IEOR 6711: Professor Whitt, Fall, 2013

Problem for Discussion, Tuesday, September 17

1. Trip to the Post Office

Five students from IEOR 6711 – Jalay Bhandari (B), Hal Cooper (C), Francois Fagan (F), Irene Lo (L) and Ni Ma (M) – simultaneously enter an empty post office, where there are three clerks ready to serve them. Jalay, Hal and Francois begin to receive service immediately, while Irene and Ni wait in a single line, ready to be served by the first free clerk, with Irene at the head of the line (to be served first when a server becomes free), and Ni last after Irene. (Service is provided in alphabetic order, not using politeness.) Suppose that the service times of the three clerks (for all customers) are independent exponential random variables, each with mean 2 minutes.

(a) What is the *conditional* probability that Francois is still in service after 10 minutes, given that Francois has not yet been served after 4 minutes?

(b) What is the probability that Jalay is the first to complete service?

(c) What is the expected time (from the moment the students enter the post office) until the first student completes service?

(d) What is the variance of the time (from the moment the students enter the post office) until the first student completes service?

(e) What is the expected time (from the moment the students enter the post office) until Irene completes service?

(f) What is the expected time (again since entering the post office) until *all* five students finish service?

(g) What is the variance of the time until *all* five students finish service?

(h) What is the probability that Irene is the *third* student to finish service?

(i) Suppose that you wanted to calculate the probability that the time required for all five students to complete service will exceed 10 minutes. What computational tool makes that calculation easy to perform? Briefly explain why.

2. **Yanan's Flashlight**

Yanan Pei's flashlight needs two batteries to be operational. Suppose that, in addition to her (empty) flashlight, Yanan has a set of 12 functioning batteries, called battery 1, battery 2, and so forth. Initially, Yanan puts batteries 1 and 2 into her flashlight, so that it starts working. Then batteries fail one by one. Whenever a battery in the flashlight fails, the flashlight stops working. Yanan then tests the two batteries in the flashlight to see which one had failed, and she removes that battery. She then puts in the next available unused battery with the remaining working battery, so that the flashlight is again working. Suppose that the batteries remain like new until installed in the flashlight. Suppose that the lifetimes of the different batteries (in use in the flashlight) are independent random variables, each with an exponential distribution having a mean of 4 months. Let T be the time that the flashlight ceases to work, i.e., the time that the flashlight fails and Yanan's supply of batteries is exhausted. At that moment, exactly one of the original 12 batteries will still be working. Let that last remaining working battery be battery N. Note that N is a random variables taking values in the set $\{1, 2, \ldots, 12\}$. (It will be the one remaining working battery in the flashlight.)

(a) What is the expected value of T?

(b) What is $P(N = 12)$?

(c) What is $P(N = 1)$?

2. **Greedy Algorithms for the Assignment Problem**

A group of n people are to be assigned to n jobs, with one person assigned to each job. A cost of $C_{i,j}$ is incurred if person i is assigned to job j. The classical assignment problem is to determine the set of assignments that minimizes the sum of the n costs incurred. We will consider a random instance of the assignment problem: Suppose that the n^2 costs $C_{i,j}$ are IID random variables, each having an exponential distribution with mean 1.

(a) Suppose that the jobs are assigned totally at random (so that each person is equally likely to be assigned each job). What are the mean and variance of the total cost?

(b) Now consider the following greedy heuristic for approximately solving the assignment problem, called Greedy Algorithm A: Assign person 1 to the least-cost job for him. Then assign person 2 to the least-cost job for him from the available jobs remaining (not counting the job already assigned to person 1), and so forth. This procedure is continued until all people are assigned jobs. What are the mean and variance of the total cost for Greedy Algorithm A?

(c) Now consider the following alternative, more global, greedy heuristic, called Greedy Algorithm B: Among all n^2 job values, choose the pair (i_1, j_1) for which $C_{i,j}$ is minimal; then assign person i_1 to job j_1. Afterwards remove this person and this job from further consideration. From the remaining $(n-1)^2$ job values, choose the pair (i_2, j_2) that is minimal; then assign person i_2 to job j_2, and so forth. What are the mean and variance of the total cost for Greedy Algorithm B? Which greedy algorithm has less expected total cost? Which greedy algorithm has the smaller variance for the total cost?

(d) What is the asymptotic behavior of the optimal expected cost?

This is Example 5.7 in the green Ross book. Part (d) is hard; the answer is $\pi^2/6$; see David Aldous, The $\zeta(2)$ limit in the random assignment problem. *Random Structures and Algorithms* **18**, pp. 381-418, 2001.

There is more related literature; e.g., see http://mikespivey.wordpress.com/2011/10/14/random-assignment-problems/