IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 11 on DTMC's

Problem 4.40 Consider a segment of a sample path beginning and ending in state i, with no visit to i in between, i.e, the vector $(i, j_1, j_2, j_3, \ldots, j_{n-1}, j_n = i)$, where $j_k \neq i$ for the non-end states j_k. Going forward in time, the probability of this segment is

$$\pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}.$$

The probability, say p, of the reversed sequence $(i, j_{n-1}, j_{n-2}, j_{n-3}, \ldots, j_1, j_0 = i)$ under the reverse DTMC with transition matrix

$$\overleftarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}$$

is

$$p = \pi_i \overleftarrow{P}_{i,j_{n-1}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i}.$$

However, successively substituting in the reverse-chain transition probabilities, we get

$$\begin{aligned}
p &= \pi_i \frac{\pi_{j_{n-1}} P_{j_{n-1},i}}{\pi_i} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\
&= P_{j_{n-1},i} \pi_{j_{n-1}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\
&= P_{j_{n-1},j_{n-2}} \pi_{j_{n-1}} \frac{\pi_{j_{n-2}} P_{j_{n-2},j_{n-1}}}{\pi_{j_{n-1}}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\
&= P_{j_{n-1},i} P_{j_{n-2},j_{n-1}} P_{j_{n-3},j_{n-2}} \ldots P_{j_1,j_2} \pi_{j_1} \overleftarrow{P}_{j_1,i} \\
&= P_{j_{n-1},i} P_{j_{n-2},j_{n-1}} P_{j_{n-3},j_{n-2}} \ldots P_{j_1,j_2} P_{i,j_1} \pi_i \\
&= \pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}.
\end{aligned}$$

Problem 4.41 (a) The reverse time chain has transition matrix

$$\overleftarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}$$

To find it, we need to first find the stationary vector π. By symmetry (or by noting that the chain is doubly stochastic), $\pi_j = 1/n,\ j = 1, \cdots, n$. Hence,

$$P^*_{ij} = \pi_j P_{ji}/\pi_i = P_{ji} = \begin{cases} p & \text{if } j = i - 1 \\ 1 - p & \text{if } j = i + 1 \end{cases}$$

(b) In general, the DTMC is not time reversible. It is in the special case $p = 1/2$. Otherwise, the probabilities of clockwise and counterclockwise motion are reversed.

Problem 4.42 Imagine that there are edges between each of the pair of nodes i and $i+1$, $i = 0, \cdots, n-1$, and let the weight on edge $(i, i+1)$ be w_i, where

$$\begin{aligned} w_0 &= 1 \\ w_i &= \prod_{j=1}^{i} \frac{p_j}{q_j} \ , \quad i \geq 1 \end{aligned}$$

where $q_j = 1 - p_j$. As a check, note that with these weights

$$P_{i,i+1} = \frac{w_i}{w_{i-1} + w_i} = \frac{p_i/q_i}{1 + p_i/q_i} = p_i \ , \quad 0 < i < n \ .$$

Since the sum of the weights on edges out of node i is $w_{i-1} + w_i$, $i = 1, \cdots, n-1$, it follows that

$$\begin{aligned} \pi_0 &= c \\ \pi_i = c\left[\prod_{j=1}^{i-1} \frac{p_j}{q_j} + \prod_{j=1}^{i} \frac{p_j}{q_j}\right] &= \frac{c}{q_i} \prod_{j=1}^{i-1} \frac{p_j}{q_j} \ , \ 0 < i < n \\ \pi_n = c \prod_{j=1}^{n-1} \frac{p_j}{q_j} \end{aligned}$$

where c is chosen to make $\sum_{j=0}^{n} \pi_j = 1$.

Problem 4.46 (a) Yes, it is a Markov chain. It suffices to construct the transition matrix and verify that the process has the Markov property. Let P^* be the new transition matrix. Then we have, for $0 \leq i \leq N$ and $0 \leq j \leq N$,

$$P^*_{i,j} = P_{i,j} + \sum_{k=N+1}^{\infty} P_{i,k} B^{(N)}_{k,j} ,$$

where $B^{(N)}_{k,j}$ is the probability of absorption into the absorbing state j in the absorbing Markov chain, where the states $N+1, N+2, \ldots$ are the transient states, while the state $1, 2, \ldots N$ are the N absorbing states. In other words, $B^{(N)}_{k,j}$ is the probability that the next state with index in the set $\{1, 2, \ldots, N\}$ visited by the Markov chain, starting with $k > N$ is in fact j. It is easy to see that the markov property is still present.

(b) The proportion of time in j is $\pi_j / \sum_{i=1}^{N} \pi_i$.

(c) Let $\pi_i(N)$ be the steady-state probabilities for the chain, only counting to visits among the states in the subset $\{1, 2, \ldots, N\}$. (This chain is necessarily positive recurrent.) By renewal theory,

$$\pi_i(N) = (E[\text{Number of}\, Y - \text{transitions between}\, Y - \text{ visits to}\, i)^{-1}$$

and

$$\begin{aligned} \pi_j(N) &= \frac{E[\text{No. } Y\text{-transitions to } j \text{ between } Y \text{ visits to } i]}{E[\text{No. } Y\text{-transitions to } i \text{ between } Y \text{ visits to } i]} \\ &= \frac{E[\text{No. } X\text{-transitions to } j \text{ between } X \text{ visits to } i]}{1/\pi_i(N)} \end{aligned}$$

(d) For the symmetric random walk, the new MC is doubly stochastic, so $\pi_i(N) = 1/(N+1)$ for all i. By part (c), we have the conclusion.

(e) It suffices to show that

$$\pi_i(N)P^*_{i,j} = \pi_j(N)P^*_{j,i}$$

for all i and j with $i \le N$ and $j \le N$. However, by above,

$$\pi_i(N)P^*_{i,j} = \pi_i(N)P_{i,j} + \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B^{(N)}_{k,j},$$

and

$$\pi_j(N)P^*_{j,i} = \pi_j(N)P_{j,i} + \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B^{(N)}_{k,i},$$

The two terms on the right are equal in these two displays. First, by the original reversibility, we have

$$\pi_i(N)P_{i,j} = \pi_j(N)P_{j,i}.$$

Second, by Theorem 4.7.2, we have

$$\pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B^{(N)}_{k,i} = \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B^{(N)}_{k,j}.$$

We see that by expanding into the individual paths, and seeing that there is a reverse path.

Problem 4.47 Intuitively, in steady state each ball is equally likely to be in any of the urns and the positions of the balls are independent. Hence it seems intuitive that

$$\pi(\underline{n}) = \frac{M!}{n_1! \cdots n_m!} \left(\frac{1}{m}\right)^M .$$

To check the above and simultaneously establish time reversibility let

$$\underline{n}' = (n_1, \cdots, n_{i-1}, n_i - 1, n_{i+1}, \cdots, n_{j-1}, n_j + 1, n_{j+1}, \cdots, n_m)$$

and note that

$$\begin{aligned}
\pi(\underline{n})P(\underline{n},\underline{n}') &= \frac{M!}{n_1!\cdots n_m!}\left(\frac{1}{m}\right)^M \frac{n_i}{M}\frac{1}{m-1} \\
&= \frac{M!}{n_1!\cdots(n_i-1)!\cdots(n_j+1)!\cdots n_m!}\left(\frac{1}{m}\right)^M \frac{n_j+1}{M}\frac{1}{m-1} \\
&= \pi(\underline{n}')P(\underline{n}',\underline{n}) \ .
\end{aligned}$$

Problem 4.48 **(a)** Each transition into i begins a new cycle. A reward of 1 is earned if state visited from i is j. Hence average reward per unit time is P_{ij}/μ_{ii}.

(b) Follows from (a) since $1/\mu_{jj}$ is the rate at which transitions into j occur.

(c) Suppose a reward rate of 1 per unit time when in i and heading for j. New cycle whenever enter i. Hence, average reward per unit time is $P_{ij}\eta_{ij}/\mu_{ii}$.

(d) Consider (c) but now only give a reward at rate 1 per unit time when the transition time from i to j is within x time units. Average reward is

$$\begin{aligned}
\frac{\mathsf{E}[\text{Reward per cycle}]}{\mathsf{E}[\text{Time of cycle}]} &= \frac{P_{ij}\mathsf{E}[\min(X_{ij},x)]}{\mu_{ii}} \\
&= \frac{P_{ij}\int_0^x \bar{F}_{ij}(y)dy}{\mu_{ii}} \\
&= \frac{P_{ij}\eta_{ij}F_{ij}^e(x)}{\mu_{ii}}
\end{aligned}$$

where $X_{ij} \sim F_{ij}$.

Problem 4.49

$$\begin{aligned}
\lim_{t\to\infty}\mathsf{P}(S(t)=j|X(t)=i) &= \frac{\lim_{t\to\infty}\mathsf{P}(S(t)=j,X(t)=i)}{\mathsf{P}(X(t)=i)} \\
&= \frac{P_{ij}\int_0^\infty \bar{F}_{ij}(y)dy/\mu_{ii}}{P_i} \quad \text{by Theorem 4.8.4} \\
&= \frac{P_{ij}\eta_{ij}}{\mu_i}
\end{aligned}$$

Problem 4.50 $\pi = (6,3,5)/14$, $\mu_1 = 25$, $\mu_2 = 80/3$, and $\mu_3 = 30$.

(a)

$$\begin{aligned}
P_1 &= \frac{6\times 25}{6\times 25 + 3\times\frac{80}{3} + 5\times 30} = \frac{15}{38} \\
P_2 &= \frac{3\times\frac{80}{3}}{6\times 25 + 3\times\frac{80}{3} + 5\times 30} = \frac{8}{38} \\
P_3 &= \frac{5\times 30}{6\times 25 + 3\times\frac{80}{3} + 5\times 30} = \frac{15}{38}
\end{aligned}$$

(b)

$$\mathsf{P}(\text{heading for 2}) = P_1 \frac{P_{12} t_{12}}{\mu_1} = \frac{15}{38} \times \frac{10}{25} = \frac{3}{19}$$

(c)

$$\text{fraction of time from 2 to 3} = P_2 \frac{P_{23} t_{23}}{\mu_2} = \frac{8}{38} \times \frac{60}{80} = \frac{3}{19}$$