IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

Lecture notes on the Black-Scholes Equation, August 20

1 Introduction

In these notes we derive the Black-Scholes equation, i.e., the Black-Scholes partial differential equation (PDE) for the price of a financial derivative of a stock price, where the stock price is assumed to evolve according to geometric Brownian motion. Let $f(t, x)$ be the unique arbitrage-free price of the financial derivative at time t when the stock price at that time is x. In addition to x and t, the Black-Scholes PDE depends on the fixed interest rate r and the volatility parameter σ. The Black-Scholes PDE is

$$rf = \frac{\partial f}{\partial t} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} \ , \tag{1}$$

where all functions are evaluated at (x, t).

As usual with PDE's, we need to specify boundary conditions in order to actually determine the financial derivative price function f. For example, the financial derivative might be a European call option, giving the opportunity to buy shares of the stock for strike price K at the expiration tine T. The value of f is specified at time T and we can work backwards to find the price $f(x, t)$ for previous times t, with $0 \le t < T$. In that special case,

$$f(x, T) = (x - K)^+ \equiv \max\{0, x - K\} \ . \tag{2}$$

For this special financial derivative, there is a closed form expression for the price as a function of the variables (K, T, x, σ, r); see (10.12) on p. 641 of Ross. With that explicit formula specified, it is easy to see that f satisfies the Black-Scholes PDE. The Black-Scholes PDE is important because it applies to general financial derivatives, not just European call options.

Most of these notes review material already covered. The derivation of the Black-Scholes PDE is done in §7. That is preceeded by a review of a few basic notions of asymptotic analysis in §6.

We start in §2 by reviewing the analysis of the one-period model. That is the essential step. With one period, we have the stock starting at price S, and then either going up to Su or down to Sd. Next in §3 we discuss how to extend to multiple periods, obtaining the binomial lattice model (BLM). Afterwards, in §4 we show how GBM can be approximated by a BLM. Once that has been done, the entire analysis can be done with the BLM; it is not really necessary to have the Black-Scholes PDE.

Of course, everything is based on GBM. In §5 we briefly discuss the strong theoretical support for GBM stemming from the central limit theorem (CLT). Ultimately, the CLT is key. It provides strong theoretical justification for using the GBM model, which in turn (by the sequence of steps discussed in these notes) is essentially equivalent to the simple one-period model in the next section, where there are only two outcomes.

Moreover, the BLM approximation - actually only the single-period version - can be used to derive the Black-Scholes PDE for the GBM model. With appropriate asymptotics, everything follows from the simple one-period model in the next section.

2 The One-Period Derivative Pricing Model

We here consider the problem of pricing a derivative of a stock in a simple highly structured single-period model. We assume that there is a stock with initial price S dollars per share. There is a single time period. We assume that the stock price will either go up to uS or go down to dS at the end of the period, as shown in Figure 1. We estimate that the probability of the stock going up is p; and the probability of it going down is $1 - p$, but this probability will turn out to play no role.

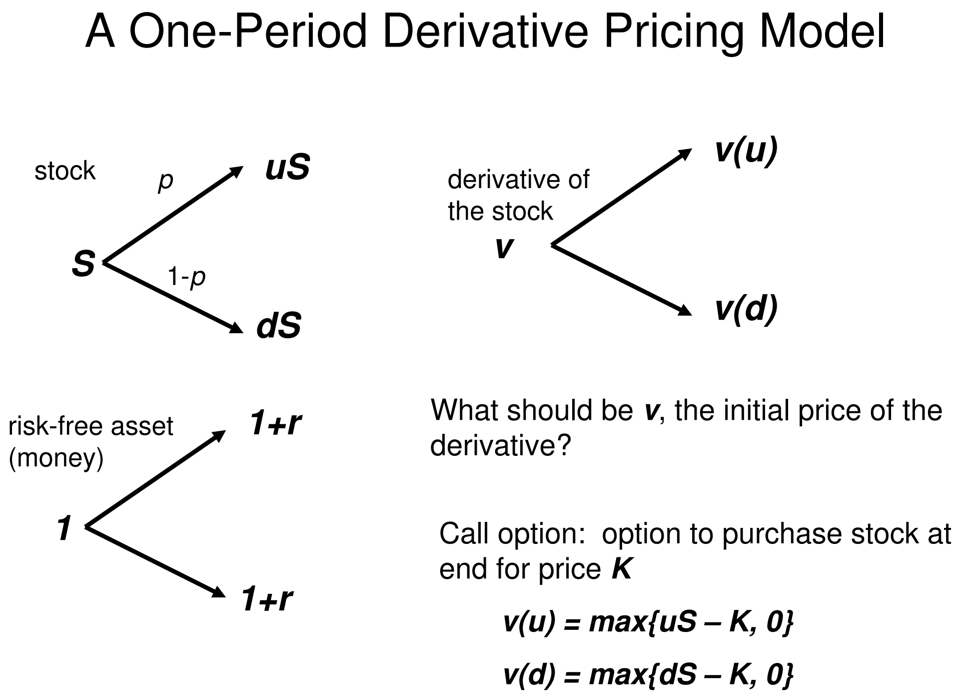

Figure 1: The problem of pricing a derivative of a stock.

There is also a risk-free asset - money. We can borrow or lend money at the fixed interest rate r. Each dollar put in the bank earns $1 + r$ dollars at the end of the period; we must pay $1 + r$ dollars at the end of the period for each dollar borrowed from the bank. We assume that $u > 1 + r > d$. That extra inequality is to avoid uninteresting cases: If $1 + r > u$, then you would never invest in the stock; if $d > 1 + r$, then you would never put money in the bank.

Now we come to the special feature we want to consider: There is also a **derivative of the stock**, such as a European call option. The price of the derivative at the end of the period depends on what happens to the stock. If the stock price goes up from S to uS, then the price of the derivative becomes $v(u)$; if the stock price goes down from S to dS, then the price of the derivative becomes $v(d)$. We want to know what is the appropriate price or value v of the derivative at time 0.

A standard example of a derivative is a **European call option** - an option to buy the stock at the end of the period for a specified price K per share. Assuming that $uS > K > dS$, if you buy this particular derivative, then you will make $uS - K$ if the stock price goes up, because you will elect to buy the stock at the end of the period when you see that the price has gone up above K. On the other hand, you will make 0 (and not lose anything either) if the stock price goes down, because you will then not elect to buy the stock. But the option costs money. The question, then, is: What is a fair or appropriate price for the option? More generally, we ask: What is a fair or appropriate price for a derivative, as a function of its specified terminal payoffs $v(u)$ and $v(d)$?

2.1 Replication

The key idea is that the derivative can be *replicated* by a strategy of buying or selling the stock and borrowing or lending the risk-free asset. This replication forces the payoffs for the derivative to be identical to the investment strategy for the stock and risk-free asset for every possible random outcome of the stock. (In this setting there are only two possible outcomes.) Hence there will be only one permissible price for the derivative (under these model assumptions). If that price does not prevail, then an arbitrage opportunity necessarily exists: We have two investment opportunities with identical outcomes; if the prices are different, then we buy a massive amount of the cheaper one and sell the same massive amount of the more expensive one, and consequently make massive profit, guaranteed, with no risk. It is significant that this argument does not depend on the transition probabilities of the stock; we do not need to know the probability p in Figure 1. For this reason, we do not use the probability p at all.

To carry out this replication, we will now construct a special portfolio (α, β) of the stock and the risk-free asset in order to replicate the derivative in both cases (if it goes up or if it goes down). The portfolio (α, β) means that we buy α shares of the stock and put β dollars in the bank. We let α and β be arbitrary real numbers, positive or negative; e.g., we can buy or sell fractional shares of the stock. If α is negative, we sell the stock; if β is negative, then we borrow from the bank (at the same interest rate r). The initial value of this portfolio is $\alpha S + \beta$.

To achieve the desired replication, the value of the portfolio at the end of our single period must match the value of the derivative at that same time, if the stock goes up or if it goes down. That leads to two equations:

$$\alpha u S + \beta(1+r) = v(u) \quad \text{if the stock price goes up} \tag{3}$$

and

$$\alpha d S + \beta(1+r) = v(d) \quad \text{if the stock price goes down} \tag{4}$$

Here these final derivative prices $v(u)$ and $v(d)$ are assumed to be known numbers. Those two equations constitute two equations in the two unknowns α and β. We can solve for these variables, obtaining the **replicating portfolio**:

$$\alpha = \frac{v(u) - v(d)}{S(u-d)} \tag{5}$$

and

$$\beta = \frac{uv(d) - dv(u)}{(1+r)(u-d)} \ . \tag{6}$$

In summary, we replicate the derivative payoffs exactly if we initially (at time 0) buy α shares of the stock and put β dollars in the bank, where these specific values are given above in (5) and (6).

2.2 The Arbitrage-Free Price of the Derivative

Since the portfolio replicates the derivative, the values of these two investments at time 0 must be identical. Hence the **initial value (unique arbitrage-free price) of the derivative** is

v, where

$$v = \alpha S + \beta = \frac{v(u) - v(d)}{u - d} + \frac{uv(d) - dv(u)}{(1 + r)(u - d)} \ . \tag{7}$$

If this value did not prevail, then we would have an arbitrage opportunity.

2.3 The Risk-Neutral Probability

It is next useful to observe that this expression for v in (7) can be simplified to yield

$$v = \left(\frac{1}{1 + r} \right) (p^* v(u) + (1 - p^*) v(d)) \ , \tag{8}$$

where

$$p^* = \frac{1 + r - d}{u - d} \quad \text{and} \quad 1 - p^* = \frac{u - (1 + r)}{u - d} \ . \tag{9}$$

Since we have assumed that $u > 1 + r > d$, we see that p^* is a probability, called the **risk-neutral probability**. Notice that this risk-neutral probability p^* in (9) need not agree with any a priori probability p specified for the stock. If we started with a probability p, then we would perform a **change of measure** to change to the risk-neutral probability distribution based on p^*.

Let V_n be the value (or price) of the derivative at time n. We know that V_1 is a random variable, equal to $v(u)$ if the stock goes up, and equal to $v(d)$ if the stock goes down. The risk-neutral probability makes $V_0 \equiv v$ equal to the expected present value (at time 0) of V_1 with respect to the probability p^*. That is, formula (8) can be restated in terms of an expected value with respect to the risk-neutral probability distribution as follows:

$$V_0 = \frac{1}{1 + r} E_*[V_1] \ , \tag{10}$$

where E_* denotes the expectation with respect to the Bernoulli distribution in (9) with probability p^*. In fact, the same is true for the stock itself. Let S_n be the value of the stock at time n. Then $S_0 = S$ and S_1 is a random variable, either equal to uS if the stock goes up or dS if the stock goes down. Then

$$\begin{aligned} \frac{1}{1 + r} E_*[S_1] &= \frac{1}{1 + r} (p^* Su + (1 - p^*) Sd) \\ &= \frac{1}{1 + r} \left(\frac{1 + r - d}{u - d} Su + \frac{u - (1 + r)}{u - d} Sd \right) \\ &= \frac{1}{1 + r} \left(\frac{(1 + r)(u - d)}{u - d} S \right) = S \ . \end{aligned} \tag{11}$$

This is a very important conclusion: **We can find the arbitrage-free price of the derivative by first finding the risk-neutral probability for the stock.** We then can compute the expected value to get the derivative price at time 0. Indeed, it is common to start by finding the risk-neutral probability for the stock. There is a general theorem, called the Arbitrage Theorem, that supports this step more generally. It states, for a more general setting, that there exists an arbitrage-free price for the derivative if and only if there exists associated risk-neutral probabilities. To have a simple clean story, we rely heavily on having only two possible outcomes. **Having only two possible outcomes enables us to obtain a unique arbitrage-free price in the one-period model** we have just considered.

3 Extending the Model to Multiple Periods

We now extend the model to multiple time periods. To do so, we first give some background.

3.1 Bernoulli Trials

A sequence of *Bernoulli trials* is a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$, where

$$P(X_n = 1) = p = 1 - P(X_n = 0), \quad n \geq 1.$$

We use Bernoulli trials to model the outcomes of successive coin tosses; we may say that $X_n = 1$ if we get heads on the n^{th} toss. Then the partial sum $S_n \equiv X_1 + \cdots + X_n$ is the number of heads on the first n tosses. The random variables S_n has a *binomial distribution* with parameters n and p:

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n \ ;$$

see Sections 2.2.1 and 2.2.2 on pages 28-29 of Ross (2006). These are bonafide probabilities because they are positive and

$$\sum_{k=0}^{n} P(S_n = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 \ .$$

To verify this last claim, we can apply the *binomial theorem*, giving the value of $(x+y)^n$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \ .$$

For the binomial distribution, the sum of the probabilities is thus a representation of $(p + (1-p))^n = 1^n = 1$.

3.2 The Binomial Lattice Model

The *binomial lattice model* is a modification of a sequence of Bernoulli trials, created to model stock prices. We again consider a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$, each assuming only two values. We make two changes: First, we allow the two possible values of X_n to be general: Instead of 1 and 0, the possible values of X_n are u and d for *up* and *down*. We now have

$$P(X_n = u) = p = 1 - P(X_n = d), \quad n \geq 1.$$

Second, we consider a multiplicative model instead of an additive model. We let the initial value be $S_0 = S$ and let

$$S_n = S_0 \times X_1 \times X_2 \times \cdots \times X_n, \quad n \geq 1 \ .$$

Pictorially, the possible values of S_k for $0 \leq k \leq n = 5$ with a binomial lattice model are depicted in Figure 3.2. This is often called a binomial tree.

Binomial Lattice Model

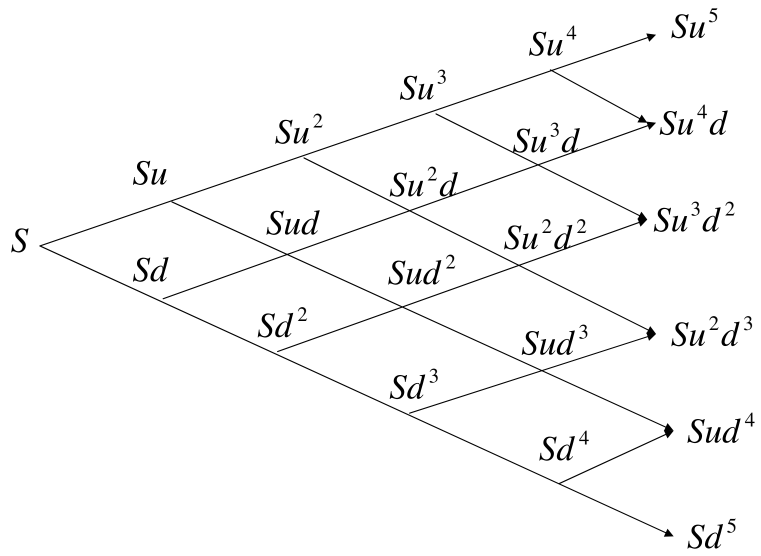

Figure 2: The general tree for the binomial lattice model with $n = 5$.

In the displayed tree there are $n = 5$ stages (beyond the initial stage 0). The random variables S_n is intended to represent the stock price at stage n. The random variable X_n is then the random multiple taking S_{n-1} into S_n:

$$S_n = S_{n-1}X_n, \quad n \geq 1 \ .$$

Since the random multipliers are assumed to be i.i.d., the stochastic process $\{S_n : n \geq 0\}$ is a Markov chain: the probability of future states conditional on the past and present depends only on the present state. The possible values of S_n are still determined by binomial(n, p) probabilities

$$P\left(S_n = Su^k d^{n-k}\right) = \left(\begin{array}{c} n \\ k \end{array}\right) p^k(1-p)^{n-k}, \quad 0 \leq k \leq n \ . \qquad (12)$$

3.3 Pricing Derivatives with Multiple Periods

We now return to the problem of pricing a derivative of a stock, but now we do so with multiple time periods. Now we assume that the stock evolves according to the binomical lattice model (BLM). We are given the values of the derivative for all possible outcomes at period n. (It is also possible to consider more general derivatives.) Let S_n be the price of the stock in period n; let V_n be the arbitrage-free value of the derivative in period n. We are given the values of the derivative for all possible outcomes at period n, but it remains to determine appropriate (arbitrage-free) values V_k of the derivative, as a function of the state then, for $k < n$. We can use the previous single-period analysis to determine the arbitrage-free prices of the derivative at the previous period $n-1$, using the logic just presented. The situation for the stock is the same in period $n-1$, except that the initial price at that time will be different. But starting from any stock price S_{n-1}, it will either go up to uS_{n-1} or go down to dS_{n-1}. The single-period analysis applies again, yielding

$$V_{n-1} = \frac{1}{1+r} E_*[V_n] \ , \tag{13}$$

where, as before, E_* denotes the expectation with respect to the Bernoulli distribution in (9) with probability p^* in (9). Notice from (9) that these risk-neutral probabilities are independent of the period n and the history up to the state at that period.

We can thus calculate all the arbitrage-free derivative prices at stage $n-1$, using the one-period analysis above. We then can proceed recursively in this way to stage $n-2$ and so on back to stage 1 and then stage 0. When we proceed in this way, we derive the replication strategy for every conceivable stock price S_k at time k for all k, $0 \le k \le n-1$.

Given the derivative payoffs at time n and the risk-neutral probability p^*, we can easily compute the expectation in (13). We can then recursively perform similar calculations until we arrive at the arbitrage-free price of the derivative at time 0, V_0, which was to be determined. We could also write

$$V_0 = \left(\frac{1}{1+r}\right)^n E_*\left[V_n\right] \ , \tag{14}$$

where E_* is the expectation with respect to a binomial probability distribution with parameters n and p^*.

For the special case in which the derivative is actually a call option having payoffs $V_n = (S_n - K)^+$, we have

$$\begin{aligned}
V_0 &= \left(\frac{1}{1+r}\right)^n E_*\left[V_n\right] \\
&= \left(\frac{1}{1+r}\right)^n E_*\left[(S_n - K)^+\right] \\
&= \left(\frac{1}{1+r}\right)^n \sum_{k=0}^{n} (u^k d^{n-k} - K)^+ P(S_n = u^k d^{n-k}) \\
&= \left(\frac{1}{1+r}\right)^n \sum_{k=0}^{n} (u^k d^{n-k} - K)^+ \binom{n}{k} (p^*)^k (1-p^*)^{n-k} \ .
\end{aligned} \tag{15}$$

However, we usually want to compute all possible derivative values V_k depending on all possible stock prices S_k, because that yields the replication strategy (for all time periods) as a byproduct.

4 An Approximation to Geometric Brownian Motion

The binomial lattice model is often introduced as a discrete approximation to geometric Brownian motion (GBM), which in turn is a commonly used continuous-time stochastic process to model security prices. After taking logarithms, this discrete approximation corresponds to the the relatively familiar approximation of Brownian motion (BM) by a simple random walk, as described at the beginning of Chapter 10 of Ross, but here we have a non-zero drift. This non-zero drift makes the analysis somewhat more complicated, but otherwise the story is just as in Section 10.1 of Ross (2006).

In fact, it is natural to think of the more elementary random walk as the basic object. When we want to understand what is Brownian motion, we think of Brownian motion as the limit of a sequence of the random walks as the time periods get short. This can be formalized as a proper limit, called a *stochastic-process limit*: a sequence of appropriately defined random walks converges to Brownian motion. From a practical perspective, to understand BM, we think of it as a random walk. Similarly, GBM can be expressed as the limit of a sequence of BLM's. To understand GBM, we think of it as a BLM. We can perform calculations by either using GBM or an associated BLM. In this section we will show how to relate them.

4.1 Brownian motion

We first give some minimal background on Brownian motion; see Chapter 10 of Ross (2006). Brownian motion is a continuous-time stochastic process with continuous real-valued sample paths. Brownian motion is a Markov process: the conditional distribution of future states, conditional on the present and past states depends only on the present state. Brownian motion is also a Gaussian process; i.e., all its joint distributions are multivariate normal. Brownian motion has independent increments. We let $\{B(t) : t \geq 0\}$ denote standard Brownian motion. The random variable $B(t)$ has a normal distribution with mean 0 and variance t. The increment $B(t) - B(s)$ is distributed as $B(t-s)$ and so has mean 0 and variance $t-s$. Thus $B(0) = 0$.

If $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion (with 0 drift coefficient and unit (1) variance coefficient), we can get an associated (ν, σ^2) Brownian motion (BM) $\{X(t) : t \geq 0\}$, with drift coefficient ν and variance coefficient σ^2 (i.e., $E[X(t)] = \nu t$ and $Var(X(t)) = \sigma^2 t$ for all $t \geq 0$) by setting

$$X(t) \equiv \nu t + \sigma B(t), \quad t \geq 0 .$$

(It is customary to use μ for the drift coefficient of BM, but we will want to use μ for another purpose below, which leads us to use different notation.) Clearly, $X(t)$ is distributed as $N(\nu t, \sigma^2 t)$, where $N(a, b)$ denotes a normal random variable with mean a and variance b. Note that $X(0) = B(0) = 0$.

4.2 Geometric Brownian Motion

Then geometric Brownian motion (GBM) with parameters ν and σ, and with initial value $Y(0)$ regarded as constant, can be defined by

$$Y(t) \equiv Y(0)e^{X(t)} = Y(0)e^{\nu t + \sigma B(t)}, \quad t \geq 0 .$$

In other words, GBM is just the exponential of a BM. Dividing by the constant $Y(0)$ and taking logarithms, we recover the BM, getting

$$\ln{(Y(t)/Y(0))} = X(t) = \nu t + \sigma B(t), \quad t \geq 0 .$$

People often characterize GBM via a *stochastic differential equation*, writing

$$dY = \mu Y dt + \sigma Y dB , \tag{16}$$

but we shall not discuss this approach. However, we do comment that μ in (16) does not coincide with ν above. In fact, $\nu = \mu - \sigma^2/2$; equivalently, $\mu = \nu + \sigma^2/2$. This can be justified by applying Ito's Lemma. We have $E[Y(t)] = Y(0)e^{\mu t}$ and $E[\ln{(Y(t)/Y(0))}] = \nu t$.

Notice that the parameters ν and σ^2 are the mean and variance, respectively, of $\ln{(Y(t)/Y(0))}$ at $t = 1$. They are not the mean and variance of $Y(t)$ or $Y(t)/Y(0)$ at $t = 1$. This construction gives $Y(t)$ a *lognormal distribution* with mean

$$E[Y(t)/Y(0)] = E[e^{X(t)}] = Y(0)E[e^{\nu t + \sigma B(t)}] = Y(0)e^{\nu t + \sigma^2 t/2} = Y(0)e^{\mu t} ,$$

second moment

$$E[(Y(t)/Y(0))^2] = E[(e^{X(t)})^2] = E[(e^{2X(t)}] = e^{2\nu t + 2\sigma^2 t} ,$$

and variance

$$Var((Y(t)/Y(0)) = E[(Y(t)/Y(0))^2] - (E[Y(t)/Y(0)])^2 = e^{2\nu t+\sigma^2 t}(e^{\sigma^2 t} - 1)$$

These calculations are based on simple properties of the exponential function and the form of the moment generating function of a normal random variable; see Example 2.42 on page 67 of Ross (2006).

4.3 Approximating GBM by a BLM

To approximate GBM by a binomial lattice model (BLM), we first simplify by letting $d = 1/u$ in our binomial lattice model. This choice would be obvious if $\nu = 0$, because then $\ln(Y(t)/Y(0)$ would have a normal distribution with mean 0. It would then be natural to have $\ln(d) = -\ln(u)$, which is equivalent to $d = 1/u$. We do this more generally because this is the standard approach.

That leaves two parameters to specify: u and p. To make the approximation, we have to decide how we are going to break up time. It is common to think of time t expressed in years, so that $t = 1$ corresponds to one year. Suppose that we do indeed measure time in years and let Δ be the time period in the discrete model. If time in the discrete model represents weeks, then $\Delta = 1/52 \approx 0.0192$; if time in the discrete model represents months, then $\Delta = 1/12 \approx 0.0833$; time in the discrete model represents days, then $\Delta = 1/365 \approx 0.002740$ (ignoring the leapyear possibility).

Given time measured in years and the parameters ν and σ measured in that time scale, and having chosen the length of a time period Δ in the approximating BLM ($\Delta = 1/n$ if $n\Delta = 1$), we can make the approximation by matching the first two moments of the distribution of $\ln(S_n/S_0)$ to $\ln(Y(n\Delta)/Y(0))$. Doing so gives

$$E[\ln(S_n/S_0)] = nE[\ln(S_1/S_0)] = n\nu\Delta$$

and

$$Var(\ln(S_n/S_0)) = nVar(\ln(S_1/S_0)) = n\sigma^2\Delta$$

Since both the left and right sides of both equations are proportional to n, we can divide through by n, which is tantamount to considering the case of $n = 1$:

$$E[\ln(S_1/S_0)] = \nu\Delta$$

and

$$Var(\ln(S_1/S_0)) = \sigma^2\Delta$$

However, letting $U = \ln(u)$, we see that these two equations are equivalent to

$$pU + (1-p)(-U) = (2p-1)U = \nu\Delta \tag{17}$$

and, working with the second moment instead of the variance,

$$pU^2 + (1-p)(-U)^2 = U^2 = \sigma^2\Delta + (\nu\Delta)^2 \ . \tag{18}$$

This gives us two equations in the two unknowns p and U. They can be combined to produce a single quadratic equation in one unknown, but a simplification follows from the observation that Δ is supposed to be small, so that Δ^2 should be much smaller than Δ. Assuming that ν is of the same order as σ (or smaller), we then can deduce that $(\nu\Delta)^2$ should be small compared to $\sigma^2\Delta$. Assuming that indeed $(\nu\Delta)^2$ is suitably small compared to $\sigma^2\Delta$, we can omit the term $(\nu\Delta)^2$ with little loss in accuracy, and that is what we do.

Hence we replace the last equation (18) by

$$U^2 \approx \sigma^2 \Delta \ , \tag{19}$$

which immediately yields

$$U = \sigma\sqrt{\Delta} \quad \text{and} \quad u = e^U = e^{\sigma\sqrt{\Delta}} \ . \tag{20}$$

Since $d = 1/u$, we also have

$$d = e^{-U} = e^{-\sigma\sqrt{\Delta}} \ . \tag{21}$$

Finally, plugging (20) into (17), we obtain

$$p = \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\Delta} \ . \tag{22}$$

Notice that (17) for the mean is satisfied exactly after this simplifying approximation, so that we have introduced no error in the mean, but there is a small error in the second moment and the variance; (18) is not satisfied exactly when we go to (19).

Summary. Given a specified time scale such as years ($t = 1$ means one year) in which the parameters ν and σ^2 are specified, let each discrete time period be of length Δ (assumed to be relatively short, such as a month, week or day in the yearly time scale), we let $S_0 = Y(0) = S$ be the initial price of the stock and then let the parameters u, d and p in the approximating binomial lattice model be given by (20), (21), and (22), i.e.,

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} \\ d &= 1/u = e^{-\sigma\sqrt{\Delta}} \\ p &= \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\Delta} \ . \end{aligned} \tag{23}$$

We now give an illustrative numerical example; it is Example 11.1 from Luenberger (1998).

Example 4.1 Consider a stock with a yearly time scale, where the parameters ν and σ are given by with $\nu = 0.15$ and $\sigma = 0.30$. (That is a relatively volatile stock; a more common parameter value would be $\sigma = 0.15$. The associated mean parameter is $\mu = \nu + \sigma^2/2 = 0.15 + 0.045 = 0.195$) Assume that the initial stock price is $S_0 = 100$. Make a binomial lattice model with weekly time periods, assuming that the stock prices follow a GBM in continuous time.

Answer: With the time specifications, $\Delta = 1/52 \approx 0.01923$. From (23), we get

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} = e^{0.30/\sqrt{52}} \approx 1.04248 \\ d &= 1/u \approx 0.95925 \\ p &= \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\Delta} = \frac{1}{2} + \frac{0.15}{20.30\sqrt{52}} = 0.534669 \ . \end{aligned} \tag{24}$$

Given that the initial stock price is $S_0 = 100$, we get the tree in Figure 4.1. We do not show the probabilities in this tree. They are as in (12) with $p = 0.534669$.

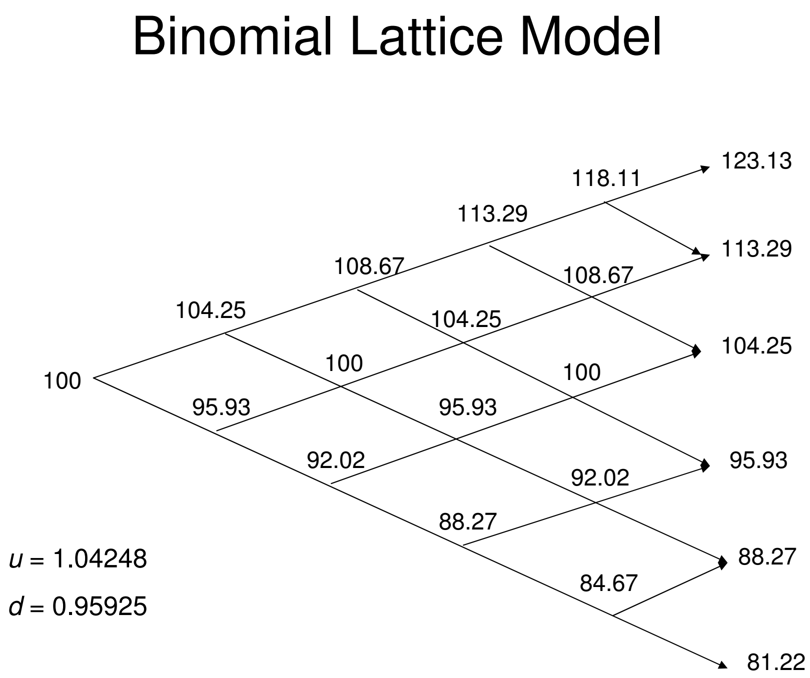

Figure 3: The binomial lattice model approximating the specified geometric Brownian motion in Example 4.1. The time periods are weeks in a yearly time scale. Time periods 0 through 5 are shown.

Example 4.2 Now return to Example 4.1, where we developed a BLM approximation for a GBM with parameters $\nu = 0.15$ and $\sigma = 0.30$ with a yearly time scale. As before, assume that the initial stock price is $S_0 = 100$. Figure 4.1 displays a binomial lattice model with weekly time periods. Now suppose that the interest rate is 8% compounded weekly. Find the arbitrage-free price of an option to buy the stock at $K = 98$ dollars per share after 5 weeks.

Answer: We previously derived the parameters of the BLM. With the time specifications, $\Delta = 1/52 \approx 0.01923$. From (23), we got

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} = e^{0.30/\sqrt{52}} \approx 1.04248 \\ d &= 1/u \approx 0.95925 \\ p &= \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\Delta} = \frac{1}{2} + \frac{0.15}{2(0.30)\sqrt{52}} = 0.534669\ . \end{aligned} \tag{25}$$

Given that the initial stock price is $S_0 = 100$, we get the tree in Figure 4.1, just as before. However, instead of the probability p just computed, we use the risk-neutral probability p^*. To compute it, we need to determine r. We have assumed that the interest is compounded weekly, which is consistent with our time scale. The interest rate per week is then

$$r = 0.08/52 = 0.001538\ ,$$

which yields an annual interest of $(1 + 0.001538)^{52} = 1.083$, corresponding to 8.3%, which exceeds the originally specified 8% because of the compounding. Over individual weeks, the interest rate does not play a big role. The risk-neutral probability then is

$$p^* = \frac{1 - r - d}{u - d} = \frac{1 + 0.001538 - 0.9593}{1.0425 - 0.9593} = \frac{0.0422}{0.0832} = 0.5072$$

Now we proceed to develop the values V_k of the option in period k depending on the state of the stock. To do so, we first put the values of the option in at the final period $n = 5$. The values there are $(S_n - K)^+ = (S_5 - 98)^+$. We then calculate the values of the option in preceding periods using equation (13). The option values are displayed in Figure 4.2. For $n = 5$, the option values depend on the stock prices that may occur at that period; i.e., we have just computed $(S_n - K)^+ = (S_5 - 98)^+$ in each case, where S_5 is given in Figure 4.1. We then calculate the arbitrage-free option prices in previous periods by computing the discounted expected present value, as in (13). The values are displayed in Figure 4.2. We see that the arbitrage-free option price initially is $V_0 = 5.28$ dollars per share.

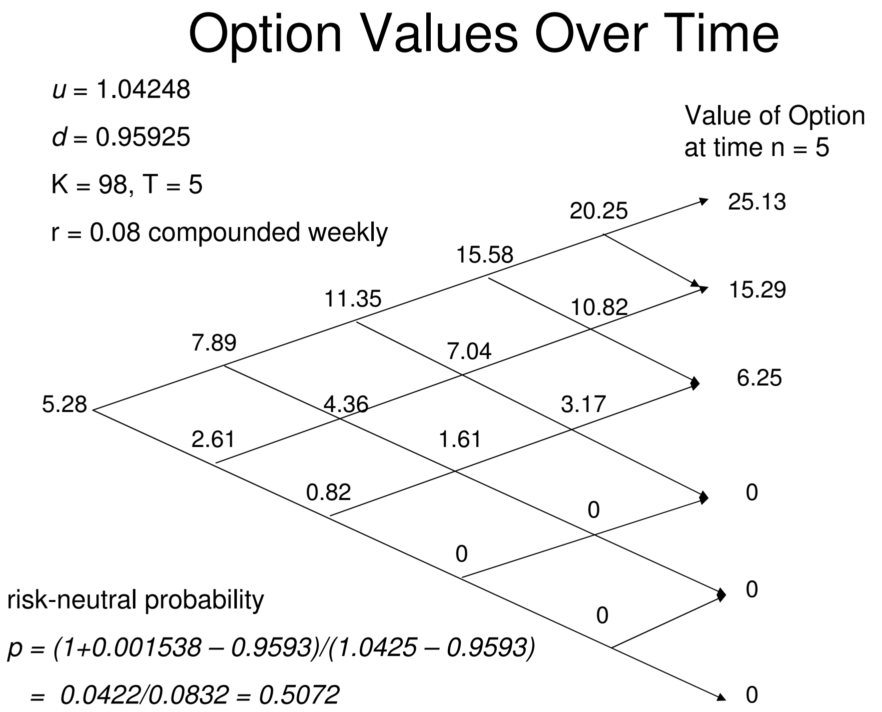

Figure 4: The option prices over time determined by the binomial lattice model for Example 4.2.

Figure 4.2 shows all the option values over time, but it does not show the replicating strategy, but it is easy to construct the replicating strategy. It is given by equations (5) and (6). To determine the strategy at any point in the tree, let $v(u)$ and $v(d)$ be the option values at the next stage if the stock goes up or down, respectively. To illustrate, suppose that we are at state Su^3 after the stock price has gone up three times. From Figures 4.1 and 4.2, we see that the stock price in this state is 113.29, while the option value is 15.57. From that node in the tree, $v(u) = 20.25$ and $v(d) = 10.82$. We use these to calculate the replicating strategy in this state. We get

$$\alpha = \frac{v(u) - v(d)}{S(u - d)} = \frac{20.25 - 10.82}{113.29(1.04248 - 0.95925)} = \frac{9.43}{9.43} = 1.00 \tag{26}$$

and

$$\beta = \frac{uv(d) - dv(u)}{(1+r)(u-d)} = \frac{1.04248(10.82) - 0.95925(20.25)}{(1.001538)(0.0832)} = -97.75\ . \tag{27}$$

Note that $1.00 \times 113.29 - 97.75 \times 1 = 113.29 - 97.75 = 15.54$. This is close to the directly computed value of the option of 15.57.

4.4 Testing Your Understanding

Example 4.3 In Example 4.1 we developed a BLM approximation for a GBM with parameters $\nu = 0.15$ and $\sigma = 0.30$ with a yearly time scale, and used it to price an option. As before, assume that the initial stock price is $S_0 = 100$. Figure 4.1 displays a binomial lattice model with weekly time periods. We now make some changes. Now suppose that the interest rate is 26% compounded weekly. Find the arbitrage-free price of an option to buy the stock at $K = 98$ dollars per share after 3 weeks. Notice that the interest rate r and the expiration time T have been changed.

(a) What are the option values over time?

(a) What is the replicating strategy to use at time 2 after the stock has gone up twice, i.e., in state Su^2?

Answers: (a) The BLM is just as in Figures 3.2 and 4.1, but now the arbitrage-free option values over time change. Note that $r = 0.26/52 = 0.005$, so that the risk-neutral probabilty is $p^* = (1.005 - d)/(u - d) = 0.5493$. We can use this risk-neutral probability to compute the option values in previous periods. They now are as shown in Figure 4.3 instead of as in Figure 4.2.

(b) Figure 4.3 shows all the option values over time, but it does not show the replicating strategy, but it is again easy to construct the replicating strategy. It is given by equations (5) and (6). To determine the strategy at any point in the tree, let $v(u)$ and $v(d)$ be the option values at the next stage if the stock goes up or down, respectively. As asked, suppose that we are at state Su^2 after the stock price has gone up two times. From Figures 4.1 and 4.2, we see that the stock price in this state is 108.67, while the option value is 11.21. From that node in the tree, $v(u) = 15.29$ and $v(d) = 6.25$. We use these to calculate the replicating strategy in this state. We get

$$\alpha = \frac{v(u) - v(d)}{S(u-d)} = \frac{15.29 - 6.25}{108.67(1.04248 - 0.95925)} = \frac{9.04}{9.04} = 1.00 \tag{28}$$

and

$$\beta = \frac{uv(d) - dv(u)}{(1+r)(u-d)} = \frac{1.04248(6.25) - 0.95925(15.29)}{(1.005)(0.0832)} = -97.49\ . \tag{29}$$

Note that $1.00 \times 108.67 - 97.49 \times 1 = 108.67 - 97.49 = 11.18$. This is close to the directly computed value of the option of 11.21. The small error seems to be due to rounding.

5 Why GBM?

We have seen that a geometric Brownian motion (GBM) can be approximated by a binomial lattice model (BLM). After that, we can carry out the analysis described above for the BLM. But we should ask if it is really appropriate to use GBM? It is a very special stochastic process characterized by only two parameters. That parsimonious description is critical for having

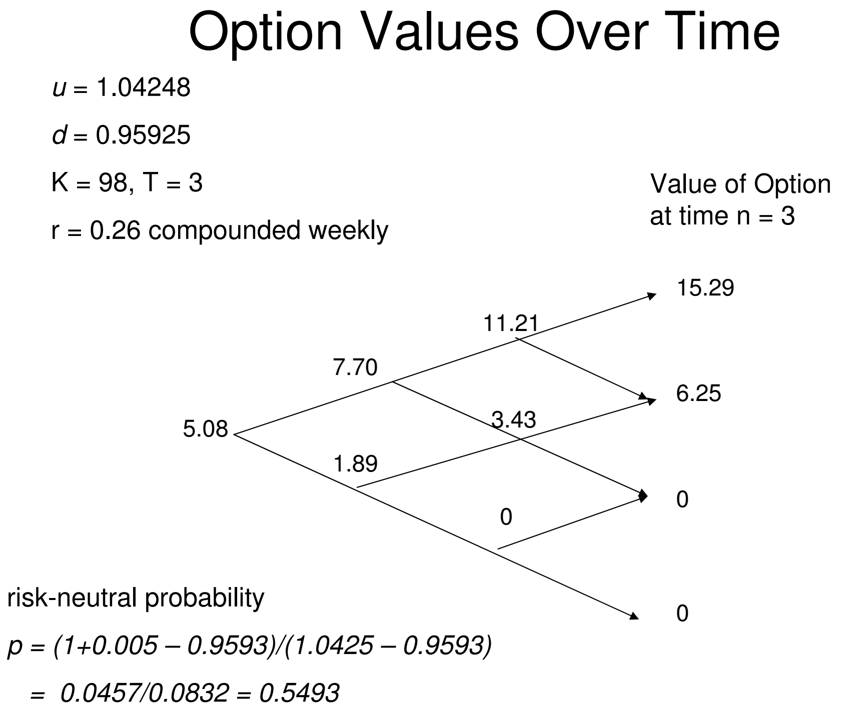

Figure 5: The option prices over time determined by the binomial lattice model for Example 4.3.

a unique arbitrage-free price for the derivative at all times. So we should ask: What is the support for the GBM model?

Fortunately, there actually is very strong theoretical support for the GBM. First, it is natural to consider a multiplicative model where the stock price is of the form

$$S_n = S_0 Y_1 \times \cdots \times Y_n \ ,$$

where the multipliers $Y_1, \ldots, Y_n$ are independent and identically distributed random variables, or nearly so, but with more general distributions. But it is not natural to assume that the distribution of Y_i concentrate on only two possible values. Fortunately, we can justify GBM from this framework, allowing the random multipliers Y_i to have general distributions.

The first step in the justification is to apply logarithms to convert the random product to a random sum. By taking logarithms, we get

$$\ln (S_n/S_0) = \ln (Y_1) + \cdots + \ln (Y_n) \ .$$

In this setting, see that the Central Limit Theorem implies that these sums should be approximately normally distributed, provided only that the random variables being added have finite variance. More generally, extensions of the Central Limit Theorem imply that the entire sequence of partial sums - the random walk - converges to Brownian motion. As a consequence, the stock-price process itself converges to GBM. So there is indeed very strong theoretical support for GBM.

6 Asymptotic Analysis

We will perform asymptotic analysis to derive the Black-Scholes partial differential equation (PDE), starting from the single-period model. We will consider the single-period model as the time interval $\Delta \to 0$. The terms of order Δ will appear in the final answer. Smaller terms, i.e., terms of order Δ^p for $p > 1$ will be asymptotically neglibible.

Definition 6.1 (*little "oh" notation*) *A function f of a real variable x is said to be asymptotically negligible as $x \to 0$, and we write $f(x) = o(x)$ as $x \to 0$, if*

$$\lim_{x \to 0} \frac{f(x)}{x} = 0 \ .$$

We will use Taylor series approximations, which are supported by Taylor's theorem. However, we will not be concerned with the exact form of the remainder.

Theorem 6.1 (*Taylor's theorem*) *If a function $f : \mathbb{R} \to \mathbb{R}$ has $n+1$ continuous derivatives $f^{(k)}$ in an interval containing x, then*

$$f(x+\Delta) = f(x) + f^{(1)}(x)\Delta + \frac{f^{(2)}(x)\Delta^2}{2!} + \cdots + \frac{f^{(n)}(x)\Delta^n}{n!} + R_n(x,\Delta)$$

where the remainder is

$$R_n(x,\Delta) = o(\Delta^n) \quad as \quad \Delta \to 0.$$

In particular, we will apply Taylor's theorem to the exponential function, expanding about $x = 0$, getting:

$$\begin{aligned} e^x &= e^0 + o(1) = 1 + o(1) \quad \text{as} \quad x \to 0 \\ e^x &= 1 + x + o(x) \quad \text{as} \quad x \to 0 \\ e^x &= 1 + x + \frac{x^2}{2} + o(x^2) \quad \text{as} \quad x \to 0 \\ e^x &= 1 + x + \cdots + \frac{x^n}{n!} + o(x^n) \quad \text{as} \quad x \to 0 \ . \end{aligned} \tag{30}$$

We will also want to use the two-dimensional Taylor's theorem for the case of $n = 2$. We can allow different increments in the two arguments. Below one is Δ and the other is Γ. Our application will involve increments in different orders of the same variable.

Theorem 6.2 (*special case of the two-dimensional Taylor's theorem*) *If $f(x,t)$ is a function $f : \mathbb{R}^2 \to \mathbb{R}$ with 3 continuous partial derivatives, denoted by*

$$f_x \equiv \frac{\partial f}{\partial x}(x,t), \quad f_t \equiv \frac{\partial f}{\partial t}(x,t), \quad f_{x,t} \equiv \frac{\partial^2 f}{\partial x \partial t}(x,t) \qquad f_{x,x} \equiv \frac{\partial^2 f}{\partial x^2}(x,t)$$

and so forth, then

$$\begin{aligned} f(x+\Delta, t+\Gamma) &= f(x,t) + f_x\Delta + f_t\Gamma + \frac{1}{2}f_{x,x}\Delta^2 \\ &\quad + f_{x,t}\Delta\Gamma + \frac{1}{2}f_{t,t}\Gamma^2 + R_2(x,t,\Delta,\Gamma) \end{aligned} \tag{31}$$

where the remainder is

$$R_n(x,t,\Delta,\Gamma) = o(\Delta^2 + \Gamma^2) \quad as \quad \Delta \to 0 \quad and \quad \Gamma \to 0 \ .$$

7 The Black-Scholes Equation

We now derive the Black-Scholes PDE for a financial derivative associated with a stock whose price follows GBM. Let $S(t)$ be the stock price at time t. Let $f(t,x)$ be the unique arbitrage-free price of the financial derivative at time t when the stock price at that time is $S(t) = x$. In addition to x and t, the Black-Scholes PDE depends on the interest rate r and the volatility parameter σ. The Black-Scholes PDE is given in (1), where all functions are evaluated at (x,t).

We start with the specified GBM as in §4.2 having parameters ν and σ^2. We then construct the approximating BLM as in §4.3. The approximation is done so that the single period approximates the GBM over a time interval of length Δ, where we are thinking of letting $\Delta \to 0$. In this framework, we let the stock price start at level x at time t. Given that the stock starts at x, the stock price then either goes up to $xu = xe^{\sigma\sqrt{\Delta}}$ at time $t+\Delta$ or the stock price goes down to $xd = x/u = xe^{-\sigma\sqrt{\Delta}}$ at time $t+\Delta$, as determined in equations (20) and (21).

We now draw on the analysis of the single-period model in §2. There we derived the risk-neutral probability p^*, the unique probability so that there is no arbitrage. In the notation of §2, we had

$$p^* = \frac{1+r-d}{u-d} \; .$$

In our setting, that translates into

$$p^*(\Delta) = \frac{1 + r\Delta - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}} \; . \tag{32}$$

With this new risk-neutral probability $p^*(\Delta)$, we have the one-period model approximating GBM shown in Figure 6.

We now must do asymptotic analysis. We get

$$\begin{aligned} p^*(\Delta) &= \frac{1 + r\Delta - [1 - \sigma\sqrt{\Delta} + (\sigma^2/2)\Delta + o(\Delta)]}{[1 + \sigma\sqrt{\Delta} + (\sigma^2/2)\Delta + o(\Delta)] - [1 - \sigma\sqrt{\Delta} + (\sigma^2/2)\Delta + o(\Delta)]} \\ &= \frac{r\Delta + \sigma\sqrt{\Delta} - (\sigma^2/2)\Delta + o(\Delta)}{[2\sigma\sqrt{\Delta} + o(\Delta)} \\ &= \frac{1}{2} + \frac{(r - (\sigma^2/2)}{2\sigma}\sqrt{\Delta} + o(\sqrt{\Delta}) \; . \end{aligned} \tag{33}$$

This is as we expect: If we had started with GBM having drift ν as in 4.2, then the associated risk-neutral drift for GBM is $r - \sigma^2/2$. In terms of the SDE in (16), we have $\mu = \nu + \sigma^2/2$ and $\nu = \mu - \sigma^2/2$. So we are replacing μ by r when we consider the risk-neutral version of GBM. In other words, the probability here in (33) coincides with the value of p previously derived in (22) after replacing ν there by $r - \sigma^2/2$.

We are now ready to derive the Black-Scholes PDE. From our one-period analysis, we know that the derivative price $f(x,t)$ at time t when the stock price is $S(t) = x$ can be characterized as the discounted present value of the derivative price at the end of the period, provided that we take the expectation with respect to the risk-neutral probability distribution. That gives us

$$f(x,t) = \frac{1}{1+r\Delta}\left(p*(\Delta)f(xe^{\sigma\sqrt{\Delta}}, t+\Delta) + (1 - p*(\Delta))f(xe^{-\sigma\sqrt{\Delta}}, t+\Delta)\right) \; . \tag{34}$$

What do we do now? We perform asymptotic analysis of $f(xe^{\sigma\sqrt{\Delta}}, t+\Delta)$ and $f(xe^{-\sigma\sqrt{\Delta}}, t+\Delta)$ using the two-dimensional Taylor's theorem. We then plug in $p^*(\Delta)$ from (33) into (34). When we do, we will see that we get the Black-Scholes PDE.

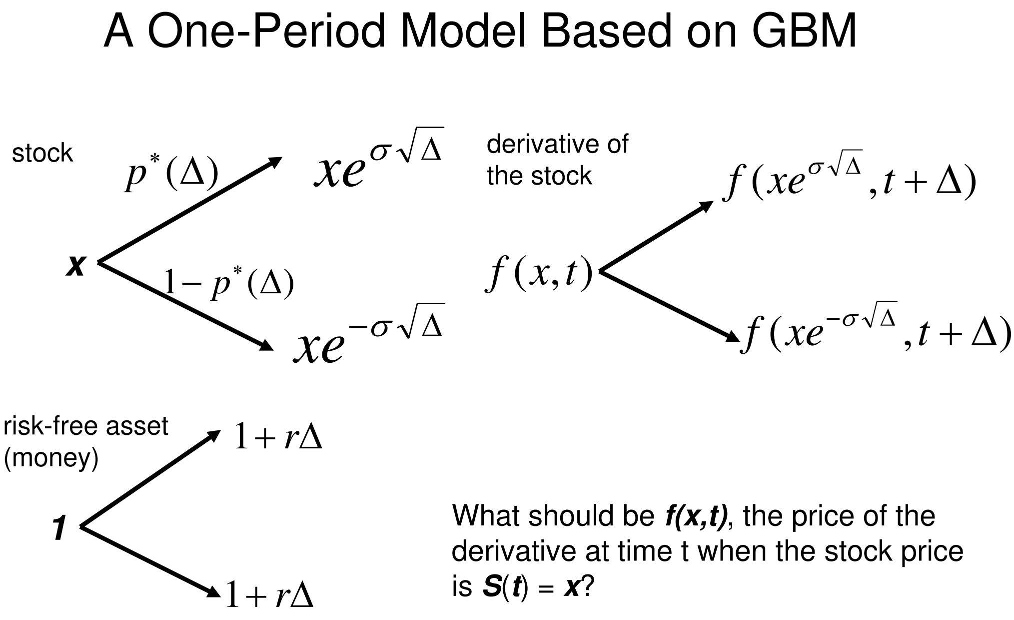

Figure 6: The one-period problem after approximating GBM.

In preparation for applying Taylor's theorem to (34), we apply Taylor's theorem to the component exponential:

$$e^{\sigma\sqrt{\Delta}} = 1 + \sigma\sqrt{\Delta} + \frac{\sigma^2\Delta}{2} + o(\Delta) \quad \text{as} \quad \Delta \to 0 \ . \tag{35}$$

Now we apply the two-dimensional Taylor's theorem. Note that the increments for the two arguments are order $\sqrt{\Delta}$ and Δ, respectively. Taylor's theorem covers that, but we have to be careful to do the accounting right. Applying the two-dimensional Taylor's theorem, we get

$$\begin{aligned} f(xe^{\sigma\sqrt{\Delta}}, t+\Delta) &= f(x,t) + xf_x(e^{\sigma\sqrt{\Delta}} - 1) + f_t\Delta + x^2\frac{f_{x,x}}{2}(e^{\sigma\sqrt{\Delta}} - 1)^2 \\ &\quad + xf_{x,t}(e^{\sigma\sqrt{\Delta}} - 1)\Delta + f_{t,t}\Delta^2 + o(\Delta) \end{aligned} \tag{36}$$

as $\Delta \to 0$. Continuing, we get

$$\begin{aligned} f(xe^{\sigma\sqrt{\Delta}}, t+\Delta) &= f(x,t) + xf_x(\sigma\sqrt{\Delta} + \frac{\sigma^2\Delta}{2} + o(\Delta)) + f_t\Delta \\ &\quad + \frac{x^2 f_{x,x}}{2}(\sigma\sqrt{\Delta} + \frac{\sigma^2\Delta}{2} + o(\Delta))^2 + xf_{x,t}(\sigma\sqrt{\Delta} + \frac{\sigma^2\Delta}{2} + o(\Delta))\Delta \\ &\quad + f_{t,t}\Delta^2 + o(\Delta) \ . \end{aligned} \tag{37}$$

Collecting terms, we get

$$\begin{aligned} f(xe^{\sigma\sqrt{\Delta}}, t+\Delta) &= f(x,t) + xf_x(\sigma\sqrt{\Delta} + \frac{\sigma^2\Delta}{2}) + f_t\Delta \\ &\quad + x^2\frac{f_{x,x}}{2}\sigma^2\Delta + o(\Delta) \ . \end{aligned} \tag{38}$$

We now evaluate the expectation in (34). We substitute the asymptotic form for $p^*(\Delta)$ from (33) and the asymptotic form of $f(xe^{\sigma\sqrt{\Delta}}, t+\Delta)$ from (38) (for both the positive exponent and the negative exponent). After multiplying through by $1+r\Delta$ in (34) and collecting terms, we get

$$(1+r\Delta)f = f + \frac{x\sigma^2}{2} f_x \Delta + f_t \Delta + \frac{x^2\sigma^2}{2} f_{x,x}\Delta + \left(\frac{(r-\sigma^2/2)}{\sigma}\right) x\sigma f_x \Delta + o(\Delta) \ . \quad (39)$$

Simplifying the penultimate term gives

$$(1+r\Delta)f = f + \frac{x\sigma^2}{2} f_x \Delta + f_t \Delta + \frac{x^2\sigma^2}{2} f_{x,x}\Delta + \left((r-\sigma^2/2)\right) x f_x \Delta + o(\Delta) \ . \quad (40)$$

Combining the second and fifth terms on the right gives

$$(1+r\Delta)f = f + f_t \Delta + \frac{x^2\sigma^2}{2} f_{x,x}\Delta + rxf_x\Delta + o(\Delta) \ . \quad (41)$$

Subtracting f from both sides, dividing by Δ and then letting $\Delta \to 0$, we get

$$rf = rxf_x + f_t + \frac{x^2\sigma^2}{2} f_{x,x} \ , \quad (42)$$

just as in (1).

7.1 Delta Hedging Strategy

Recall that the entire analysis for the one-period model was based on the concept of replication; i.e., we create a replicating portfolio of the stock and the risk-free asset that exactly matches the payoff of the financial derivative, whatever is the outcome - whether the stock price goes up or down. We can apply that analysis here too. What we do in continuous time can be derived from what we do in discrete time for the approximating BLM.

Using the same analysis, the number of shares of stock we should hold at time t when the stock price is x is

$$\begin{aligned}
\alpha &\equiv \alpha(\Delta) = \alpha(\Delta, x, t) \\
&= \frac{f(xe^{\sigma\sqrt{\Delta}}, t+\Delta) - f(xe^{-\sigma\sqrt{\Delta}}, t+\Delta)}{xe^{\sigma\sqrt{\Delta}} - xe^{-\sigma\sqrt{\Delta}}} \\
&= \frac{2xf_x(\sigma\sqrt{\Delta} + o(\sqrt{\Delta}))}{x(2\sigma\sqrt{\Delta} + o(\Delta))} \quad \text{as} \quad \Delta \to 0 \\
&\to f_x \quad \text{as} \quad \Delta \to 0
\end{aligned}$$

The partial derivative $f_x \equiv \frac{\partial f}{\partial x}(x,t)$ is called Δ (Delta). Hence the replicating strategy is called delta hedging.

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