IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 13

Problem 5.12 **(a)** Since

$$P_0 = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\mu}{\lambda + \mu}$$

it follows that

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{\alpha_0 \mu}{\lambda + \mu} + \frac{\alpha_1 \lambda}{\lambda + \mu} .$$

(b) The expected total time spent in state 0 by t is

$$\int_0^t P_{00}(s) ds = \frac{\mu}{\lambda + \mu} t + \frac{\lambda}{(\lambda + \mu)^2} \left(1 - e^{-(\lambda+\mu)t}\right) .$$

Calling the above $\mathsf{E}[T(t)]$ we have

$$\mathsf{E}[N(t)] = \alpha_0 \mathsf{E}[T(t)] + \alpha_1 (t - \mathsf{E}[T(t)]) .$$

Problem 5.19 Let R_0 be the time required to return to state 0 (starting in state 0), once it has been left (i.e., starting from the moment that the CTMC first leaves state 0). What is the expected value, $E[R_0]$?

The times spent in state 0 and away from state 0 constitute an alternating renewal process. So this can be answered by an elementary application of renewal theory. For that purpose, Let $\nu_0 = \sum_{j, j \neq 0} Q_{0,j} \equiv -Q_{0,0}$ and let α_j be the stationary or steady-state probability of state j. Then

$$\alpha_0 = \frac{1/\nu_0}{E[R_0] + (1/\nu_0)} ,$$

from which we see that

$$E[R_0] = \frac{1 - \alpha_0}{\alpha_0 \nu_0} .$$

Let Y be the time of the first transition, and condition on that time, to get

$$\begin{aligned} E[T] &= te^{-\nu_0 t} + \int_0^t E[T | Y = s] \nu_0 e^{-\nu_0 s} \, ds \\ &= te^{-\nu_0 t} + \int_0^t (s + E[R_0] + E[T]) \nu_0 e^{-\nu_0 s} \, ds . \end{aligned}$$

We can thus solve for $E[T]$, getting

$$E[T] = \frac{te^{-\nu_0 t} + [(1 - \alpha_0)/(\alpha_0 \nu_0)](1 - e^{-\nu_0 t}) + \int_0^t s \nu_0 e^{-\nu_0 s} \, ds}{e^{-\nu_0 t}} .$$

It would be OK to stop there, but you could go on to evaluate the integral. That leads to some simplification. Using the fact that $\nu_0^2 s e^{-\nu_0 s}$ is the density of the Erlang E_2 (special form of gamma, with shape parameter 2) distribution, we have

$$\int_0^t s\nu_0 e^{-\nu_0 s}\, ds = \frac{1-e^{-\nu_0 t}}{\nu_0} - \frac{\nu_0 t e^{-\nu_0 t}}{\nu_0}$$

or

$$\int_0^t s\nu_0 e^{-\nu_0 s}\, ds = \frac{1-e^{-\nu_0 t}}{\nu_0} - te^{-\nu_0 t} \ .$$

When we apply the algebra, we get

$$E[T_0] = \frac{e^{\nu_0 t}-1}{\alpha_0 \nu_0} \ .$$

Problem 5.22 The analysis on page 153-154 with

$$\begin{aligned} \lambda_n &= \quad \lambda \quad n \geq 0 \\ \mu_n &= \begin{cases} n\mu & 1 \leq n \leq s \\ s\mu & n > s \end{cases} \end{aligned} \ .$$

We need $\lambda < s\mu$.

Problem 5.24 Let $X_i(t)$ denote the number of customers at server i, $i = 1, 2$, when there is unlimited waiting room. The, in steady state,

$$\mathsf{P}(n_i \text{ at server } i) = \prod_{i=1}^{2} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \left(1 - \frac{\lambda_i}{\mu_i}\right) \ .$$

Now the model under consideration is just a truncation of the above, which is time reversible by problem 5.23. The truncation is to the set

$$A \triangleq \{(n,m) : n = 0, m \leq N+1 \text{ or } m = 0, n \leq N+1 \text{ or } n > 0, m > 0, n+m \leq N+2\} \ .$$

Hence, for $(n,m) \in A$,

$$P(n,m) = C\left(\frac{\lambda_1}{\mu_1}\right)^n \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_2}{\mu_2}\right)^m \left(1 - \frac{\lambda_2}{\mu_2}\right) \ .$$

Problem 5.25 In steady state it has the same probability structure as the arrival process. Hence if we include in the departure process those arrivals that do not enter, then it is a Poisson process.

Problem 5.26 (a) Follows from results of section 6.2 by writing $\underline{n}' = D_j \underline{n}$ and so $\underline{n} = B_{j-1}\underline{n}'$.

(b) A Poisson process by time reversibility. If $D(0) = 0$, it is a nonhomogeneous Poisson process.

Problem 5.28 For $\underline{n} = (n_1, \cdots, n_r)$, let

$$P(\underline{n}) = C \prod_{i=1}^{r} \alpha_i^{n_i} \ .$$

If

$$\underline{n}' = (n_1, \cdots, n_i - 1, \cdots, n_j + 1, \cdots, n_r)$$

with $n_i > 0$ then

$$\begin{aligned} & P(\underline{n})q(\underline{n}, \underline{n}') = P(\underline{n}')q(\underline{n}', \underline{n}) \\ \Leftrightarrow \quad & \alpha_i \frac{\mu_i}{r-1} = \alpha_j \frac{\mu_j}{r-1} \\ \Leftrightarrow \quad & \alpha_i \mu_i = \alpha_j \mu_j \ . \end{aligned}$$

So, setting $\alpha_i = 1/\mu_i$, $i = 1, \cdots, r$ and letting C be such that $\sum_{\underline{n}} P(\underline{n}) = 1$ the time reversibility equations are satisfied.

Problem Extra

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Program = Calculations for the M/M/s/r=M Model following 1999 Improving paper
WhoWhen = Ward Whitt 11/17/03

Step1 = Model Parameters
s = number of servers, s = 100
lambda = arrival rate, lambda =   100
mu = individual service rate, mu = 1
alpha = individual abandonment rate (exponential abandonments), alpha =    1
r = number of extra waiting spaces, r =   100

Step2 = Key Steady-state Probabilities
ProbNoWait = Probability of not having to wait before beginning service ProbNoWait = 0.48670120172085
ProbAllBusy = Probability that all servers are busy upon arrival ProbAllBusy = 0.51329879827915
ProbWaitandServed = 0.47343780147000
ProbServed = 0.96013900319085
ProbLoss = Probability that an arrival is lost (blocked) at arrival ProbLoss = 4.716970602792618e-019
ProbAban = Probability that a customer eventually abandons ProbAban = 0.03986099680915

Step3 = Mean Values - Counting
MeanInQueue =   3.98609968091471
MeanBusyServers =  96.01390031908522
MeanNumberinSys = 99.99999999999993

Step4 = Second Moments and Variances - Counting
SecondMomInQueue = 51.32987982791486
VarInQueue =  35.44088916172650
SecondMomBusyServers =    9.251450183989135e+003
VarBusyServers = 32.78112950590003
SecondMomNumberInSys =    1.010000000000000e+004
VarNumberInSys =    1.000000000000109e+002

Step5 = Response-Time Moments
MeanRespTime =   0.99750428450446
ConditMeanRespTimeServed =   1.03891653311596
SecMomRespTime = 2.94660609401905
VarRespTime =   1.95159129641430
ConditSecMomRespTimeServed =   3.06893698123555
ConditVarRespTimeServed =   1.98958941845386

Step6 = Waiting-Time Moments
Step6a = Waiting Times for Customers who are Served
MeanWaitServed =   0.03736528131360
ConditMeanWaitServed =   0.03891653311596
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SecMomWaitandServed = 0.00472192207013
VarWaitandServed =   0.00332575782249
ConditSecMomWaitServed =   0.00491795672756
ConditVarWaitServed = 0.00340346017779

Step6b = Waiting Times for Customers who Abandon
MeanWaitAbandon = 0.00249571549554
ConditMeanWaitAban =   0.06261046374463
SecMomWaitAbandon =    2.695089209549688e-004
ConditSecMomWaitAbandon =   0.00676121879855
ConditVarWaitAban =   0.00284114862823

Step6c = Waiting Time for All Customers, Served or Abandoning
MeanWaitTimeAll =   0.03986099680915
SecMomWaitTimeAll =   0.00499143099109
VarWaitTimeAll =   0.00340253192447

Step7 = Conditional Waiting Time CCDF Values For Served Customers
t =   0.05000000000000 ProbOKWaitifServed =   0.70144791424012
t =   0.10000000000000 ProbOKWaitifServed =   0.84743410401854
t =   0.20000000000000 ProbOKWaitifServed =   0.97658087743471
t =   0.40000000000000 ProbOKWaitifServed =   0.99993119489496

Step8 = Conditional Waiting Time CDF Values For Customers Who Abandon
t =   0.05000000000000 ProbOKWaitifAbandon =   0.50953673688750
t =   0.10000000000000 ProbOKWaitifAbandon =   0.79190403053181
t =   0.20000000000000 ProbOKWaitifAbandon =   0.97645578331304
t =   0.40000000000000 ProbOKWaitifAbandon =   0.99995439602867

Step9 = Waiting Time CDF Values For All Customers
t =   0.05000000000000 ProbOKWait =   0.69379814341203
t =   0.10000000000000 ProbOKWait =   0.84522061993647
t =   0.20000000000000 ProbOKWait =   0.97657589105832
t =   0.40000000000000 ProbOKWait =   0.99993211971528
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(a) MeanInQueue = 3.98609968091471, VarInQueue = 35.44088916172650

(b) arrival rate $\times$ ProbAban = 100 $\times$ 0.03986099680915= 3.986

(c) ProbNoWait = 0.48670120172085

(d) ConditMeanWaitServed + service time = 0.03891653311596 + 1

(e) Conditional Waiting Time CDF Values For Served Customers at $t = 0.1$ = ProbOK-WaitifServed = 0.84743410401854

(f) Since the abandon rate α equals the service rate μ, the model simplifies. But note that the simplification and following solution only works for the special case of $\alpha = \mu$. First, in the view point of *the number of customers in the system*, the system is equivalent to $M/M/s + r/0$ since the departure rate from the system including the abandoned customers is $n\mu$ even if $n > s$ because of $(n-s)\alpha = (n-s)\mu$ abandonment rate. However, we can anticipate that blocking is negligible with such a large waiting space. Thus the $M/M/s + r/0$ model should be essentially equivalent to a $M/M/\infty$ model, for which the steady-state distribution is exactly Poisson, and approximately normal. Hence, the probability all servers are busy in the given $M/M/s/r+M$ model is approximately equal to the probabiity that at least s servers are busy in the $M/M/\infty$ model. Clearly, it is possible to compute the probability that at least s servers are busy in the $M/M/\infty$ mode very quickly. (It might not be regarded as "simple," however. When you read, "it is easy to see that ...," in a paper, take that as an invitation to check very carefully.)

Now we describe how to proceed with the more special, but exact, representation in terms of the $M/M/s+r/0$ model: For $n < s+r = 200$

$$\begin{aligned} P_n &= \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{k=1}^{s+r} \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}\right)} \\ &= \frac{\lambda^n}{n! \left(\sum_{k=0}^{s+r} \frac{\lambda^k}{k!}\right)} \\ &= \frac{e^{-\lambda} \frac{\lambda^n}{n!}}{\left(\sum_{k=0}^{s+r} e^{-\lambda} \frac{\lambda^k}{k!}\right)} \end{aligned}$$

and we have (for Poisson random variable N with parameter $\lambda = 100$)

$$\begin{aligned} \mathsf{P}(\text{All servers are busy}) &= \sum_{n=0}^{s-1} P_n \\ &= \frac{\mathsf{P}(N \le 99)}{\mathsf{P}(N \le 200)} \\ &= 0.48670120172087 \quad \text{from MATLAB} \end{aligned}$$

(g) MeanInQueue$\times$ abandon rate per customer in queue
$= 3.98609968091471 \times 1 = 3.98609968091471$