(t)\ .$$

Thus we have verified the formula for the integral.

Thus, if BM had nice differentiable sample paths (which it does not!), then instead of (1) we would have the corresponding formula without the last term in (1), as in (2). However, since the paths of BM are *not* differentiable, the standard calculus rules do not apply, and we need to do something else, which ends up with (1). That last term needs to be explained.

2.2 A General Stochastic Integral

The left side of formula (1) is a special case of the **stochastic integral**

$$\int_0^t X(s)\,dM(s), \quad t \ge 0\ , \tag{3}$$

where $X \equiv \{X(t) : t \ge 0\}$ and $M \equiv \{M(t) : t \ge 0\}$ are both allowed to be **stochastic processes**. Formula (1) is the special case in which *both* of these stochastic processes are standard Brownian motion (BM).

Our goal is to give a brief explanation of expression (3). The story is actually quite involved, but it is not difficult to understand some of the main features. For more details, see Karatzas and Shreve (1988) or Steele (2001), for example. Karatzas is right here at Columbia and offers advanced courses related to this material and its application to finance.

What do we assume about the integrand X and the integrator M? It is critically important that we allow these two stochastic processes to be dependent in order to treat many intended applications. We will not try to be excessively general. We will assume that X and M both are stochastic processes with **continuous sample paths**. That is the case for various smooth functions of BM. We note that this smoothness condition can be generalized.

We will assume that M is a **martingale (MG)** with respect to some family of histories, called a filtration $\{\mathcal{F}_t : t \ge 0\}$. If M is defined to be a martingale with respect to some other stochastic process $Y \equiv \{Y(t) : t \ge 0\}$, for which the critical condition is

$$E[M(t)|Y(u), 0 \le u \le s] = M(s) \quad \text{for all} \quad s, \quad 0 \le s < t\ ,$$

then $\mathcal{F}_s$ is the collection of events generated by $\{Y(u), 0 \le u \le s\}$, and we often rewrite the MG condition as

$$E[M(t)|\mathcal{F}_s] = M(s) \quad \text{for all} \quad s, \quad 0 \le s < t\ .$$

Coupled with the MG property for M, we also assume that $X(t)$ is determined by the filtration up to time t for each $t \ge 0$. We say that X **is adapted** to the filtration $\{\mathcal{F}_t : t \ge 0\}$, written succinctly as $X(t) \in \mathcal{F}_t$. In other words, $X(t)$ is regarded as a function of $\{Y(u), 0 \le u \le t\}$ for each $t \ge 0$. These assumptions on the pair (X, M) make increments $M(t+u) - M(t)$ have conditional expectation 0 given the history of both X and M up to time t, and whatever else, if anything, is in the filtration $\{\mathcal{F}_t : t \geq 0\}$.

These MG properties of M and X can be generalized too, but they cannot be removed entirely. Many generalizations have been developed over the last 50 years. There is a rich and complicated theory.

2.3 Approximating Sums

We define the left side of (3) in essentially the same way that we define the Riemann integral in calculus: We define it as the limit of approximating sums over a discrete time grid, as the discrete grid gets finer and finer. The difficulty is that one has to be careful taking this limit because the usual assumptions for the Riemann integral are not satisfied here.

Following the procedure for the Riemann integral, we pick $n+1$ ordered points t_i in the interval $[0,t]$ satisfying $0 = t_0 < t_1 < \cdots < t_n = t$ and let the approximating sum be

$$S_n \equiv \sum_{i=0}^{n-1} X(s_i)(M(t_{i+1}) - M(t_i)) \ , \tag{4}$$

where s_i is some point in the interval $[t_i, t_{i+1}]$, i.e., where $t_i \leq s_i \leq t_{i+1}$. We want to define the integral as the limit as we take more and more points, letting the maximum interval $t_{i+1} - t_i$ go to 0 as $n \to \infty$.

There are two problems:

(1) We have to be careful how we define the limit. We do not want to try to have convergence with probability one for each sample point. (Instead we shall have mean-squared convergence, but we will not elaborate.)

(2) We have to be careful how we select the point s_i within the interval $[t_i, t_{i+1}]$. We will pay careful attention to this detail because that choice can affect the answer. Indeed, we can get different answers if (i) we let $s_i = t_i$ or if $s_i = t_{i+1}$. Indeed, if we let $s_i = \alpha t_i + (1-\alpha)t_{i+1}$, then in general we get different answers for each value of α with $0 \leq \alpha \leq 1$. We will use the Ito integral, which lets $s_i = t_i$. The case $\alpha = 1/2$ leads to what is called the Fisk-Stratonovich integral. We will see that this choice makes a difference in the relatively elementary setting of (1).

Even if we address these two problems, there are technical issues. We still need the MG structure for M and X briefly outlined above.

2.4 The Ito Integral for our BM Example

Having made the choice $s_i = t_i$ in the last subsection, we have the following approximating sums for our initial stochastic integral in (1):

$$\int_0^t B(s)\, dB(s) \approx \sum_{i=0}^{n-1} B(t_i)(B(t_{i+1}) - B(t_i)) \ . \tag{5}$$

To see what happens, it is convenient to rewrite each summand as

$$B(t_i)(B(t_{i+1}) - B(t_i)) = \frac{1}{2}(B(t_{i+1})^2 - B(t_i)^2) - \frac{1}{2}(B(t_{i+1}) - B(t_i))^2 \ . \tag{6}$$

(verify that by expanding the final quadratic term.) That leads to the single sum in (5) being replaced by two sums, but now the first sum is telescoping, i.e., there is massive cancellation when we add, yielding

$$\sum_{i=0}^{n-1}(B(t_{i+1})^2 - B(t_i)^2) = B(t_n)^2 - B(0)^2 = B(t)^2 - 0 = B(t)^2 \ . \tag{7}$$

When we put the whole thing together, we get

$$\sum_{i=0}^{n-1} B(t_i)(B(t_{i+1}) - B(t_i)) = \frac{B(t)^2}{2} - \frac{1}{2}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 \tag{8}$$

We now need to know what happens to the final sum of squares of the increments of BM. The limit is called the **quadratic variation** of BM. It turns out that, with probability 1,

$$\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 \to t \quad \text{as} \quad n \to \infty \quad \text{with} \quad \max\{t_{i+1} - t_i\} \to 0 \ . \tag{9}$$

We now present some partial supporting evidence. We show a weaker form of convergence (convergence in mean square): To quickly see why the limit should be valid, calculate the mean and variance, first for a single summand and then for each term:

$$E[(B(t_{i+1}) - B(t_i))^2] = E[(B(t_{i+1} - t_i))^2] = t_{i+1} - t_i \ ,$$

so that, by another telescoping argument,

$$E\left[\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2\right] = \sum_{i=0}^{n-1}(t_{i+1} - t_i) = t_n - t_0 = t \tag{10}$$

for all partitions.

We now show that the variance is getting negligible:

$$\begin{aligned}
Var[(B(t_{i+1}) - B(t_i))^2] &= Var[(B(t_{i+1} - t_i))^2] \\
&= Var(N(0, t_{i+1} - t_i)^2) \\
&= Var((\sqrt{t_{i+1} - t_i})^2 N(0,1)^2) \\
&= Var((t_{i+1} - t_i) N(0,1)^2) \\
&= (t_{i+1} - t_i)^2 Var(N(0,1)^2) \\
&= (t_{i+1} - t_i)^2 E[(N(0,1)^2 - 1)^2) \\
&= 2(t_{i+1} - t_i)^2
\end{aligned}$$

However,

$$(t_{i+1} - t_i)^2 \le (t_{i+1} - t_i)\max\{t_{i+1} - t_i\} \ ,$$

where $\max\{t_{i+1} - t_i\} \to 0$ as $n \to \infty$, as part of our conditions. Hence

$$\begin{aligned}
Var\left(\sum_{i=0}^{n-1}((B(t_{i+1}) - B(t_i))^2)\right) &= \sum_{i=0}^{n-1} Var((B(t_{i+1}) - B(t_i))^2) \\
&= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \\
&\le 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)\max\{t_{i+1} - t_i\} \\
&= 2t\max\{t_{i+1} - t_i\} \to 0 \quad \text{as} \quad n \to \infty \ . \qquad (11)
\end{aligned}$$

Hence the mean of the second sum goes to t and the variance goes to 0. That directly shows convergence of the approximating sums in the sense of mean square (i.e., $Z_n \to Z$ in mean square if $E[|Z_n - Z|^2] \to 0$), which implies convergence in probability and convergence in distribution ($Z_n \Rightarrow Z$). The limit actually holds with probability one, but that extension is not too critical. Careful treatment of the modes of convergence become important in this setting.

2.5 Using Grid Points at the Other End: the Backwards Ito Integral

In the last subsection we let $s_i = t_i$. Now we want to see what happens if instead we were to let $s_i = t_{i+1}$. We get a different answer. We now have the following approximating sums for our stochastic integral in (1):

$$\int_0^t B(s)\, dB(s) \approx \sum_{i=0}^{n-1} B(t_{i+1})B(t_{i+1}) - B(t_i) \ . \tag{12}$$

By adding and subtracting $B(t_i)$, i.e., by writing $B(t_{i+1}) = B(t_i) + B(t_{i+1}) - B(t_{i+1})$, for each summand, we can obtain the previous approximation plus an additional term: We get

$$\sum_{i=0}^{n-1} B(t_{i+1})(B(t_{i+1}) - B(t_i)) = \sum_{i=0}^{n-1} B(t_i)(B(t_{i+1}) - B(t_i)) + \sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 \ . \tag{13}$$

From the previous reasoning, we get convergence to the previous value $(1/2)(B(t)^2 - t)$ for the first term plus t for the new second term. Hence we get the alternative expression

$$\int_0^t B(s)\, dB(s) = \frac{B(t)^2}{2} + \frac{t}{2}, \quad t \ge 0 \ , \quad \text{(backwards Ito integral)} \tag{14}$$

If instead we let $s_i = (t_i + t_{i+1})/2$, then we get the Fisk-Stratonovich integral

$$\int_0^t B(s)\, dB(s) = \frac{B(t)^2}{2}, \quad t \ge 0 \ , \quad \text{(Fisk-Stratonovich integral)} \ , \tag{15}$$

agreeing with (2). The standard approach is to use (1).

3 Properties of General Stochastic Integrals

The general integral in (3) is defined as the limit of approximating sums as in (4) with $s_i = t_i$ for all i:

$$\int_0^t X(s)\, dM(s) \approx \sum_{i=0}^{n-1} X(t_i)(M(t_{i+1}) - M(t_i)) \ . \tag{16}$$

As before, we require that M and X have continuous sample paths, we require that M be a MG relative to $\{\mathcal{F}_t : t \ge 0\}$, and we require that $X(t)$ be adapted to $\mathcal{F}_t$ for each t.

With this definition and these assumptions, the stochastic integral in (4) itself becomes a **martingale** stochastic process (as a function of the time argument t, relative to the filtration $\{\mathcal{F}_t : t \ge 0\}$ governing the integrator M).

First, given that M is a MG, the discrete-time processes $\{M(t_i) : \imath \ge 0\}$ obtained through the approximation in (16) is itself a discrete-time martingale for each partition we form. Moreover, given that we let $s_i = t_i$ as we have stipulated, the approximating sum in (16) becomes a martingale as well, which is often called a **martingale transform**. (For this, it is critical that we chose $s_i = t_i$.) One immediate consequence of these MG properties is that the stochastic integral and the approximating sums have mean 0 for each t.

Our analysis of the special Brownian stochastic integral in (1) suggests that a key role is played by the **quadratic variation** process. In this general martingale setting, it is defined as

$$\langle M \rangle \equiv \lim \sum_{i=0}^{i=n-1} (M(t_{i+1}) - M(t_i))^2 \ , \tag{17}$$

where the limit is as $n \to \infty$ with the partitions of the interval $[0, t]$ getting finer and finer. It turns out that $\langle M \rangle \equiv \{\langle M \rangle(t) : t \geq 0\}$ can also be characterized (in our setting) as the unique nonnegative nondecreasing (and predictable, not to be explained) process such that $M^2 - \langle M \rangle$ is a martingale. (There are some regularity conditions here.)

For Brownian motion, $\langle M \rangle(t) = t$ and this characterization of $\langle M \rangle$ says that $B(t)^2 - t$ is a MG, which we have already seen to be the case. The stochastic process $B(t)^2 - t$ is the **quadratic martingale** associated with BM. An immediate consequence of this quadratic MG representation for the stochastic integral is that the variance of the MG $M(t)$ is $E[\langle M \rangle(t)]$ for each t.

For the stochastic integral, we have the related property that

$$I(t)^2 - \langle I \rangle(t) \equiv \left(\int_0^t X(s)\, dM(s) \right)^2 - \int_0^t X(s)^2\, d\langle M \rangle(t) \tag{18}$$

is a martingale, so that the variance of the stochastic integral is

$$Var(I(t)) = E\left[\left(\int_0^t X(s)\, dM(s) \right)^2 \right] = E\left[\int_0^t X(s)^2\, d\langle M \rangle(t) \right] \ . \tag{19}$$

4 SDE's and Ito's Lemma

We first specify what we mean by a stochastic process satisfying a stochastic differential equation (SDE). We then state Ito's lemma, which provides the SDE of a smooth function of a process satisfying an SDE.

4.1 Stochastic Differential Equations

Suppose that X is a stochastic process satisfying the **stochastic differential equation (SDE)**

$$dX = adt + bdB \ , \tag{20}$$

where B is BM, by which we mean

$$dX(t) = a(X(t), t)dt + b(X(t), t)dB(t) \ , \tag{21}$$

where a and b are real-valued functions on $\mathbb{R}^2$, by which we mean the X satisfies the integral equation

$$X(t) = \int_0^t a(X(s), s)ds + \int_0^t b(X(s), s)dB(s) \ , \tag{22}$$

where the last integral is defined as an Ito integral. Such a process X is often called an **Ito process**. Note that the process X appears on both sides of the equation, but the value at t given on the left depends only on the values at times s for $s \leq t$. Assuming that X has continuous paths, it suffices to know $X(s)$ for all $s < t$ on the right. on the right. Nevertheless, there is a need for supporting theory (which has been developed) about the existence and uniqueness of asolution to the integral equation (or equivalently the SDE).

An elementary example arises when $X(t) = \mu t + \sigma B(t)$, where μ and σ are constants. Then we have (20) with $a(x,t) = \mu$ and $b(x,t) = \sigma$, independent of x and t. Then we can directly integrate the SDE to see that the process is BM with drift μ and diffusion coefficient σ^2.

Another important example is standard **geometric Brownian motion (GBM)**. Then we have (20) with $a(x,t) = \mu x$ and $b(x,t) = \sigma x$. Letting the stock price at time t be $S(t)$, we write the classical GBM SDE as

$$dS = \mu S dt + \sigma S dB \ , \tag{23}$$

where again μ and σ are constants. Note that S appears in both terms on the right.

4.2 Ito's Lemma

Now we see what happens when we consider a smooth function of an Ito process. We assume given the Ito process X in (20)-(22). Now suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, with continuous second derivatives. Ito's lemma concludes that $Y(t) \equiv f(X(t),t)$ has an SDE representation with

$$dY = \left(f_t + a f_x + \frac{1}{2} b^2 f_{x,x}\right) dt + b f_x dB \tag{24}$$

or, in more detail,

$$dY(t) \equiv df(X(t),t) = \left(\frac{\partial f}{\partial t} + a(X(t),t)\frac{\partial f}{\partial x} + \frac{1}{2} b(X,t)^2 \frac{\partial^2 f}{\partial x^2}\right) dt + b(X(t),t)\frac{\partial f}{\partial x} dB(t); \tag{25}$$

or, in even more detail

$$\begin{aligned} dY(t) \equiv df(X(t),t) \quad &= \quad \left(\frac{\partial f}{\partial t}(X(t),t) + a(X(t),t)\frac{\partial f}{\partial x}(X(t),t) + \frac{1}{2} b(X,t)^2 \frac{\partial^2 f}{\partial x^2}(X(t),t)\right) dt \\ &\qquad + b(X(t),t)\frac{\partial f}{\partial x}(X(t),t) dB(t) \ . \end{aligned} \tag{26}$$

In other words, Y satisfies the associated SDE with coefficients a_Y and b_Y that are functions of $(X(t),t)$ and the function f. In particular,

$$dY = a_Y dt + b_Y dB \ ,$$

where

$$a_Y(X(t),t,f) = \frac{\partial f}{\partial t}(X(t),t) + a(X(t),t)\frac{\partial f}{\partial x}(X(t),t) + \frac{1}{2} b(X,t)^2 \frac{\partial^2 f}{\partial x^2}(X(t),t)$$

and

$$b_Y((X(t),t,f) = b(X(t),t)\frac{\partial f}{\partial x}(X(t),t)$$

The surprising part is the term

$$\frac{1}{2} b(X,t)^2 \frac{\partial^2 f}{\partial x^2}(X(t),t)$$

appearing in a_Y. That would not be there if B had differentiable sample paths. An intuitive proof of Ito's lemma follows by applying a second-order Taylor series expansion. In that expansion we need to replace the $(dB)^2$ term by dt. This stems from the quadratic variation of BM.

4.3 Two Examples

Example 4.1 (*logarithm of GBM*) In (23) we have the standard SDE representation of GBM. Suppose that we now consider the logarithm: $\ln(S(t)/S(0)) = \ln(S(t)) - \ln(S(0))$. We can apply the function $f(x,t) = \ln(x)$, for which $f_x = 1/x$, $f_{x,x} = -1/x^2$ and $f_t = 0$. Hence we get

$$d\ln(S(t)/S(0)) = (\mu - \frac{\sigma^2}{2})dt + \sigma dB \ , \tag{27}$$

from which we immediately see that

$$\ln(S(t)/S(0)) = (\mu - \frac{\sigma^2}{2})t + \sigma B(t), \quad t \geq 0 \ . \tag{28}$$

Note that the drift of this Brownian motion is not μ. The drift terms in the two specifications do not agree. Given that

$$\ln(S(t)/S(0)) = \nu t + \sigma B(t), \quad t \geq 0 \ , \tag{29}$$

we get

$$E[S(t)] = S(0)e^{(\nu + \sigma^2/2)t}, \quad t \geq 0 \ , \tag{30}$$

whereas from the SDE it would be $E[S(t)] = S(0)e^{\mu t}$. The parameters μ and ν in these two representations should be related by

$$\mu = \nu + \frac{\sigma^2}{2} \quad \text{or} \quad \nu = \mu - \frac{\sigma^2}{2} \ . \tag{31}$$

For more discussion, see pages 307-313 of Luenberger (1998).

Example 4.2 (*our initial Brownian stochastic integral example*) Suppose we apply Ito's lemma to ordinary BM with $f(x) = x^2$. We start with the SDE in (20) being dB (having $a = 0$ and $b = 1$). When we apply Ito's lemma with $f(x) = x^2$, we get

$$dB(t)^2 = dt + 2B(t)dB(t) \tag{32}$$

or

$$B(t)^2 = t + 2\int_0^t B(s)\,dB(s) \ , \tag{33}$$

just as in our initial BM stochastic integral.

5 The Black-Scholes Equation

The Black-Scholes equation is a **partial differential equation (PDE)** satisfied dy a derivative (financial) of a stock price process that follows GBM, under the no-arbitrage condition. We start with GBM represented as an SDE. We then apply Ito's lemma to describe the derivative as an SDE. However, the arbitrage-free condition serves to eliminate the stochastic BM component, leaving only a deterministic PDE for the time-dependent and state-dependent price of the security.

In more detail, we start with the stock price process satisfying the GBM SDE in (23). We represent the price of the derivative of the stock price process as $f(S(t),t)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. We then apply Ito's lemma and consider the consequences of the arbitrage-free condition. That forces the Black-Scholes PDE. See Hull (2006) and Luenberger (1998) for quick informal proofs. See Steele (2001) for a more careful treatment, with insightful discussion.

Before we apply the arbitrage-free argument, we simply apply Ito's Lemma to obtain an SDE for the stock derivative. Given that S follows the GBM SDE in (23), we can apply (25) to see that the derivative, say $Y(t) = f(S(t), t))$, satisfies the associated SDE

$$dY(t) \equiv df(S(t), t) = \left(\frac{\partial f}{\partial t} + \mu S(t) \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma S(t) \frac{\partial f}{\partial x} dB(t). \tag{34}$$

It is understood that the partial derivatives in (34) are evaluated at $(S(t), t)$, as made explicit in (26).

An additional finance argument is needed to go from this SDE to the Black-Scholes PDE. Suppose that the interest rate is r. Risk neutrality is achieved by setting $\mu = r$, using the SDE representation in (23). The resulting Black-Scholes PDE is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \ . \tag{35}$$

In more detail, we have

$$\frac{\partial f}{\partial t}(S(t), t) + \frac{\partial f}{\partial S}(S(t), t) rS(t) + \frac{1}{2} \sigma^2 S(t)^2 \frac{\partial^2 f}{\partial S^2}(S(t), t) = rf(S(t), t) \ . \tag{36}$$

References

Hull, J. C. 2006. *Options, Futures and Other Derivatives*, sixth edition, Prentice Hall, Englewood Cliffs, NJ.

Karatzas, I. and S. E. Shreve. 1988 *Brownian Motion and Stochastic Calculus*, Springer, New York.

Luenberger, D. G. 1998. *Investment Science*, Oxford University Press, New York.

Steele, J. M. 2001. *Stochastic Calculus and Financial Applications*, Springer, New York.