IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt

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The Renewal Function, The Renewal Equation and Renewal Theorems

Renewal theory involves only a few key ideas. First, renewal theory is about **renewal processes**. A key quantity is the **renewal function**. The renewal function is important because it is a key component of the solution of the **renewal equation**. In applications, we are often interested in the asymptotic form of the renewal function and the solution of the renewal equation. Several theorems describe this asymptotic behavior: (i) the **elementary renewal theorem**, (ii) **Blackwell's renewal theorem**, and (iii) **the key renewal theorem**. These notes explain.

1. a renewal process

Let $\{X_n : n \ge 1\}$ be a sequence of IID nonnegative real-valued random variables with cdf F having finite mean EX. Avoid trivialities by assuming that $P(X > 0) > 0$. Let

$$S_n \equiv X_1 + \cdots + X_n, \quad n \ge 1 \; , \tag{1}$$

with $S_0 \equiv 0$ and

$$N(t) \equiv \max\{n \ge 0 : S_n \le t\}, \quad t \ge 0 \; . \tag{2}$$

Then $N \equiv \{N(t) : t \ge 0\}$ is a *renewal (counting) process*, for which X_n is the length of the interval between the $(n-1)^{\text{st}}$ point and the n^{th} point, for $n \ge 1$. (We do not put a point at 0 unless $X_1 = 0$.)

In addition, for the asymptotic results we will want to assume that the distribution of X_1 is non-lattice. A distribution is a lattice distribution if it concentrates all mass (has support) on the sequence $\{cn : n \ge 0\}$ for some constant $c > 0$. The most common lattice distributions have mass concentrated on the integers (the case $c = 1$). For lattice distributions, there are corresponding limits, along the lattice of support.

2. the renewal function

Then the *renewal function* is

$$m(t) \equiv E[N(t)], \quad t \ge 0 \; .$$

We have the following expression for the renewal function:

$$m(t) \equiv E[N(t)] = \sum_{n=1}^{\infty} P(N(t) \ge n) = \sum_{n=1}^{\infty} P(S_n \le t) = \sum_{n=1}^{\infty} F_n(t) \; ,$$

where $S_n \equiv X_1 + \cdots + X_n$ and $F_n(t) \equiv P(S_n \le t)$. Note that the Laplace transform of $F_n(t)$ is

$$\hat{F}_n(s) \equiv \int_0^\infty e^{-st} F_n(t)\, dt = \frac{\hat{f}_n(s)}{s} = \frac{\hat{f}(s)^n}{s} \; ,$$

where $\hat{f}_n(s)$ is the Laplace transform of f_n, the pdf associated with the cdf F_n, and $\hat{f}(s) = \hat{f}_1(s)$ is the Laplace transform of the pdf f_1. Hence, we can compute the Laplace transform of the renewal function m:

$$\hat{m}(s) \equiv \int_0^\infty e^{-st} m(t)\, dt = \sum_{n=1}^{\infty} \hat{F}_n(s) = \sum_{n=1}^{\infty} (\hat{f}_n(s)/s) = \sum_{n=1}^{\infty} (\hat{f}(s)^n/s) \; .$$

Hence,

$$\hat{m}(s) = \sum_{n=1}^{\infty}(\hat{f}(s)^n/s) = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} = \frac{s\hat{F}(s)}{s - s^2\hat{F}(s)} \ . \tag{3}$$

As a consequence, we see that there is a *one-to-one correspondence* between cdf's F and renewal functions m. In particular, we can express $\hat{f}$ in terms of $\hat{m}$, as well as vice versa: In addition to (3), we have

$$s\hat{F}(s) = \hat{f}(s) = \frac{s\hat{m}(s)}{1 + s\hat{m}(s)} \ . \tag{4}$$

As another consequence, you can compute any renewal function $m(t)$ by numerically inverting its Laplace transform, provided that you can evaluate the Laplace transform of F, which is $\hat{F}(s) = \hat{f}(s)/s$. That was made part of one of the homework assignments.

We can also obtain the expression for the Laplace transform $\hat{m}$ from the renewal equation for $m(t)$,

$$m(t) = F(t) + \int_0^t m(t-y)\,dF(y) = F(t) + \int_0^t m(t-y)f(y)\,dy \ .$$

(More on the renewal equation below.) By taking Laplace transforms in this integral equation, we get

$$\hat{m}(s) = \hat{F}(s) + \hat{m}(s)\hat{f}(s) = \frac{\hat{f}(s)}{s} + \hat{m}(s)\hat{f}(s) = \hat{F}(s) + s\hat{F}(s)\hat{m}(s) \ ,$$

from which we derive the same formula above.

3. convolution and transforms

In this section we digress to provide some general technical background.

(a) definition of convolution

Consider two sequences of real numbers, $a \equiv \{a_k : k \geq 0\}$ and $b \equiv \{b_k : k \geq 0\}$. We can form a new sequence $c \equiv \{c_k : k \geq 0\}$ that is the *convolution* of the first two sequences, denoted by $c = a * b$, by letting

$$c_n = \sum_{k=0}^{k=n} a_k b_{n-k}, \quad n \geq 0 \ .$$

Similarly, consider two functions of a nonnegative real variable, $f \equiv \{f(t) : t \geq 0\}$ and $g \equiv \{g(t) : t \geq 0\}$. We can form a new function $h \equiv \{h(t) : t \geq 0\}$ that is the *convolution* of the first two functions, denoted by $h = f * g$, by letting

$$h(t) = \int_0^t f(y)g(t-y)\,dy, = \int_0^t f(t-y)g(y)\,dy, \quad t \geq 0 \ .$$

(b) associated transforms

For the sequences a, b and c, (under appropriate regularity conditions), we can form associated *generating functions*, by letting

$$\hat{a}(z) \equiv \sum_{k=0}^{\infty} z^k a_k$$

and similarly for $\hat{b}(z)$ and $\hat{c}(z)$. It is easy to see that the convolution property for the sequences a, b and c is equivalent to the transform (generating function) equation

$$\hat{c}(z) = \hat{a}(z)\hat{b}(z)$$

for all (allowed) z.

Similarly, for the functions f, g and h, (under appropriate regularity conditions), we can form associated *Laplace transforms*, by letting

$$\hat{f}(s) \equiv \int_0^\infty e^{-st} f(t)\, dt \ ,$$

for all complex variables s with positive real part, and similarly for $\hat{g}(s)$ and $\hat{h}(s)$. It is easy to see that the convolution property for the functions f, g and h is equivalent to the transform (Laplace transform) equation

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s)$$

for all (allowed) s.

(c) probability applications

The above convolution and transform relations are general properties of sequences and functions, without probability playing a role. There are important applications to probability. However, there can be confusion if you are not careful in your treatment of pmf's, pdf's, cdf's and random variables. Almost all difficulty comes from not being careful about the definitions of these basic probability model elements.

Let X and Y be nonnegative random variables with cdf's F and G, respectively. Assume that X and Y are independent. Suppose that F and G have pdf's f and g. Let $\hat{f}_X(s)$ be the Laplace transform of X, which means

$$\hat{f}_X(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} dF(x) = \int_0^\infty e^{-sx} f(x)\, dx \ .$$

Let H be the cdf and h be the pdf of $X+Y$. Then the pdf h of $X+Y$ is the convolution of the pdf's of f and g

$$h(t) = \int_0^t f(t-x)g(x)\, dx = \int_0^t f(x)g(t-x)\, dx \ .$$

If you take Laplace transforms, you get

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s) \ ,$$

as above.

A similar equation holds for cdf's, but you have to be careful:

$$H(t) = \int_0^t F(t-x)dG(x) \int_0^t F(t-x)g(x)\, dx = \int_0^t F(x)g(t-x)\, dx \ .$$

When we consider Laplace transforms, you have to remember that the Laplace transform of the cdf is not the same as the Laplace transform of the pdf. (This is an important issue in renewal theory.) In particular, applying integration by parts (p. 150 of Feller II), we see that

$$\hat{F}(s) \equiv \int_0^\infty e^{-st} F(t)\, dt = \frac{\hat{f}(s)}{s} \ .$$

Then

$$\hat{H}(s) = \frac{\hat{h}(s)}{s} = \hat{F}(s)\hat{g}(s) = \frac{\hat{f}(s)}{s}\hat{g}(s) \ .$$

4. rough properties of the renewal function

We now return to considering the renewal function $m(t)$. This section is rough and intuitive. We have the following rough properties:

$$m(t) \equiv E[N(t)] \approx \lambda t \quad \text{for} \quad \text{large} \quad t \; , \tag{5}$$

where

$$\lambda \equiv \frac{1}{\mu} \equiv \frac{1}{E[X]} \; . \tag{6}$$

and

$$P(N(t+h) - N(t) = 1) \approx \lambda h \quad \text{for} \quad \text{small} \quad h \quad \text{and large} \quad t \tag{7}$$

and for λ in (6). This last approximation (7) actually requires $P(X = 0) = 0$ to avoid multiple points. It also requires additional regularity conditions. We cannot have X be integer valued. Otherwise,

$$P(N(t+h) - N(t) = 1) = 0 \quad \text{for all} \quad t = n + (1/2) \quad \text{and} \quad h < 1/2 \; , \tag{8}$$

for all n integer. There are regularity conditions. In particular, we assume that the cdf F of an inter-renewal time X is a *non-lattice* distribution. (A distribution is a lattice distribution if it concentrates all mass (has support) on the sequence $\{cn : n \geq 0\}$ for some constant $c > 0$. The most common lattice distributions have mass concentrated on the integers (the case $c = 1$). For lattice distributions, there are corresponding limits, along the lattice of support.

Note that (5) holds as an equality (exactly) in the special case in which the inter-renewal cdf F is the exponential cdf, i.e., in which N is a *Poisson process*. Then property (7) corresponds to Definition 2.1.2 (iii) on p. 60. Since there is a one-to-one correspondence between the cdf F and the renewal function m, because each can be expressed in terms of the other (see equations (3) and (4)), the Poisson process is the only renewal process for which $m(t) = \lambda t$ for $t \geq 0$. We will be interested in having the approximation $m(t) \approx \lambda t$ for large t.

Theorem 3.3.4 (the elementary renewal theorem) on page 107 makes (5) precise. Blackwell's Theorem, Theorem 3.4.1 on p. 110, makes (7) precise. Note that, as a rough property, (7) is roughly equivalent to

$$E[N(t+h) - N(t)] \approx \lambda h \quad \text{for} \quad \text{small} \quad h \quad \text{and large} \quad t \tag{9}$$

and for λ in (6). Blackwell's theorem states that

$$E[N(t+a) - N(t)] \to \lambda a \quad \text{for} \quad a > 0 \quad \text{as} \quad t \to \infty \; . \tag{10}$$

As a further rough property, related to the two above, we have that (under regularity conditions) the renewal function (which is obviously nondecreasing) has a positive derivative,

$$\nu(t) \equiv m'(t) \equiv \frac{dm}{dt}(t) \; , \tag{11}$$

which approaches a limit as $t \to \infty$, i.e.,

$$\nu(t) \equiv m'(t) \to \lambda \equiv \frac{1}{\mu} \equiv \frac{1}{E[X]} \quad \text{as} \quad t \to \infty \; . \tag{12}$$

5. the renewal equation for the alternating renewal process

In applications with renewal processes, we can condition on the time of the *first* renewal (the time of X_1) and use the fact that the process *starts over* or *renews* at that random time, having a future from that point on that looks the same as the future from time 0. When we do that, we obtain a renewal equation, which always has a solution expressed in terms of the renewal function. The renewal equation involves the cdf F of the time until the first renewal, while the solution of the renewal equation replaces the cdf F by the renewal function m.

In the textbook, Ross takes a different approach, looking at the *last* renewal before time t. By invoking his Lemma 3.4.3 on page 113, he goes directly in one step to the solution of the renewal equation. In these notes, we follow the more conventional approach and condition on the time of the first renewal and create the renewal equation. The present approach is easier to understand, but it is good to be aware of the other approach too.

We develop the approach through an example. We consider an **alternating renewal process**, as in §3.4.1. This is determined by a sequence of i.i.d. random vectors $\{U_k, D_k) : k \geq 1\}$, where U_k is the k^{th} "up time" and D_k is the k^{th} "down time." (We allow D_k and U_k to be dependent.) Let $Y(t) \equiv 1$ if the system is on at time t; let $Y(t) \equiv 0$ otherwise. Let

$$P(t) \equiv P(Y(t) = 1), \quad t \geq 0.$$

Assume that the process starts at the beginning of an up period at time 0. The first up time occurs in the interval $[0, U_1]$, the first down time occurs in the interval $[U_1, U_1 + D_1]$, the second up time occurs in the interval $[U_1 + D_1, U_1 + D_1 + U_2]$ and so forth.

Let $H(t) \equiv P(U \leq t)$, $G(t) \equiv P(D \leq t)$ and $F(t) \equiv P(X \leq t)$, where $X = U + D$ is a generic inter-renewal time. The idea is to condition on whether or not $X_1 > t$ or $X_1 \leq t$ and then uncondition. We then wrote down a renewal (integral) equation (**the renewal equation**)

$$\begin{aligned} P(t) &\equiv P(Y(t) = 1) = P(Y(t) = 1, X_1 > t) + P(Y(t) = 1, X_1 \leq t) \\ &= P(U_1 > t) + \int_0^t P(t-s)\, dF(s) \\ &= H^c(t) + \int_0^t P(t-s)\, dF(s) = H^c(t) + \int_0^t P(t-s) f(s)\, ds \ , \end{aligned} \tag{13}$$

where $H^c(t) \equiv 1 - H(t) = P(U_1 > t)$ is the complementary cdf (ccdf). (Ross write $\bar{H}$, where we write H^c.) The first term in (13) describes the probability associated with the first interval. The second term describes what happens after time s and before time t when s is the time that the first cycle ends, and allows s to range over all possible values in $[0, t]$.

We want to apply the renewal equation in (13) to compute $P(t)$ for any fixed t and to determine if a limit exists and, if so, what it is. We can use **Laplace transform inversion** for the first problem and **the key renewal theorem** for the second problem. For both, we first need to **solve the renewal equation**.

Equation (13) has (unique) solution (as can be easily seen by taking transforms; see the final section below)

$$P(t) = H^c(t) + \int_0^t H^c(t-s)\, dm(s) \ . \tag{14}$$

We go from (13) to (14) by replacing the two terms inside the integral: Inside the integral, we replace P by H^c and we replace dF by dm.

To understand the integral, note that in the *common case* the renewal function $m(t)$ has the density $\nu(t)$ in (11) and (12). Hence we can rewrite (14) (roughly) as

$$P(t) \approx H^c(t) + \int_0^t H^c(t-s)\nu(s)\, ds \; . \tag{15}$$

When the integral is put in this form, it is easy to understand. Then the key renewal theorem is also easy to understand. Since $\nu(t) \to \lambda$ as $t \to \infty$, and since the integral of H^c is its mean (EU in the up-down alternating-renewal-process application), which is finite, we get $H^c(t) \to 0$ as $t \to \infty$ and

$$\int_0^t H^c(t-s)\, ds = \int_0^t H^c(s)\, ds \to \int_0^\infty H^c(s)\, ds = E[U] \quad \text{as} \quad t \to \infty \; . \tag{16}$$

Thus we see that the key renewal theorem states what we expect:

$$\int_0^t H^c(t-s)\nu(s)\, ds = \int_0^t H^c(s)\nu(t-s)\, ds \to \lambda \int_0^\infty H^c(s)\, ds \; . \tag{17}$$

As a consequence, we get

$$\begin{aligned} P(t) &= H^c(t) + \int_0^t H^c(t-s)\, dm(s) = H^c(t) + \int_0^t H^c(s)\, dm(t-s) \\ &\to \lambda \int_0^\infty H^c(s)\, ds = \frac{E[U]}{E[U]+E[D]} \quad \text{as} \quad t \to \infty \; . \end{aligned}$$

Many applications can be treated by this argument. The challenging careful mathematical treatment finds precise conditions for all of the above to be fully rigorous. The book, *Applied Probability and Queues* by S. Asmussen, second edition, Springer, 2003, does a nice job on that.

6. the solution of the renewal equation in general

Equations (13) and (14) are special cases. In general we have the *renewal equation*

$$g(t) = h(t) + \int_0^t g(t-s)\, dF(s) = h(t) + \int_0^t g(t-s)f(s)\, ds \; , \tag{18}$$

where F is the cdf of the inter-renewal time X and h is an arbitrary function (satisfying regularity conditions). In applications, the challenge is to set up this equation properly, which means identifying the appropriate function $h(t)$ to use in (18). (Here the function h is an arbitrary function, chosen appropriately for the particular application. It is *not* intended to be the pdf of the cdf H in the previous section. The function H^c there is a special case of the function h here.)

Once we have equation (18), it suffices to solve it for g. (In equation (18) the desired function g appears on both sides; we need to find an explicit expression for g.

Equation (18) has solution

$$g(t) = h(t) + \int_0^t h(t-s)\, dm(s) = h(t) + \int_0^t h(t-s)\nu(s)\, ds \; . \tag{19}$$

It is understood that the final expression in each case depends on extra regularity conditions. As with (13) and (14), we go from (18) to (19) by replacing the two terms inside the integral: We replace g by h and we replace dF by dm.

To justify going from (18) to (19), we can take Laplace transforms. Starting from (18), we get

$$\hat{g}(s) = \hat{h}(s) + \hat{g}(s)\hat{f}(s) = \frac{\hat{h}(s)}{1 - \hat{f}(s)} \tag{20}$$

On the other hand, starting from (19), and applying (3), we have

$$\begin{aligned} \hat{g}(s) &= \hat{h}(s) + \hat{h}(s)\hat{\nu}(s) \\ &= \hat{h}(s) + \hat{h}(s)s\hat{m}(s) \\ &= \hat{h}(s) + \hat{h}(s)\frac{\hat{f}(s)}{1 - \hat{f}(s)} = \frac{\hat{h}(s)}{1 - \hat{f}(s)} \ , \end{aligned} \tag{21}$$

agreeing with (20).

7. the key renewal theorem

The key renewal theorem on p. 112 of Ross provides a limit as $t \to \infty$, given the solution of the renewal equation in (19). We need the inter-renewal cdf F to be non-lattice and we need the function h to be directly Riemann integrable (DRI). A sufficient condition for h to be DRI is for h to be nonnegative, non-increasing and integrable (over $[0, \infty)$).

Given (19),

$$g(t) \to \frac{\int_0^\infty h(x)\, dx}{E[X]} \quad \text{as} \quad t \to \infty \ .$$

We have applied the argument in (18) above.

Note that, by conditioning on the time of the last renewal before time t, $S_{N(t)}$, and invoking Lemma 3.4.3 on page 113, Ross goes directly in one step to the solution of the desired renewal equation. In both cases, the key renewal theorem is applied to get the limit as $t \to \infty$.

For a good account of all this, see Asmussen (2003).