IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt

Numerical Transform Inversion Homework: Tuesday, September 10

You have FIVE WEEKS: Due Tuesday, October 15.

1. Write a program implementing the algorithm "Euler," which does the Fourier-series algorithm for numerically inverting Laplace transforms with Euler summation, as in the 1995 Abate-and-Whitt paper. (See the Computational Tools web page.) You may use MATLAB in the Computer Lab in 301 Mudd, but you need not. You will need the facility to work with complex numbers. Thus $C++$ is a natural alternative. See Bjarne Stroustrup's homepage and links

http://www.research.att.com/∼bs

Please turn in a printout of your code plus the numerical results for the problems below. It is good to be able to produce tables of numerical results as well as plots. The detailed numerical results enable you to see the accuracy.

2. The exponential pdf is

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

and the exponential ccdf is

$$F^c(t) \equiv 1 - F(t) \equiv e^{-\lambda t}, \quad t \geq 0 \ .$$

The associated Laplace transforms are

$$\hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\, dt = \frac{\lambda}{\lambda + s}$$

and

$$\hat{F^c}(s) \equiv \int_0^\infty e^{-st} F^c(t)\, dt = \frac{1 - \hat{f}(s)}{s} = \frac{1}{\lambda + s} \ .$$

Check your algorithm on the Laplace transform of the exponential pdf and ccdf above in the case of $\lambda = 2$, using the discretization-error parameter $A = 19$. Estimate your error using $E(m, n+1) - E(m, n)$. Use the Euler summation parameters $n = 38$ and $m = 11$. Each instance of the inversion program computes the function at one argument ("time" value); compute the function at several time values, e.g., ten of them. (Of course, this problem is just to check your program, because we know the exponential pdf and ccdf as well as their transforms.)

3. For $i = 1, \ldots, 4$, let X_i be an exponential random variable with mean i. (Remember that the mean is the reciprocal of the rate!) Let the four random variables be mutually independent. Use numerical transform inversion to calculate the probabilities $P(X_1 + X_2 + X_3 + X_4 > 8)$ and $P(X_1 + X_2 + X_3 + X_4 > 40)$.

(a) Find the Laplace transform of the function $P(X_1 + X_2 + X_3 + X_4 > t)$, $t \geq 0$.

(b) Calculate $P(X_1 + X_2 + X_3 + X_4 > 8)$ and $P(X_1 + X_2 + X_3 + X_4 > 40)$ using your inversion code.

4. The first steps of the numerical inversion algorithm find an expression for the inversion in terms of an infinite series. The final step is to compute the sum of that infinite series. Instead of just truncating the series at some point, we use a "convergence acceleration technique." Specifically, the algorithm EULER uses Euler summation. Write a program to implement Euler summation for calculating the approximate sum of the infinite series

$$s_\infty = \sum_{k=1}^{\infty} (-1)^k a_k$$

where a_k are positive real numbers for all $k \geq 1$. As standard parameters, use $n = 38$ and $m = 11$. Compare with simple truncation ($m = 0$) using truncation levels $n = 100$, $n = 1000$ and $n = 10,000$ for the examples

(a) $a_k = (3 + k)^{-2}$

(b) $a_k = (3 + k)^{-1}$.

If you also do $m = 1$, you can see that already provides great improvement over $m = 0$.