IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 12

Problem 5.3 **(a)** Let $N(t)$ denote the number of transitions be t. It is easy to show in this case that

$$\mathsf{P}(N(t) \geq n) \leq \sum_{j=n}^{\infty} e^{-Mt} \frac{(Mt)^j}{j!}$$

and thus $\mathsf{P}(N(t) < \infty) = 1$.

(b) Let X_{n+1} denote the time between the n-th and $(n+1)$-st transition and let J_n denote the n-th state visited. Also if we let

$$N(t) \triangleq \sup\{n : X_1 + \cdots + X_n \leq t\}$$

then $N(t)$ denotes the number of transitions by t. Now let j be the first recurrent state that is reached and suppose it was reached at the n_0-th transition (n_0 must be finite by assumption). Let $n_1, n_2, \cdots$ be the successive integers n at which $J_n = j$. (Such integers exist since j is recurrent.) Set $T_0 = X_1 + \cdots + X_{n_0}$, and

$$T_k \triangleq X_{n_{k-1}+1} + \cdots + X_{n_k} \ .$$

In other words, T_k denote the amount of time between the k-th and $(k+1)$-th visit to j. Therefore it follows that $\{T_k, k \geq 1\}$ forms a renewal process, and so $\sum_{k=1}^{\infty} T_k = \infty$ with probability 1. Since

$$\sum_{n=1}^{\infty} X_n = \sum_{k=0}^{\infty} T_k$$

it follows that $\sum_{n=1}^{\infty} X_n = \infty$.

Problem 5.4 Let T_i denote the time to go from i to $i+1$, $i \geq 0$. Then $\sum_{i=0}^{N-1} T_i$ is the time to go from 0 to N. Now T_i is exponential with rate λ_i and the T_i are independent. Hence

$$\mathsf{E}\left[e^{s \sum_{i=0}^{N-1} T_i}\right] = \prod_{i=0}^{N-1} \frac{\lambda_i}{\lambda_i - s} \ .$$

We may use it to compute the mean and variance or we can do directly and mean $= 1/\lambda_0 + \cdots + 1/\lambda_{N-1}$, variance $= 1/\lambda_0^2 + \cdots + 1/\lambda_{N-1}^2$.

Problem 5.9

$$\begin{aligned} P_{ij}(t+s) &= \sum_k \mathsf{P}(X(t+s) = j \mid X_0 = i, X(t) = k)\mathsf{P}(X(t) = k \mid X_0 = i) \\ &= \sum_k P_{kj}(s) P_{ik}(t) \ . \end{aligned}$$

Problem 5.10 **(a)**

$$\lim_{t\to 0}\frac{1-P(t)}{t} = v_0 \ .$$

(b) The first inequality follows from exercise 5.9 and

$$\begin{aligned} P(t+s) &= \mathsf{P}(X(t+s)=0|X(0)=0, X(s)=0)P(s) \\ &\quad +\mathsf{P}(X(t+s)=0|X(0)=0, X(s)\neq 0)(1-P(s)) \\ &\leq P(t)P(s)+1-P(s) \ . \end{aligned}$$

(c) From (b)

$$P(s)P(t-s) \leq P(t) \leq P(s)P(t-s)+1-P(t-s)$$

or

$$P(s)+P(t-s)-1 \leq P(t) \leq P(s)+1-P(t-s)$$

where the left hand inequality follows from

$$P(s)(1-P(t-s)) \leq 1-P(t-s) \ .$$

$\lim_{s\to t} P(s-t) = 1$ implies the continuity of P.

Problem 5.13

$$\prod_{j=i}^{i+k-1} \frac{\lambda_j}{\lambda_j+\mu_j} \ .$$

Problem 5.15 **(a)** Birth and death process.

(b) $\lambda_n = n\lambda + \theta$, $\mu_n = n\mu$.

(c) Set $M(t) = \mathsf{E}[X(t)|X(0)=i]$. Then

$$\mathsf{E}[X(t+h)|X(t)] = X(t) + (\lambda X(t)+\theta)h - \mu X(t)h + o(h)$$

and so $M(t+h) = M(t)+(\lambda-\mu)M(t)h+\theta h+o(h)$. Therefore, $M'(t) = (\lambda-\mu)M(t)+\theta$ or $e^{-(\lambda-\mu)t}[M'(t)-(\lambda-\mu)M(t)] = \theta e^{-(\lambda-\mu)t}$. Integrating both sides gives

$$e^{-(\lambda-\mu)t}M(t) = -\frac{\theta}{\lambda-\mu}e^{-(\lambda-\mu)t} + C$$

or

$$M(t) = Ce^{-(\lambda-\mu)t} - \frac{\theta}{\lambda-\mu} \ .$$

As $M(0) = i$ we obtain

$$M(t) = \frac{\theta}{\lambda-\mu}\left(e^{-(\lambda-\mu)t}-1\right) + ie^{-(\lambda-\mu)t} \ .$$

Problem 5.21 With the number of customers in the shop as the state, we get a birth and death process with $\lambda_0 = \lambda_1 = 3$, $\mu_1 = \mu_2 = 4$. Therefore $P_1 = \frac{3}{4}P_0$, $P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0$. And since $P_0 + P_1 + P_2 = 1$, we get $P_0 = 16/37$.

(a)

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left(\frac{3}{4}\right)^2\right] P_0 = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{37} = \frac{28}{37} .$$

(c) $\mu = 8$, and so $P_0 = \frac{64}{97}$. So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left(\frac{3}{8}\right)^2 \frac{64}{97} = \frac{88}{97} .$$

The rate of added customers is therefore

$$\lambda\frac{88}{97} - \lambda\frac{28}{37} \simeq 0.45 .$$

The business he does would improve by 0.45 customers per hour.