IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt, Tuesday, October 8

Renewal Theory: Renewal Reward Processes

Elementary Renewal-Reward Theory

Suppose that we have a sequence of i.i.d. random vectors $\{(X_n, R_n) : n \geq 1\}$, where $P(X_1 > 0) = 1$. The random variable variable X_n is regarded as the *interrenewal time* between the $(n-1)^{\rm st}$ and $n^{\rm th}$ renewals in a renewal process; i.e., there is an associated *renewal counting process* $\{N(t) : t \geq 0\}$ with

$$N(t) \equiv \sup\{n \geq 0 : S_n \leq t\}, \quad t \geq 0,$$

where S_n is the $n^{\rm th}$ partial sum from $\{X_n\}$, i.e.,

$$S_n \equiv X_1 + \cdots X_n, \quad n \geq 1 \ ,$$

with $S_0 \equiv 0$; see Chapter 3 in Ross. We think of the random variables R_n as rewards. We "earn" reward R_n at time S_n. We define an associated continuous-time *renewal-reward stochastic process* by setting

$$R(t) \equiv \sum_{n=1}^{N(t)} R_n, \quad t \geq 0 \ .$$

The stochastic process $\{R(t) : t \geq 0\}$ is the renewal-reward process. The random variable $R(t)$ is the cumulative reward earned up to time t; see Section 3.6 of Ross.

We can do a surprising amount with the law of large numbers (LLN) for renewal-reward processes. This is perhaps the easiest part of renewal theory and yet it is perhaps the most useful part. So let us start by studying this part.

Theorem 0.1 (*SLLN for renewal-reward processes*) *If* $E[X_1] < \infty$ *and* $E[|R_1|] < \infty$, *then*

$$\frac{R(t)}{t} \to \frac{E[R_1]}{E[X_1]} \ .$$

It has an easy proof, which we will go through. See Theorem 3.6.1 of Ross. It draws on the SLLN for the renewal counting process $\{N(t) : t \geq 0\}$; see Proposition 3.3.1 in Ross.

Theorem 0.2 (*SLLN for renewal counting process*) *If* $E[X_1] < \infty$, *then*

$$\frac{N(t)}{t} \to \frac{1}{E[X_1]} \ .$$

Problems with Solutions

Source: Reference: Problems 37 and 38 in Chapter 7 of the Green Ross book.

1. **A Computer with Three Parts**

A computer has three parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts

1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time required to receive an order for a new part of type 1 is exponentially distributed with mean 1 week; the time required for a part of type 2 is uniformly distributed between 1 week and 3 weeks; and the time required for a part of type 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) What is the probability that part 1 is the first part to fail?

(b) Assuming that all parts are initially working, what is the expected time until the first failure?

(c) Assuming that all parts are initially working, what is the probability that the computer is still working (without having required to be shut down) after 5 weeks?

(d) What is the long-run proportion of time that the computer is working?

(e) What is the long-run proportion of time that the computer is down waiting for an order of a new part 1?

(f) What is the long-run proportion of time that part 2 is in a state of suspended animation (part 2 is working, but the computer is shut down)?

(g) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

2. **Trucks**

A truck driver regularly drives round trips from Atlanta (A) to Boston (B) and then back. Each time he drives from A to B he drives at a fixed speed (in miles per hour) that is uniformly distributed between 40 and 60. Each time he drives from B to A, he drives at a fixed speed that is equally likely to be 40 or 60 miles per hour.

(a) In the long-run, what proportion of his driving time is spent going to B?

(b) In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour?

SOLUTIONS

Solution to Number 1.

(a) What is the probability that part 1 is the first part to fail?

Let N be the index of the first part to fail. Since the failure times are mutually independent exponential random variables (see Concise notes on exponential and Poisson),

$$P(N=1) = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} = \frac{(1/10)}{(1/10)+(1/20)+(1/30)} = \frac{6}{11} \ .$$

(b) Assuming that all parts are initially working, what is the expected time until the first failure?

Let T be the time until the first failure. Then T is exponential with rate equal to the sum of the rates; i.e.,

$$ET = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1/10) + (1/20) + (1/30)} = \frac{1}{(11/60)} = \frac{60}{11} = 5.45 \text{ weeks} .$$

(c) Assuming that all parts are initially working, what is the probability that the computer is still working (without having required to be shut down) after 5 weeks?

Continuing from the last part,

$$P(T > 5) = e^{-(11/60)5} = e^{-55/60} \approx 0.40 .$$

(d) What is the long-run proportion of time that the computer is working?

Now for the first time we need to consider the random times it takes to get the replacement parts. Actually these distributions beyond their means do not affect the answers to the questions asked here. Only the means matter here. Use elementary renewal theory. The successive times that the computer is working and shut down form an alternating renewal process. Let T be a time until a failure (during which the computer is working) and let D be a down time. Then the long-run proportion of time that the computer is working is $ET/(ET+ED)$. By part (b) above, $ET = 60/11$. It suffices to find ED. To find ED, we consider the three possibilities for the part that fails:

$$\begin{aligned} ED &= P(N=1)E[D|N=1] + P(N=2)E[D|N=2] + P(N=3)E[D|N=3] \\ &= (6/11)E[D|N=1] + (3/11)E[D|N=2] + (2/11)E[D|N=3] \\ &= (6/11)1 + (3/11)2 + (2/11)3 \\ &= 18/11 . \end{aligned}$$

Hence,

$$ET/(ET + ED) = \frac{(60/11)}{(60/11) + (18/11)} = \frac{60}{78} = \frac{30}{39} \approx 0.769$$

(e) What is the long-run proportion of time that the computer is down waiting for an order of a new part 1?

Again apply renewal theory. In particular, apply the renewal reward theorem. The long-run proportiuon of time can be found by the renewal reward theorem: We want ER/EC, where ER is the expected reward per cycle and EC is the expected length of a cycle. Here a cycle is an up time plus a down time; i.e., $EC = ET + ED = 78/11$ from the last part. Here $ER = P(N = 1)E(D|N = 1) = (6/11) \times 1 = 6/11$. So

$$\frac{ER}{EC} = \frac{(6/11)}{(78/11)} = \frac{6}{78} = \frac{3}{39} \ .$$

(f) What is the long-run proportion of time that part 2 is in a state of suspended animation (part 2 is working, but the computer is shut down)?

This is a minor modification of the last part. We again use the renewal reward theorem, but now with a different reward. Now

$$ER = P(N = 1)E(D|N = 1) + P(N = 3)E(D|N = 3) = (6/11)1 + (2/11)3 = 12/11 \ .$$

So

$$\frac{ER}{EC} = \frac{(12/11)}{(78/11)} = \frac{12}{78} = \frac{6}{39} \ .$$

(g) Suppose that new parts of type 1 each cost \$50; new parts of type 2 each cost \$100; and new parts of type 3 each cost \$400. What is the long-run average cost of replacement parts per week?

This is yet another application of the renewal reward theorem. The cycle is the same as before, but now we have a new reward R. Now

$$\begin{aligned}
\frac{ER}{EC} &= \frac{ER}{ET + ED} \\
&= \frac{P(N = 1)50 + P(N = 2)100 + P(N = 3)400}{78/11} \\
&= \frac{(6/11)50 + (3/11)100 + (2/11)400}{78/11} \\
&= \frac{(300/11) + (300/11) + (800/11)}{78/11} \\
&= \frac{1400/11}{78/11} = \frac{1400}{78} \approx 17.95 \text{ dollars per week}
\end{aligned}$$

Solution to Problem 2.

(a) The proportion of his driving time spent driving from A to B is

$$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} \ ,$$

where $E[T_{A,B}]$ is the expected time to drive from A to B, while $E[T_{B,A}]$ is the expected time to drive from B to A.

To find $E[T_{A,B}]$ and $E[T_{B,A}]$, we use the elementary formula $d = rt$ (distance = rate $\times$ time). Let S be the driver's random speed driving from A to B. Then

$$\begin{aligned}
E[T_{A,B}] &= \frac{1}{20}\int_{40}^{60} E[T_{A,B}|S=s]\,ds \\
&= \frac{1}{20}\int_{40}^{60} \frac{d}{s}\,ds \\
&= \frac{d}{20}(ln(60) - ln(40)) \\
&= \frac{d}{20}(ln(3/2)\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[T_{B,A}] &= \frac{1}{2}E[T_{B,A}|S=40] + \frac{1}{2}E[T_{B,A}|S=60] \\
&= \frac{1}{2}(\frac{d}{40} + \frac{d}{60}) \\
&= \frac{d}{48}
\end{aligned}$$

(b) Assume that a reward is earned at rate 1 per unit time whenever he is driving at a rate of 40 miles per hour, we can again apply the renewal reward approach. If p is the long-run proportion of time he is driving 40 miles per hour,

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{1/80}{\frac{1}{20}ln(3/2) + 1/48}\ .$$
