# IEOR 6711: Stochastic Models I

## Fall 2013, Professor Whitt

## Homework Assignment 1: Tuesday, September 3

## Due before class on Tuesday, September 10. To be discussed in the recitation, Sunday, September 8, 7:00-9:00pm in 303 Mudd

During the first two weeks, read Chapter 1 of the course textbook, *Stochastic Processes*, second edition, by Sheldon Ross. We start with a review of basic probability theory, emphasizing technical details.

The assignment consists of the following ten problems from Chapter 1: Problems 1.1-1.4, 1.8, 1.11, 1.12, 1.15, 1.22 and 1.37. You need not turn in problems with answers in the back. (Here those are 1.22 and 1.37.) Since you may not have the textbook yet, the problems are given again here and expanded upon. (However, there seemed to be plenty of copies of the textbook in the Columbia bookstore.) You are to do the extra parts added in this expansion, as well as the problems from the book. The expansion illustrates that there may be more going on than you at first think.

In all homework and exams, show your work.

1. Elaboration of Problem 1.1 on p. 46 of Ross.

(a) Application

This first problem in Ross is about the *tail-integral formula for expected value*,

$$EX = \int_0^\infty P(X > t)\, dt = \int_0^\infty \bar{F}(t)\, dt \ ,$$

where $F \equiv F_X$ is the *cumulative distribution function* (cdf) of the random variable $X$, i.e., $F(t) \equiv P(X \le t)$, under the assumption that $X$ is a nonnegative random variable. Here $\equiv$ denotes "equality by definition." Here $\bar{F} \equiv 1 - F$ is the *complementary cumulative distribution function* (ccdf) or tail probability function. Before considering the proof, let us see why the formula is interesting and useful.

Apply the tail integral formula for the expected value to compute the expected value of an exponential random variable with rate $\lambda$ and mean $1/\lambda$, i.e., for the random variable with tail probabilities (ccdf)
$$\bar{F}(t) \equiv P(X > t) \equiv e^{-\lambda t}, \quad t \ge 0 \ ,$$
and density (probability density function or pdf)

$$f(t) \equiv f_X(t) \equiv \lambda e^{-\lambda t}, \quad t \ge 0 \ .$$

(b) Alternative Approaches

Compute the expected value of the exponential distribution above in two other ways:

(i) Exploit the structure of the *gamma distribution*: The gamma density with *scale parameter* $\lambda$ and *shape parameter* $\nu$ is

$$f_{\lambda,\nu}(t) \equiv \frac{1}{\Gamma(\nu)}\lambda^{\nu}t^{\nu-1}e^{-\lambda t}, \quad t \geq 0,$$

where $\Gamma$ is the *gamma function*, i.e.,

$$\Gamma(t) \equiv \int_0^{\infty} x^{t-1}e^{-x}\,dx \ ,$$

which reduces to a factorial at integer arguments: $\Gamma(n+1) = n!$. The important point for the proof here is that the gamma density is a proper probability density, and so integrates to 1.

(ii) Use *integration by parts*: Suppose that $u$ is bounded and has continuous derivative $u'$. Suppose that $f$ is a pdf of a nonnegative random variable with associated ccdf $\bar{F}$. (The derivative of $\bar{F}$ is $-f$.) Then, for any $b$ with $0 < b < \infty$,

$$\int_0^b u(t)f(t)\,dt = -u(b)\bar{F}(b) + u(0)\bar{F}(0) + \int_a^b u'(t)\bar{F}(t)\,dt \ .$$

(c) Interchanging the order of integrals and sums.

Relatively simple proofs of the results to be proved in Problem 1.1 of Ross follow from interchanging the order of integrals and sums. We want to use the following relations:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}$$

and

$$\int_0^{\infty}\int_0^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{\infty} f(x,y)\,dy\,dx$$

It is important to know that these relations are *usually valid*. It is also important to know that these relations are *not always valid*: In general, there are *regularity conditions*. The complication has to do with infinity and limits. There is no problem at all for finite sums:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j} \ .$$

That is always true. Clearly, the sum of $mn$ numbers is independent of the order in which you add them. However, infinite sums involve limits, and integrals are defined as limits of sums. There are some subtleties there.

The interchange property is often used with expectations. In particular, it is used to show that the expectation can be taken inside sums and integrals:

$$E[\sum_{i=1}^{\infty} X_i] = \sum_{i=1}^{\infty} EX_i$$

and

$$E[\int_0^{\infty} X(s)\,ds] = \int_0^{\infty} E[X(s)]\,ds \ .$$

These relations are of the same form because the expectation itself can be expressed as a sum or an integral. However, there are regularity conditions, as noted above.

(i) Compute the two iterated sums $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}$ for

$$a_{i,j} = 2 - 2^{-i} \quad \text{for} \quad i = j, \quad a_{i,j} = -2 + 2^{-i} \quad \text{for} \quad i = j + 1, \quad j \geq 1 \;,$$

and $a_{i,j} = 0$ otherwise. What does this example show?

(ii) Look up Tonelli's theorem and Fubini's theorem in a book on measure theory and integration, such as *Real Analysis* by H. L. Royden. Or just Google these! What do they say about this problem?

(d) Do the three parts to Problem 1.1 in Ross (specified below here).

Hint: Do the proofs by interchanging the order of sums and integrals. The first step is a bit tricky. We need to get an iterated sum or integral; i.e., we need to insert the second sum or integral. To do so for sums, write:

$$EN = \sum_{i=1}^{\infty} iP(N = i) = \sum_{i=1}^{\infty}[\sum_{j=1}^{i} 1]P(N = i) \;.$$

(i) For a nonnegative integer-valued random variable $N$, show that

$$EN = \sum_{i=1}^{\infty} P(N \geq i) \;.$$

(ii) For a nonnegative random variable $X$ with cdf $F$, show that

$$EX = \int_0^{\infty} P(X > t)\, dt = \int_0^{\infty} \bar{F}(t)\, dt \;.$$

(iii) For a nonnegative random variable $X$ with cdf $F$, show that

$$E[X^n] = \int_0^{\infty} nt^{n-1}\bar{F}(t)\, dt \;.$$

(e) What regularity property justifies the interchanges used in part (d)?

(f) What happens to the tail-integral formula for expected value

$$EX = \int_0^{\infty} P(X > t)\, dt = \int_0^{\infty} \bar{F}(t)\, dt \;,$$

when the random variable $X$ is integer valued?

2. Elaboration of Problems 1.2 on p. 46 and 1.15 on p. 49.

(a) Application to simulation

(i) Suppose that you know how to generate on the computer a random variable $U$ that is uniformly distributed on the interval $[0, 1]$. (That tends to be easy to do, allowing for the necessary approximation due to discreteness.) How can you use $U$ to generate a random

variable $X$ with an cdf $F$ that is an arbitrary continuous and strictly increasing function? (Hint: Use the inverse $F^{-1}$ of the function $F$.)

(ii) Suppose that you know how to generate on the computer a random variable $U$ that is uniformly distributed on the interval $[0, 1]$. How can you use $U$ to generate a random variable $X$ with an exponential distribution with mean $1/\lambda$?

(b) Problem 1.2 for a cdf $F$ having a positive density $f$.

Do the two parts of Problem 1.2 in Ross (specified here below) under the assumption that the random variable $X$ concentrates on the interval $(a, b)$ for $-\infty \le a < b \le +\infty$ and has a strictly positive density $f$ there. That is, $P(a \le X \le b) = 1$,

$$F(x) \equiv P(X \le x) = \int_a^x f(y)\, dy \ ,$$

where $f(x) > 0$ for all $x$ such that $a \le x \le b$.

The two parts of Problem 1.2 are:

(i) If $X$ is a random variable with a continuous cdf $F$, show that $F(X)$ is uniformly distributed on the interval $[0, 1]$.

(ii) If $U$ is uniformly distributed on $(0, 1)$, then $F^{-1}(U)$ has cdf $F$.

Note that the positive-density condition is added to Ross's conditions. Note that we use that extra condition so far. We explore how that condition can be relaxed below.

(c) Right continuity.

A cdf $F$ is a right-continuous nondecreasing function of a real variable such that $F(x) \to 1$ as $x \to \infty$ and $F(x) \to 0$ as $x \to -\infty$.
A real-valued function $g$ of a real variable $x$ is *right-continuous* if

$$lim_{y \downarrow x} g(y) = g(x)$$

for all real $x$, where $\downarrow$ means the limit from above (or from the right). A function $g$ is *left-continuous* if

$$lim_{y \uparrow x} g(y) = g(x)$$

for all real $x$, where $\uparrow$ means the limit from below (or from the left).
A function $g$ has a limit from the right at $x$ if the limit $lim_{y \downarrow x} g(y)$ exists. Suppose that a function $g$ has limits everywhere from the left and right. Then the *right-continuous version* of $g$, say $g_+$ is defined by

$$g_+(x) \equiv \lim_{y \downarrow x} g(y)$$

for all $x$. The *left-continuous version* of $g$, $g_-$ is defined similarly.
Suppose that $P(X = 1) = 1/3 = 1 - P(X = 3)$. Let $F$ be the cdf of $X$. What are $F(1)$, $F(2)$, $F(3)$ and $F(4)$? What are the values of the left-continuous version $F_-$ at the arguments 1, 2, 3 and 4?

(d) The left-continuous inverse of a cdf $F$.

Given a (right-continuous) cdf $F$, let

$$F^{\leftarrow}(t) \equiv \inf\{x : F(x) \ge t\}, \quad 0 < t < 1 \ .$$

Fact: $F^{\leftarrow}$ is a left-continuous function on the interval $(0, 1)$.

Fact: In general, $F^{\leftarrow}$ need not be right-continuous.

Fact:
$$F^{\leftarrow}(t) \le x \quad \text{if and only if} \quad F(x) \ge t$$
for all $t$ and $x$ with $0 < t < 1$.

Suppose that $X$ is a random variable with a continuous cdf $F$. Show that $F(X)$ is uniformly distributed on the interval $(0, 1)$. (Hint: use the fact that $F(F^{\leftarrow}(t)) = t$ for all $t$ with $0 < t < 1$ because $F$ is continuous.)

(e) The right-continuous inverse of a cdf $F$.

Given a (right-continuous) cdf $F$, let
$$F^{-1}(t) \equiv \inf\{x : F(x) > t\}, \quad 0 < t < 1 .$$

It can be shown that $F^{-1}$ is right-continuous, but in general is not left-continuous. You should know how to prove that. That is the kind of proof that is covered in Introduction to Modern Analysis, Math 4061.

Show that the relation
$$F^{-1}(t) \le x \quad \text{if and only if} \quad F(x) \ge t$$
for all $t$ and $x$ with $0 < t < 1$ does *not hold* in general.

(f) The two inverses. Let $X$ be the discrete random variable defined above with $P(X = 1) = 1/3 = 1 - P(X = 3)$. Draw pictures (graphs) of the cdf $F$ of $X$ and the two inverses $F^{\leftarrow}$ and $F^{-1}$.

(g) Suppose that you know how to generate on the computer a random variable $U$ that is uniformly distributed on the interval $[0, 1]$. How can you use $U$ to generate a random variable $X$ with an *arbitrary* cdf $F$?

(h) It can be shown that $P(F^{-1}(U) = F^{\leftarrow}(U)) = 1$. Given that result, what is the distribution of $F^{-1}(U)$?

3. Problem 1.3 in Ross:

Let the random variable $X_n$ have a binomial distribution with parameters $n$ and $p_n$, i.e.,
$$P(X_n = k) = \frac{n!}{k!(n-k)!} p_n^k (1 - p_n)^{n-k} .$$

(a) Show that
$$P(X_n = k) \to \frac{e^{-\lambda} \lambda^k}{k!} \quad \text{as} \quad n \to \infty \quad \text{for each } k$$
if $np_n \to \lambda$ as $n \to \infty$. Hint: Use the fact that $(1 + (c_n/n))^n \to e^c$ as $n \to \infty$ if $c_n \to c$ as $n \to \infty$.

The limit above is an example of convergence in distribution. We say that random variables $X_n$ with cdf's $F_n$ *converge in distribution* to a random variable $X$ with cdf $F$, and write $X_n \Rightarrow X$ or $F_n \Rightarrow F$, if
$$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x \quad \text{that are continuity points of } F .$$

A point $x$ is a continuity point of $F$ if $F$ is continuous at $x$.

(b) Show that the limit above is indeed an example of convergence in distribution. What is the limiting distribution?

(c) Suppose that $P(X_n = 1 + n^{-1}) = P(X = 1) = 1$ for all $n$. Show that $X_n \Rightarrow X$ as $n \to \infty$ and that the continuity-point condition is needed here.

(d) Use the techniques of Problem 2 above to prove the following *representation theorem*:

**Theorem 0.1** *If* $X_n \Rightarrow X$, *then there exist random variables* $\tilde{X}_n$ *and* $\tilde{X}$ *defined on a common probability space such that* $\tilde{X}_n$ *has the same distribution* $F_n$ *as* $X_n$ *for all* $n$, $\tilde{X}$ *has the same distribution* $F$ *as* $X$, *and*
$$P(\tilde{X}_n \to \tilde{X} \quad as \quad n \to \infty) = 1 \ .$$

4. Problem 1.4 in Ross:

Derive the mean and variance of a binomial random variable with parameters $n$ and $p$. Hint: Use the relation between Bernoulli and binomial random variables.

5. Problem 1.8 in Ross:

Let $X_1$ and $X_2$ be independent Poisson random variables with means $\lambda_1$ and $\lambda_2$, respectively.

(a) Find the distribution of the sum $X_1 + X_2$.

(b) Compute the conditional distribution of $X_1$ given that $X_1 + X_2 = n$.

6. Problem 1.11 in Ross: generating functions.

Let $X$ be a nonnegative integer-valued random variable. Then the generating function of $X$ is
$$\hat{P}(z) \equiv E[z^X] = \sum_{j=0}^{\infty} z^j P(X = j) \ .$$

(a) Show that the $k^{th}$ derivative of $\hat{P}(z)$ evaluated at $z = 0$ is $k! P(X = k)$.

(b) Show that (with 0 being considered even)
$$P(X \text{ is even }) = \frac{\hat{P}(1) + \hat{P}(-1)}{2} \ .$$

(c) Calculate $P(X$ is even) when

(i) $X$ is binomial $(n, p)$;
(ii) $X$ is Poisson with mean $\lambda$;
(iii) $X$ is geometric with parameter $p$, i.e., $P(X = k) = p(1 - p)^{k-1}$ for $k \geq 1$.

7. Problem 1.12 in Ross:

If $P(0 \leq X \leq a) = 1$, show that
$$Var(X) \leq a^2/4.$$

8. Problem 1.22 in Ross (answer in back):

The *conditional variance* of $X$ given $Y$ is defined as

$$Var(X|Y) \equiv E[(X - E[X|Y])^2|Y] \ .$$

Prove the *conditional variance formula*:

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]) \ .$$

9. Problem 1.37 in Ross (answer in back):

Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed (IID) random variables with a probability density function $f$. Say that a peak occurs at index $n$ if $X_{n-1} < X_n > X_{n+1}$. Show that the long-run proportion of indices at which a peak occurs is, with probability 1, equal to $1/3$. (Hint: Use the strong law of large numbers for partial sums of IID random variables, but observe it does not apply directly.)