Solutions to Homework Assignment 6

Problem 3.1 (a) Yes since it is a contrapositive statement of (3.2.1).

  (b) No. Note that we can have $X_{n+1} = 0$, so that $S_n = S_{n+k} = 0$ for some $k \geq 1$. Hence We can have $S_n = t$, but $N(t) > n$.

  (c) No. Follows from (b) above.

Problem 3.2

$$\mathsf{P}(\text{exactly } n \text{ renewals}) = F(\infty)^n (1 - F(\infty)).$$

Problem 3.3 One might simply say that it is the length of the renewal interval that contains t, but that is not quite correct. Or, at least that answer is at best only partially correct. More precisely, it is the length of the first renewal interval that ends *after* time t. Suppose that $S_n = t < S_{n+1}$. Then the interval $(S_{n-1}, S_n]$ contains t. It is natural to define the renewal intervals as open on the left and closed on the right. Then $N(t) = n$. Then $X_{N(t)+1}$ is the length of the interval $(S_n, S_{n+1}]$, which does not contain t. The point $S_{N(t)+1}$ is the first renewal after $(>) \; t$. But we can only say that $S_{N(t)} \leq t$; we could have $S_{N(t)} = t$. When we write $X_{N(t)+1}$, we only discuss the length of that interval; the nature of the endpoints does not come up; so it is ambiguous.

$$
\mathsf{P}(A(t) \leq x) = \begin{cases} 1 & \text{if } x \geq t \\ \mathsf{P}(\text{At least one event in } [t-x,t]) & \text{if } x < t \end{cases}
$$
$$
= \begin{cases} 1 & \text{if } x \geq t \\ 1 - e^{-\lambda x} & \text{if } x < t \end{cases}
$$

which is a *truncated* exponential distribution. The momoryless property implies that $Y(t)$ is an exponential distribution with rate λ and so the distribution of $X_{N(t)+1}$ is the convolution of $A(t)$ and an exponential with rate λ.

Problem 3.4

$$\mathsf{E}[N(t)] = \int_0^\infty \mathsf{E}[N(t)|X_1 = x] dF(x) \; .$$

As

$$
\mathsf{E}[N(t)|X_1 = x] = \begin{cases} 1 + \mathsf{E}[N(t-x)] & \text{if } x \leq t \\ 0 & \text{if } x > t \end{cases}
$$

the result follows.

Or following purely analytic way:

$$
\begin{aligned}
m(t) &= \sum_{n=1}^{\infty} F_n(t) \\
&= F_1(t) + \sum_{n=2}^{\infty} F_n(t) \\
&= F(t) + \sum_{n=1}^{\infty} \int_0^{\infty} F_n(t-x)dF(x) \\
&= F(t) + \int_0^{\infty} \left(\sum_{n=1}^{\infty} F_n(t-x) \right) dF(x) \\
&= F(t) + \int_0^{\infty} m(t-x)dF(x) \\
&= F(t) + \int_0^{t} m(t-x)dF(x)
\end{aligned}
$$

where the interchange of integral and sum is allowed by the non-negativity of the integrand and $m(t) = 0$ if $t \le 0$.

Problem 3.5 There is a simple derivation here (to be presented below), but there is a source of confusion: We need to properly distinguish between the Laplace transform (LT) of a function and the Laplace-Stieltjes transform (LST) of an increasing function (or measure). The LT of the cdf F is

$$
\hat{F}(s) = \int_0^{\infty} e^{-sx} F(x) \, dx \ ,
$$

while the LST of F is

$$
\hat{f}(s) = \int_0^{\infty} e^{-sx} dF(x) \ .
$$

If the cdf F has a pdf (density) f, then the LST of F is the LT of f:

$$
\hat{f}(s) = \int_0^{\infty} e^{-sx} dF(x) = \int_0^{\infty} e^{-sx} f(x) \, dx \ .
$$

The same distinction applies to the renewal function $m(t)$, but the notation is confusing, because $m(t)$ plays the role of the cdf F, being increasing. We call $\hat{m}(s)$ the LT of m:

$$
\hat{m}(s) = \int_0^{\infty} e^{-sx} m(x) \, dx \ ,
$$

But we could consider the LST of m.

It is next important to be able to relate the LST and the LT. They are related by

$$
\hat{F}(s) = \frac{\hat{f}(s)}{s} \ .
$$

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Note that this is a straightforward relation between LT's if the cdf F has a pdf f, but it applies more generally. Then, using properties of the Laplace transform of a convolution, we know that if $f_n(\cdot)$ represents the density of the sum of n IID random variables, then

$$\hat{f}_n(s) = (\hat{f}(s))^n \ ,$$

so that

$$\hat{F}_n(s) = \frac{\hat{f}_n(s)}{s} = \frac{(\hat{f}(s))^n}{s} \ .$$

Now using the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \ ,$$

take Laplace transform to obtain

$$
\begin{aligned}
\hat{m}(s) &= \sum_{n=1}^{\infty} \hat{F}_n(s) \\
&= \sum_{n=1}^{\infty} \frac{\hat{f}(s)^n}{s} \\
&= \frac{\hat{f}(s)}{s(1 - \hat{f}(s))}
\end{aligned}
$$

or, equivalently,

$$\hat{F}(s) = \frac{\hat{f}(s)}{s} = \frac{\hat{m}(s)}{1 + s\,\hat{m}(s)} \ .$$

Since, a function is determined by its Laplace transform, this implies the result.

Or we may derive it by applying the Laplace transform to the result of Problem 3.3.4:

$$
\begin{aligned}
\hat{m}(s) &= \hat{F}(s) + \hat{m}(s)\hat{f}(s) \\
&= \frac{\hat{f}(s)}{s} + \hat{m}(s)\hat{f}(s) \ .
\end{aligned}
$$

Are you sure above derivations? For the first one, we need $|\hat{f}(s)| < 1$ and for the second one, $\hat{f}(s) \neq 1$.

$s = a + ib$ with $a > 0$ implies

$$
\begin{aligned}
|\hat{f}(s)| &\leq \int_0^{\infty} |e^{-ax - ibx}| dF(x) \\
&= \int_0^{\infty} e^{-ax} dF(x) \\
&= \int_0^{x^*} e^{-ax} dF(x) + \int_{x^*}^{\infty} e^{-ax} dF(x) \\
&\leq \int_0^{x^*} dF(x) + e^{-ax^*} \int_{x^*}^{\infty} dF(x) \\
&= F(x^*) + e^{-ax^*}(1 - F(x^*)) \\
&< 1
\end{aligned}
$$

for any $x^* > 0$ satisfying $F(x^*) < 1$, which supports our previous derivations.

Added Problem (a)

$$\hat{f}_{X+Y}(s) \;=\; \hat{f}_X(s)\hat{f}_Y(s) = \frac{1/2}{(s+1/2)}\frac{1/3}{(s+1/3)} = \frac{1}{(2s+1)(3s+1)} \;.$$

Hence

$$\hat{m}(s) = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} = \frac{1}{s^2(6s+5)} \;.$$

(b) $m(10) \simeq 1.7601$, $m(20) \simeq 3.7600$.

(c) Adopting the assumption $m(t) \le c + dt$, let's start at the equation (11) of Whitt's 1995 paper:

$$
\begin{aligned}
e_d \;&=\; \sum_{k=1}^{\infty} e^{-kA} m((2k+1)t) \\
&\le\; \sum_{k=1}^{\infty} e^{-kA}(c + d(2k+1)t) \\
&=\; (c+dt)\sum_{k=1}^{\infty} e^{-kA} + 2dt\sum_{k=1}^{\infty} ke^{-kA} \\
&=\; (c+dt)\frac{e^{-A}}{1-e^{-A}} + 2dt\lim_{n\to\infty}\frac{(n+1)e^{-(n+1)A} - ne^{-nA} + 1}{(1-e^{-A})^2}e^{-A} \\
&=\; (c+dt)\frac{e^{-A}}{1-e^{-A}} + 2dt\frac{e^{-A}}{(1-e^{-A})^2} \\
&\simeq\; (c+dt)e^{-A} + 2dte^{-A} \\
&=\; (c+3dt)e^{-A} \;.
\end{aligned}
$$

Hence to achieve $\epsilon = 10^{-8}$ precision,

$$
\begin{aligned}
A \;&\ge\; -\ln\left(\frac{\epsilon}{c+3dt}\right) \\
&=\; 8\ln 10 + \ln(c+3dt) \;.
\end{aligned}
$$

$$d = \frac{1}{\mu} = \frac{1}{2+3} = \frac{1}{5} = 0.2$$

$$
\begin{aligned}
c \;&\simeq\; \frac{\mathsf{E}[X^2]}{2\mu^2} - 1 = \frac{2^2 + 3^2 + 5^2 2 \times 5^2}{2 \times 2 \times 5^2} \\
&=\; \frac{4+9-25}{50} = -\frac{6}{25} \;.
\end{aligned}
$$

So

$$
\begin{aligned}
A \;&\ge\; 8\ln 10 + \ln\left(-\frac{6}{25} + 0.6t\right) \\
&=\; 8\ln 10 + \ln(5.76 \text{ or } 11.76) \\
&=\; 20.1716 \text{ or } 20.8854 \\
&\simeq\; 21 \;.
\end{aligned}
$$

With this new parameter, no change of the result up to four decimal digits.

Problem 3.6 For $s \le t$,

$$\mathsf{E}[N(s)|N(t)] = \frac{s}{t}N(t)$$

and so

$$\mathsf{E}\left[\frac{N(s)}{s}\right] = \mathsf{E}\left[\frac{N(t)}{t}\right]$$

implying $\mathsf{E}[N(t)/t]$ is a constant. That is $\mathsf{E}[N(t)] = ct$. But since this is also the renewal function for a Poisson process with rate c, and the renewal function uniquely determines the interarrival distribution (by Problem 3.3.5), we conclude that the renewal process is a Poisson process.

Problem 3.9 Set $\mu = \int_0^\infty \bar{G}(t)dt$. The underlying model is M/G/1/1. The first step is converting the model into an alternating renewal process. We may define the alternating cycles like (server is idle, server is busy), which is (exponential interarrival time, service time with $G(\cdot)$) in distributional sense. This is a renewal process since the arrival follows Poisson or the interarrival time doesn't have any memory. Then is is easy to get the followings:

(a)

$$\left(\frac{1}{\lambda} + \mu\right)^{-1}$$

(b)

$$\frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mu} = \frac{1}{1 + \lambda\mu}$$

(c)

$$\frac{\mu}{\frac{1}{\lambda} + \mu}$$

Problem 3.10 First I will do the computations, without thinking carefully about justifying the steps:

(a)

$$\frac{S_1 + \cdots + S_m}{N_1 + \cdots + N_m} = \frac{\sum_{i=1}^{N_1 + \cdots + N_m}}{N_1 + \cdots + N_m} \xrightarrow[m \to \infty]{} \mathsf{E}[X] \ .$$

(b)

$$\frac{S_1 + \cdots + S_m}{m} \xrightarrow[m \to \infty]{} \mathsf{E}[S] = \mathsf{E}\left[\sum_1^{N_1} X_i\right] \ ,$$

$$\frac{N_1 + \cdots + N_m}{m} \xrightarrow[m\to\infty]{} E[N] = E[N_1]$$

imply

$$\frac{S_1 + \cdots + S_m}{N_1 + \cdots + N_m} = \frac{S_1 + \cdots + S_m}{b} \frac{m}{N_1 + \cdots + N_m} \xrightarrow[m\to\infty]{} \frac{E\left[\sum_1^{N_1} X_i\right]}{E[N_1]} .$$

(c)

$$\frac{E\left[\sum_1^{N_1} X_i\right]}{E[N_1]} = E[X] .$$

Now where did we make bold computations? Are S_i IID? Why? If they are IID, that means

$$X_{N_1+i} \overset{d}{=} X_i$$

which looks similar to the famous *strong Markov* property. First noting that

$$\begin{aligned}
P(X_{N_1+i} \le x | N_1 = n) &= P(X_{n+i} \le x | N_1 = n) \\
&= P(X_{n+i} \le x) \quad \text{since } N_1 \text{ is a stopping time.} \\
&= P(X_i \le x) ,
\end{aligned}$$

$$\begin{aligned}
P(X_{N_1+i} \le x) &= E[P(X_{N_1+i} \le x | N_1)] \\
&= E[P(X_i \le x)] \\
&= P(X_i \le x) .
\end{aligned}$$

We can also conclude that S_i are IID using similar arguments.

Problem 3.11 (a) X_i is the travel time on the ith choice; N is the number of choices until freedom is reached.

(b) $E[T] = E[N]E[X]$, $E[X] = \frac{1}{3}(2 + 4 + 8) = \frac{14}{3}$, $E[N] = 3$ since N is geometric with $p = 1/3$. Hence $E[T] = 14$.

(c)

$$\begin{aligned}
E\left[\sum_1^N X_i \,\middle|\, N = n\right] &= \sum_1^n E[X_i | N = n] \\
&= \sum_1^{n-1} E[X_i | \text{Incorrect door}] + E[X_n | \text{Correct door}] \\
&= \sum_1^{n-1} \frac{1}{2}(4 + 8) + 2 \\
&= 6n - 4 \\
&\ne \frac{14}{3}n \\
&= E\left[\sum_1^n X_i\right] .
\end{aligned}$$

(c)

$$
\begin{aligned}
\mathsf{E}[T] \;&=\; \mathsf{E}\!\left[\sum_{1}^{N} X_i\right] \\
&=\; \mathsf{E}\!\left[\left[\sum_{1}^{N} X_i \,\middle|\, N\right]\right] \\
&=\; \mathsf{E}[6N - 4] \\
&=\; 6\mathsf{E}[N] - 4 = 18 - 4 = 14 \; .
\end{aligned}
$$