IEOR 4701: Stochastic Models in Financial Engineering

Midterm Exam, Monday, July 30, in class

Open Book: but only the Ross textbook plus class lecture notes and one extra 8×11 page of notes

Justify your answers; show your work.

Honor Code: Students are expected to behave honorably, following the accepted code of academic honesty. After completing your exam, please affirm that you have done so by writing "I have neither given not received improper help on this examination," on your blue book and sign your name.

1. Value at Risk (VAR). (12 **points)**

Let the *value at risk* (VAR) of an investment (at confidence level $0.99 = 1 - 0.01$) be the value v such that there is only a one percent chance that the loss from the investment will exceed v. If the gain is represented by the random variable G, then the random variable $-G$ represents the associated loss. Hence, the VAR is the value v such that

$$P(-G > v) = 0.01 \ ,$$

assuming that $-G$ has a continuous distribution. The VAR criterion for choosing among different investments selects the investment with the smallest VAR.

(a) Consider three alternative investments, where the gain G_i of investment i is *normally distributed* for each i. Let the gain of investment i have mean m_i and variance σ_i^2 for $i = 1, 2, 3$, where

$$m_1 = 10, \quad m_2 = 5, \quad m_3 = 1$$

while

$$\sigma_1^2 = 100, \quad \sigma_2^2 = 49, \quad \sigma_3^2 = 36 \ .$$

Which investment would be preferred by the VAR criterion, and what is the value at risk for that investment?

(b) Suppose that the gain G_i of each investment i in part (a) is *uniformly distributed* on the interval $[m_i - \sigma_i, m_i + \sigma_i]$ instead of being normally distributed. Which investment would be preferred by the VAR criterion now, and what is the value at risk for that investment?

2. The Capital Asset Pricing Model (CAPM). (15 points)

The Capital Asset Pricing Model (CAPM) attempts to relate the one-period return of a specified security i, denoted by R_i, to the one-period return of the entire market, denoted by R, and the risk-free interest rate r. For example, the market return R might be measured by the Standard and Poor's index of 500 stocks, while r might be taken to be the current rate of a U.S. Treasury Bill. Here we regard R_i and R as random variables, but we regard r as a constant.

The model then assumes that

$$R_i = r + \beta_i(R - r) + N_i \ ,$$

where β_i is a constant, while N_i is a random variable with mean 0 and variance σ_i^2, which is assumed to be independent of R. We also assume that R has mean m and variance σ^2.

(a) Give an expression for the covariance $Cov(R_i, R)$ in terms of the parameters r, β_i, m, σ^2 and σ_i^2.

(b) Suppose that the current risk-free interest rate r is 0.06 (six percent), $m = 0.10$ and $\sigma = 0.20$, respectively. If $Cov(R_i, R) = 0.05$ for stock i, then what is the expected value $E[R_i]$?

(c) Suppose, in addition to the assumptions in part (b), that R and N_i are normally distributed and $\sigma_i^2 = 0.0275$. What is the approximate probability $P(R_i > 0.41)$?

3. Coin Tossing in Two Stages (10 points)

We consider a two-stage coin tossing experiment. In the first stage we toss a coin 6 times and count the number of heads. If the number of heads in the first stage is k, then in the second stage we toss a coin k times and count the number of heads. (We assume that successive coin tosses are independent experiments, with the probability of heads being 1/2 in each toss.)

(a) What is the probability that 5 heads come up in the *second* stage?

(b) What is the conditional probability that 5 heads came up in the *first* stage, given that 4 heads came up in the second stage?

4. Proportional Betting and the Kelly Criterion. (15 points)

Suppose that you play a series of games (gambles) in which you either win or lose in each game. In each game you can bet any amount from 0 up to your current wealth. If you bet x on game n, then your return from game n is xZ_n, where $\{Z_n : n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with

$$P(Z_n = 1) = p = 1 - P(Z_n = -1)$$

for some fixed $p > 1/2$. Since $p > 1/2$, $E[Z_n] = 2p - 1 > 0$. Since $E[Z_n] > 0$, these are *favorable games.* Suppose that you decide to bet a proportion f of your current wealth in each game; i.e., if your wealth is W_{n-1} before game n, you bet fW_{n-1} and your return is $fW_{n-1}Z_n$, and you do this for all $n \geq 1$. That is called the strategy of *proportional betting.*

(a) Let W_n be your wealth after game n, starting with some fixed value W_0, assuming that you are using proportional betting with some proportion f. Show that W_n can be expressed as the direct product of two independent random variables, one of them being W_{n-1}.

(b) Find an expression for W_n in terms of your initial wealth W_0, the betting proportion f and the random variables $Z_1, \ldots, Z_n$.

(c) Let $\ln(x)$ be the natural logarithm of x, i.e., the inverse of the exponential function: $\ln(e^y) = y$ for all real y and $e^{\ln(x)} = x$ for all positive real x. If a deterministic wealth function can be expressed as $w(t) = w(0)e^{rt}$ for $t \geq 0$, then r is the *growth rate*, which can be recovered from

$$r = \frac{1}{t} \ln\left(w(t)/w(0)\right) .$$

Analogously, for our positive stochastic wealth process $\{W_n : n \geq 0\}$, we say that

$$R_n \equiv \frac{1}{n} \ln\left(W_n/W_0\right)$$

is the *random growth rate.* We say that the constant r is the *asymptotic growth rate* of $\{W_n : n \geq 0\}$ if R_n converges to r as $n \to \infty$.

Explain why an asymptotic growth rate $r \equiv r(f)$ (depending on f) is well defined for our wealth stochastic process $\{W_n : n \geq 0\}$ under an extra regularity condition. State the extra regularity condition and give the value of the asymptotic growth rate.

(d) What is the approximate distribution of the random growth rate R_n for large n (assuming an additional regularity condition)? Explain.

(e) What is the approximate distribution of the wealth W_n for large n (assuming the additional regularity condition in part (d))? Explain.

(f) What proportion f for the proportional betting maximizes the asymptotic growth rate of $\{W_n : n \geq 0\}$? Proportional betting with this optimal proportion f^* is called the *Kelly criterion.*

(g) How would the approximate distributions change in parts (d) and (e) above if the probability distribution of the random variables Z_n are changed to another probability distribution on the interval $[-1, \infty)$?

5. Markov Processes. (16 points)

Consider the stochastic process $\{Y_n : n \geq 1\}$ for Y_n defined in different ways below in terms of the random variables Z_n and W_n defined in Problem 4 above. Indicate which of the definitions below always make the stochastic process $\{Y_n : n \geq 1\}$ so defined a Markov process. Briefly explain.

(a) $Y_n = Z_n, \quad n \geq 1$.

(b) $Y_n = (Z_n - 2)^2, \quad n \geq 1$.

(c) $Y_n = Z_{n+1}, \quad n \geq 1$.

(d) $Y_n = Z_n + Z_{n+1}, \quad n \geq 1$.

(e) $Y_n = Z_1 + \cdots + Z_n, \quad n \geq 1$.

(f) $Y_n = W_n, \quad n \geq 1$.

(g) $Y_n = (W_n - 2)^2, \quad n \geq 1$.

(h) $Y_n = W_n + W_{n+1}, \quad n \geq 1$.

(i) $Y_n = (W_n, W_{n+1}), \quad n \geq 1$.

6. A Gambler's Ruin Problem. (22 points)

Consider a gambler who plays a sequence of independent games, either winning one dollar or losing one dollar on each play of the game. However, suppose that the probability of winning depends on the gambler's wealth. Specifically, suppose that the probability of winning 1 dollar is $1/(n+1)$ when the gambler's wealth is n dollars. Suppose that the gambler starts out with 6 dollars and plays until his wealth reaches either 8 dollars or he is ruined (goes broke, i.e., until he has 0 dollars).

(a) Make a Markov chain model describing the gambler's wealth over time.

(b) What are the transient states and recurrent states of this Markov chain?

(c) Find the three-step transition probability $P^{(3)}_{4,5}$.

(d) Identify as many probability vectors π as possible that satisfy $\pi P = \pi$.

(e) What is the canonical form for the transition matrix of the Markov chain?

(f) Indicate how to calculate the probability of ruin (without doing the actual computation).

7. An Exotic Option. (10 points)

Suppose that AAA Corporation, BBB Corporation and CCC Corporation are three companies with publicly traded stock. Initially, the stock prices for AAA, BBB and CCC are 50, 60 and 70 dollars per share, respectively. Suppose that these are good times, so that the stock prices only go up; the only question is by how much. Suppose that the change (the amount of increase) in these stock prices over a year are distributed as A, B and C, respectively, where A, B and C are independent exponential random variables with means 10, 20 and 20 dollars, respectively. An *exotic option* is available for 5 dollars per share providing the opportunity, but not the obligation, at the end of the year to buy a share of the stock (among these three) that has *increased the least* (among these three) over the year at the initial price of that particular stock. (For this problem, please ignore the time value of money, i.e., interest.)

(a) What is the probability that the stock of AAA Corporation increases the least among these three stocks over the year?

(b) What is the expected value of the option per share, i.e., the expected return minus the cost, per share?

(c) What is the probability that buying the option will produce a profit?