IEOR 6711: Stochastic Models I, Professor Whitt

SOLUTIONS to Homework Assignment 7

Problem 3.12 Take $h(t) = \mathbf{1}_{(0,a]}(t)$.

Problem 3.13 Since the state circulates the state space $\{1, 2, \cdots, n\}$ in the same order. Hence we can define the alternating renewal process by *on* when it is in the state i and *off* when it is among $\{1, \cdots, i-1, i+1, \cdots, n\}$. Define $\mu_i = \int \bar{F}_i(t)dt$. Then

$$\mathsf{P}(\text{process is in } i) \to \frac{\mathsf{E}[on]}{\mathsf{E}[on] + \mathsf{E}[off]} = \frac{\mu_i}{\sum_{j=1}^n \mu_j} .$$

Problem 3.14 **(a)** $[t-x, t]$

(b) $[t, t+x]$

(c) $\mathsf{P}(Y(t) > x) = \mathsf{P}(A(t+x) > x)$

(d) See Problem 3.3.

Problem 3.15 **(a)**

$$\begin{aligned} \mathsf{P}(Y(t) > x | A(t) = s) &= \mathsf{P}(X_{N(t)+1} > x + s | \text{time at } t \text{ since the last renewal} = s) \\ &= \frac{\bar{F}(x+s)}{\bar{F}(s)} . \end{aligned}$$

(b) Using (a),

$$\mathsf{P}(Y(t) > x | A(t + x/2) = s) = \begin{cases} 0 & \text{if } s < \frac{x}{2} \\ \frac{\bar{F}(s+x/2)}{\bar{F}(s)} & \text{if } s \geq \frac{x}{2} \end{cases} .$$

(c)

$$\mathsf{P}(Y(t) > x | A(t+x) > s) = \begin{cases} 1 & \text{if } s \geq x \\ \mathsf{P}(\text{no events in } [t, t+x-s]) = e^{-\lambda(x-s)} & \text{if } s < x \end{cases} .$$

(d)

$$\mathsf{P}(Y(t) > x,\ A(t) > y) = \mathsf{P}(Y(t-y) > x+y) = \mathsf{P}(A(t+x) > x+y) .$$

(e)

$$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{N(t)} \frac{N(t)}{t} \to 1 - \mu \frac{1}{\mu} = 0 .$$

Problem 3.16

$$\begin{aligned}\lim_{t\to\infty} \mathsf{E}[Y(t)] &= \frac{\mathsf{E}[X^2]}{2\mu} \\ &= \frac{n\left(\frac{1}{\lambda}\right)^2 + \left(\frac{n}{\lambda}\right)^2}{2\frac{n}{\lambda}} \\ &= \frac{1+n}{2\lambda}\ .\end{aligned}$$

To get it without any computations, consider a Poisson process with rate λ and say the a *renewal* occurs at the Poisson events numbered $n, 2n, \cdots$. Now at time t, t large, it is equally likely that the most recent event was an event of the form $i+kn$, $i = 0, 1, 2, \cdots, n-1$. That is, modulo n, the number of the most recent Poisson event is equally likely to be $n, 1, \cdots, n-1$. Conditioning on the value of this quantity gives that for the renewal process

$$\lim_{t\to\infty} \mathsf{E}[Y(t)] = \frac{1}{n}\left(\frac{1}{\lambda} + \cdots + \frac{n}{\lambda}\right) = \frac{n+1}{2\lambda}\ .$$

Problem 3.18 **(a)** Delayed renewal process.

(b) Neither.

If F is exponential,

(a) Delayed renewal process.

(b) Renewal process.

Problem 3.21 Let X_i equal 1 if the gambler wins bet i, and let it be 0 otherwise. Also, let N denote the first time the gambler has won k consecutive bets. Then $X = \sum_{i=1}^N X_i$ is equal to the number of bets that he wins, and $X - (N - X) = 2X - N$ is his winnings. By Wald's equation

$$\mathsf{E}[X] = p\mathsf{E}[N] = p\sum_{i=1}^{k} p^{-i}\ .$$

Thus

(a) $\mathsf{E}[2X - N] = 2\mathsf{E}[X] - \mathsf{E}[N] = (2p-1)\mathsf{E}[N] = (2p-1)\sum_{i=1}^k p^{-i}$

(b) $\mathsf{E}[X] = p\sum_{i=1}^k p^{-i}$

Problem 3.22 **(a)**

$$\begin{aligned}\mathsf{E}[T_{HHTTHH}] &= \mathsf{E}[T_{HH}] + p^{-4}(1-p)^{-2} \\ &= \mathsf{E}[T_H] + p^{-2} + p^{-4}(1-p)^{-2} \\ &= p^{-1} + p^{-2} + p^{-4}(1-p)^{-2}\end{aligned}$$

(b) $\mathsf{E}[T_{HTHTT}] = p^{-2}(1-p)^{-3}$

$\mathsf{E}[N_{B|A}] = \mathsf{E}[N_{HTHTT|H}] = \mathsf{E}[N_{HTHTT}] - \mathsf{E}[N_H] = 32 - 2 = 30$, $\mathsf{E}[N_{A|B}] = \mathsf{E}[N_A] = 64 + 4 + 2 = 70$ and $\mathsf{E}[N_B] = 32$.

(c) $P_A = (32 + 70 - 70)/(30 + 70) = 0.32$

(d) $\mathsf{E}[M] = 32 - 30(0.32) = 22.4$

Problem 3.23 Let H denote the first k flips and Ω is the set of all possible H. Conditioning on H gives:

$$\begin{aligned}\mathsf{E}[\text{number until repeat}] &= \sum_{H\in\Omega} \mathsf{E}[\text{number until repeat}|H]\mathsf{P}(H) \\ &= \sum_{H\in\Omega} \frac{1}{\mathsf{P}(H)}\mathsf{P}(H) = |\Omega| = 2^k\end{aligned}$$

Problem 3.25 **(a)** First note that

$$\mathsf{E}[N_D(t)|X_1 = x] = \begin{cases} 1 + \mathsf{E}[N(t-x)] & \text{if } x \le t \\ 0 & \text{if } x > t \end{cases}.$$

$$\begin{aligned} m_D(t) = \mathsf{E}[N_D(t)] &= \int_0^\infty \mathsf{E}[N_D(t)|X_1 = x]dG(x) \\ &= \int_0^t (1 + \mathsf{E}[N(t-x)])dG(x) \\ &= G(t) + \int_0^t m(t-x)dG(x)\end{aligned}$$

(b)

$$\begin{aligned}\mathsf{E}[A_D(t)] &= \mathsf{E}[A_D(t)|S_{N_D(t)} = 0]\bar{G}(t) + \int_0^t \mathsf{E}[A_D(t)|S_{N_D(t)} = s]\bar{F}(t-s)dm_D(s) \\ &= t\bar{G}(t) + \int_0^t (t-s)\bar{F}(t-s)dm_D(s) \\ &\xrightarrow{t\to\infty} \frac{1}{\mu}\int_0^\infty t\bar{F}(t)dt \quad \text{By key renewal theorem (Proposition 3.5.1(v)} \\ &= \frac{1}{\mu}\int_0^\infty t\int_t^\infty dF(s)dt \\ &= \frac{\int_0^\infty s^2 dF(s)}{2\int_0^\infty s dF(s)}\end{aligned}$$

(c) $t\bar{G}(t) = t\int_t^\infty dG(x) \le \int_t^\infty s\ dG(s) \xrightarrow{t\to\infty} 0$ since $\int_0^\infty s\ dG(s) < \infty$.
(Here we used the so-called *dominated convergence theorem.*

$$\begin{aligned}\int_n^\infty s\ dG(s) &= \int_0^\infty s\mathbf{1}_{[n,\infty)}(s)\ dG(s) \\ &\underset{\text{dominated convergence thorem}}{\xrightarrow{n\to\infty}} \int_0^\infty \lim_{n\to\infty} s\mathbf{1}_{[n,\infty)}(s)\ dG(s) = \int_0^\infty 0\ dG(s)\end{aligned}$$

since $s\mathbf{1}_{[n,\infty)}(s) \le s$ and s is integrable with respect to $G(\cdot)$ from $\int_0^\infty s\, dG(s) < \infty$ and $s\mathbf{1}_{[n,\infty)}(s) \to 0$ for each s in *pointwise* sense. (Check the conditions for the dominated convergence theorem.) Now we extend n to t using monotonicity of the integral. Wow! This is a good example showing that if you are familiar with a little rigorous *analysis*, then it's O.K. with only one line. But if not, you should practice the underlying logic whenever you encounter them.)

Problem 3.28 Using the *uniformity* of each Poisson arrival under given $N(t)$,

$$\mathsf{E}[\text{Cost of a cycle}|N(T)] = K + N(T) \times c \times \frac{T}{2}$$

and so

$$\frac{\mathsf{E}[\text{Cost}]}{\mathsf{E}[\text{Time}]} = \frac{K + \lambda c T^2/2}{T} = \frac{K}{T} + \frac{\lambda c T}{2}$$

which is minimized at $T^* = \sqrt{2K/\lambda c}$ and minimal average cost is thus $\sqrt{2\lambda K c}$. On the other hand the optimal value of N is (using calculus) $N^* = \sqrt{2\lambda K/c}$ and the minimal average cost is $\sqrt{2\lambda c K} - \frac{c}{2}$.