for some $\theta_j \in (0, 1)$, and this differs from $h_j(D_jf)(\mathbf{x})$ by less than $|h_j|\varepsilon/n$, using (41). By (42), it follows that

$$\left| f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \sum_{j=1}^{n} h_j(D_jf)(\mathbf{x}) \right| \le \frac{1}{n} \sum_{j=1}^{n} |h_j|\varepsilon \le |\mathbf{h}|\varepsilon$$

for all $\mathbf{h}$ such that $|\mathbf{h}| < r$.

This says that f is differentiable at $\mathbf{x}$ and that $f'(\mathbf{x})$ is the linear function which assigns the number $\Sigma h_j(D_jf)(\mathbf{x})$ to the vector $\mathbf{h} = \Sigma h_j \mathbf{e}_j$. The matrix $[f'(\mathbf{x})]$ consists of the row $(D_1f)(\mathbf{x}), \ldots, (D_nf)(\mathbf{x})$; and since $D_1f, \ldots, D_nf$ are continuous functions on E, the concluding remarks of Sec. 9.9 show that $f \in \mathscr{C}'(E)$.

THE CONTRACTION PRINCIPLE

We now interrupt our discussion of differentiation to insert a fixed point theorem that is valid in arbitrary complete metric spaces. It will be used in the proof of the inverse function theorem.

9.22 Definition Let X be a metric space, with metric d. If φ maps X into X and if there is a number $c < 1$ such that

(43) $$d(\varphi(x), \varphi(y)) \le c\, d(x, y)$$

for all $x, y \in X$, then φ is said to be a *contraction* of X into X.

9.23 Theorem *If X is a complete metric space, and if φ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\varphi(x) = x$.*

In other words, φ has a unique fixed point. The uniqueness is a triviality, for if $\varphi(x) = x$ and $\varphi(y) = y$, then (43) gives $d(x, y) \le c\, d(x, y)$, which can only happen when $d(x, y) = 0$.

The *existence* of a fixed point of φ is the essential part of the theorem. The proof actually furnishes a constructive method for locating the fixed point.

Proof Pick $x_0 \in X$ arbitrarily, and define $\{x_n\}$ recursively, by setting

(44) $$x_{n+1} = \varphi(x_n) \qquad (n = 0, 1, 2, \ldots).$$

Choose $c < 1$ so that (43) holds. For $n \ge 1$ we then have

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \le c\, d(x_n, x_{n-1}).$$

Hence induction gives

(45) $$d(x_{n+1}, x_n) \le c^n d(x_1, x_0) \qquad (n = 0, 1, 2, \ldots).$$

If $n < m$, it follows that

$$\begin{aligned} d(x_n, x_m) &\le \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \\ &\le (c^n + c^{n+1} + \cdots + c^{m-1})\, d(x_1, x_0) \\ &\le [(1-c)^{-1} d(x_1, x_0)]c^n. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\lim x_n = x$ for some $x \in X$.

Since φ is a contraction, φ is continuous (in fact, uniformly continuous) on X. Hence

$$\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

THE INVERSE FUNCTION THEOREM

The inverse function theorem states, roughly speaking, that a continuously differentiable mapping $\mathbf{f}$ is invertible in a neighborhood of any point $\mathbf{x}$ at which the linear transformation $\mathbf{f}'(\mathbf{x})$ is invertible:

9.24 Theorem *Suppose $\mathbf{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into R^n, $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then*

(*a*) *there exist open sets U and V in R^n such that $\mathbf{a} \in U$, $\mathbf{b} \in V$, $\mathbf{f}$ is one-to-one on U, and $\mathbf{f}(U) = V$;*

(*b*) *if $\mathbf{g}$ is the inverse of $\mathbf{f}$ [which exists, by (a)], defined in V by*

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x} \qquad (\mathbf{x} \in U),$$

then $\mathbf{g} \in \mathscr{C}'(V)$.

Writing the equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$ in component form, we arrive at the following interpretation of the conclusion of the theorem: The system of n equations

$$y_i = f_i(x_1, \ldots, x_n) \qquad (1 \le i \le n)$$

can be solved for $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_n$, if we restrict $\mathbf{x}$ and $\mathbf{y}$ to small enough neighborhoods of $\mathbf{a}$ and $\mathbf{b}$; the solutions are unique and continuously differentiable.

Proof

(*a*) Put $\mathbf{f}'(\mathbf{a}) = A$, and choose λ so that

(46) $$2\lambda \|A^{-1}\| = 1.$$