IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt, Thursday, October 17

Renewal Theory: Patterns

1. Patterns: see Example 3.5 A, p. 125

Consider successive independent flips of a biased coin. On each flip, the coin comes up heads (H) with probability p or tails (T) with probability $q = 1 - p$, where $0 < p < 1$. A given segment of finitely many consecutive outcomes is called a *pattern*. The pattern is said to occur at flip n if the pattern is completed at flip n. For example, the pattern $A \equiv HTHTHT$ occurs at flips 8 and 10 in the sequence $TTHTHTHTHTTTTHHHT\ldots$ and at no other times among the first 17 flips.

WARMUP

For parts (a) and (b) below, assume that $p = 1/2$, but for later parts do not make that assumption.

(a) Which pattern occurs more frequently in the long run: $A \equiv HHH$ or $B \equiv HTH$?

(b) For patterns A and B in part (a), let N_A and N_B be the numbers of flips until the patterns A and B, respectively, first occur. Is $E[N_A] = E[N_B]$?

MAIN PROBLEM

Now we revert to general probabilities p and $q = 1 - p$.

(c) What is the probability that pattern $A \equiv HTHTHT$ occurs at flip 72?

(d) Suppose that pattern A from part (c) does indeed occur at flip 72. What is the expected number of flips until pattern A occurs again?

(e) Let $N_A(n)$ be the number of occurrences of pattern A in the first n flips, where A is again the pattern in part (c). Does

$$\frac{N_A(n)}{n} \to x \quad \text{as} \quad n \to \infty \quad \text{w.p.1?}$$

If so, what is the limit x?

(f) What is $E[N_A]$, the expected number of flips until pattern $A \equiv HTHTHT$ *first* occurs?

(g) What is the probability that pattern A occurs before pattern $B \equiv TTH$? That is, what is $P(N_A < N_B)$?

(h) As before, let $N_A(n)$ be the number of occurrences of pattern A in the first n flips. What is the approximate distribution of $N_A(n)$ for large n?

(i) Justify your answer in part (h).

2. Answers

(a) Which pattern occurs more frequently in the long run: $A \equiv HHH$ or $B \equiv HTH$?

Since $p = q = 1/2$, we have $P(A(n)) = P(B(n)) = 1/8$ for all $n \geq 3$. Thus the two patterns occur equally often in the long run.

(b) For patterns A and B in part (a), let N_A and N_B be the numbers of flips until the pattern first occurs. Is $E[N_A] = E[N_B]$?

No, we do *not* have $E[N_A] = E[N_B]$. See below and at the very end.

(c) What is the probability that pattern $A \equiv HTHTHT$ occurs at flip 72?

For any pattern C, let $C(n)$ be the event that pattern C occurs at time (flip) n. Then $P(C(n))$ is the probability of event $C(n)$, i.e., the probability that pattern C occurs at flip n. This question is very easy to answer: With general probabilities p and $q \equiv 1 - p$,

$$P(A(n)) = p^3 q^3, \quad n \geq 6 \ .$$

That is because the specified outcomes must occur at flips n, $n-1$, $n-2$, $n-3$, $n-4$ and $n-5$. We simply multiply the probabilities for independent events. We require $n \geq 6$, because this pattern is of length 6; it cannot occur before flip 6. Observe that the limiting value as $n \to \infty$ already occurs at $n = 6$; we have a common value for all $n \geq 6$. The limit is attained at a finite value of n.

(d) Suppose that pattern A does indeed occur at flip 72. What is the expected number of flips until pattern A occurs again?

We invoke renewal theory. We observe that **the times (flips) when the event occurs are renewals.** (Of course that is why we are discussing this problem while we are reading Chapter 3.) Note that here we have a **delayed renewal process**. The times between successive renewals are IID. We have a *delayed renewal process* because the time until the first pattern occurrence in general has a distribution that is *different* from the distribution of the number of flips between renewals. Let $N_A(n)$ be the number of times pattern A has occurred in the first n flips.

First we observe that

$$E[N_A(n)] = \sum_{k=1}^{n} P(A(n)) \ ,$$

so that, by the reasoning above for part (c),

$$\frac{E[N_A(n)]}{n} \to p^3 q^3 \quad \text{as} \quad n \to \infty \ .$$

Let T_A be the time between successive occurrences of event A. By Proposition 3.5.1 (ii) of Ross,

$$\frac{E[N_A(n)]}{n} \to \frac{1}{E[T_A]} \quad \text{as} \quad n \to \infty \ .$$

Moreover, by the SLLN for delayed renewal processes, we have

$$\frac{N_A(n)}{n} \to \frac{1}{E[T_A]} \ .$$

see Proposition 3.5.1 (i) in Section 3.5 of Ross. As a consequence, we must have

$$E[T_A] = \frac{1}{P(A(n))} \quad \text{for} \quad n \quad \text{suitably large} .$$

Here, in our specific context,

$$E[T_A] = p^{-3}q^{-3}$$

.

(e) Let $N_A(n)$ be the number of occurrences of pattern A in the first n flips, where A is the pattern in part (c). Does

$$\frac{N_A(n)}{n} \to x \quad \text{as} \quad n \to \infty \quad \text{w.p.1?}$$

If so, what is the limit x?

We already used this result to answer the last question.

$$\frac{N_A(n)}{n} \to \frac{1}{E[T_A]} = p^3q^3 \quad \text{as} \quad n \to \infty \quad \text{w.p.1}$$

by the SLLN for delayed renewal processes; Proposition 3.5.1 on page 125.

(f) What is $E[N_A]$, the expected number of flips until pattern $A \equiv HTHTHT$ first occurs?

Like question (b), this is a tricky question. To understand this, it is useful to reconsider the mean of T_A. When we consider $E[T_A]$, the time between occurrences of $A \equiv HTHTHT$, we do not start with nothing, but we start already having had the partial pattern $HTHT$. Let $N_{C \to D}$ be the number of flips to get pattern D after observing pattern C. (Our notation $N_{C \to D}$ corresponds to $N_{D|C}$ in Ross; we use the arrow to emphasize which pattern comes first.)

In particular, our strategy is to relate $E[N_A]$ to $E[T_C]$ for various patterns C. (That is useful because, by above, we know how to compute $E[T_C]$ for any pattern C.) First note that, for $A \equiv HTHTHT$,

$$T_A \equiv N_{A \to A} \equiv N_{HTHTHT \to HTHTHT} \stackrel{\text{d}}{=} N_{HTHT \to HTHTHT},$$

so that

$$E[T_A] \equiv E[T_{HTHTHT}] \equiv E[N_{HTHTHT \to HTHTHT}] = E[N_{HTHT \to HTHTHT}].$$

Similarly

$$E[T_{HTHT}] \equiv E[N_{HTHT \to HTHT}] = E[N_{HT \to HTHT}]$$

and

$$E[T_{HT}] \equiv E[N_{HT \to HT}] = E[N_{\cdot \to HT}] \equiv E[N_{HT}].$$

Putting this all together, we get

$$\begin{aligned} E[N_A] &= E[N_{HT}] + E[N_{HT \to HTHT}] + E[N_{HTHT \to HTHTHT}] \\ &= E[T_{HT}] + E[T_{HTHT}] + E[T_{HTHTHT}] = \frac{1}{pq} + \frac{1}{p^2q^2} + \frac{1}{p^3q^3} . \end{aligned}$$

(g) What is the probability that pattern A occurs before pattern $B \equiv TTH$?

This is another tricky question; see page 127 of Ross for a detailed explanation. We set up two equations in two unknowns and solve them. One unknown is the probability $P_A \equiv P(N_A < N_B)$ that A occurs before B. The other unknown is $E[M_{A,B}]$, where $M_{A,B} \equiv \min\{N_A, N_B\}$ is the first time that one of the patterns A or B first occurs. These variables are expressed in terms of four computable means:

$$E[N_A], \quad E[N_B], \quad E[N_{A\to B}] \quad \text{and} \quad E[N_{B\to A}] \ .$$

We first show how to set up the two equations in two unknowns. Following Ross, we have

$$\begin{aligned} E[N_A] &= E[M_{A,B}] + E[N_A - M_{A,B}] \\ &= E[M_{A,B}] + E[N_A - M_{A,B} | B \text{ before } A](1 - P_A) \\ &= E[M_{A,B}] + E[N_{B\to A}](1 - P_A) \ . \end{aligned}$$

Similarly,

$$E[N_B] = E[M_{A,B}] + E[N_{A\to B}]P_A \ .$$

Solving these two equations, we obtain

$$P_A = \frac{E[N_B] + E[N_{B\to A}] - E[N_A]}{E[N_{B\to A}] + E[N_{A\to B}]}$$

and

$$E[M_{A,B}] = E[N_B] - E[N_{A\to B}]P_A \ .$$

We have seen how to derive $E[N_A]$ and $E[N_B]$. From part (f),

$$E[N_A] = E[T_{HT}] + E[T_{HTHT}] + E[T_{HTHTHT}] = \frac{1}{pq} + \frac{1}{p^2q^2} + \frac{1}{p^3q^3} \ .$$

On the other hand, the occurrence of B gives no head start toward having B occur again; i.e., we have

$$N_B \stackrel{\mathrm{d}}{=} T_B \quad \text{and} \quad E[N_B] = E[T_B] = \frac{1}{pq^2} \ .$$

So now we are ready to consider $E[N_{A\to B}]$ and $E[N_{B\to A}]$. Note that $N_{A\to B} \stackrel{\mathrm{d}}{=} N_{T\to TTH}$ and

$$E[N_{TTH}] = E[N_T] + E[N_{T\to TTH}] \ ,$$

so that

$$E[N_{T\to TTH}] = E[N_{TTH}] - E[N_T] = E[T_{TTH}] - E[T_T] = \frac{1}{pq^2} - \frac{1}{q} \ .$$

Next note that $N_{B\to A} \stackrel{\mathrm{d}}{=} N_{H\to HTHTHT}$ and

$$E[N_{HTHTHT}] = E[N_H] + E[N_{H\to HTHTHT}] \ ,$$

so that

$$\begin{aligned} E[N_{H\to HTHTHT}] &= E[N_{HTHTHT}] - E[N_H] = E[T_{HT}] + E[T_{HTHT}] + E[T_{HTHTHT}] - E[T_H] \\ &= \frac{1}{pq} + \frac{1}{p^2q^2} + \frac{1}{p^3q^3} - \frac{1}{p} \ . \end{aligned}$$

(h) As before, let $N_A(n)$ be the number of occurrences of pattern A in the first n flips. What is the approximate distribution of $N_A(n)$ for large n?

(i) Justify your answer in part (h).

Answer to (h) and (i).

The approximate distribution is normal. We could apply the CLT for renewal processes on page 108. That yields the limit

$$\frac{N_A(n) - (n/m)}{\sqrt{n\sigma^2/m^3}} \Rightarrow N(0,1) \ ,$$

where $m \equiv m_A \equiv E[T_A]$ and $\sigma^2 \equiv \sigma_A^2 \equiv Var(T_A)$, with T_A being a random variable with the distribution of the times between renewals. That yields the approximation

$$N_A(n) \approx N(n/m, n\sigma^2/m^3)$$

for large n. It is often convenient to rewrite the variance term as

$$\frac{\sigma^2}{m^3} = \lambda c^2 \ ,$$

where

$$\lambda \equiv \frac{1}{m} \quad \text{and} \quad c^2 \equiv \frac{\sigma^2}{m^2} \ ;$$

i.e., c^2 is the squared coefficient of variation (SCV, variance divided by the mean) of the time between renewals.

In part (b), we derived the mean, getting

$$E[T_A] = \frac{1}{p^3q^3} \ .$$

Hence $1/m = p^3q^3$. However, we have no convenient formula for the variance $\sigma^2 \equiv Var(T_A)$.

However, to get at the variance, we can draw upon a different CLT, a CLT for sums of *dependent* random variables. We can write

$$N_A(n) = \sum_{k=1}^{n} I_k \ ,$$

where I_k is the indicator function, with $I_k = 1$ if pattern A occurs at flip k, and $I_k = 0$ otherwise. (Note that I_k is a random variable for each k.) For $k \geq 6$,

$$E[I_k] = P(I_k = 1) = p^3q^3 \ .$$

The limit we will get is

$$\frac{N_A(n) - E[N_A(n)]}{\sqrt{Var(N_A(n))}} \Rightarrow N(0,1) \quad \text{as} \quad n \to \infty \ ,$$

where

$$\frac{Var(N_A(n))}{n} \to \gamma > 0$$

for some constant γ. (We want to identify that constant γ.)

We calculate the variance by

$$Var(N_A(n)) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(I_i, I_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}(E[I_iI_j] - E[I_i]E[I_j]) \ .$$

It is not difficult to calculate the expected values $E[I_iI_j]$. If $|i-j| \geq 6$, then I_i and I_j are independent and $E[I_iI_j] = E[I_i]E[I_j] = p^6q^6$. The other cases are easy to compute too, so we can obtain the desired formula. In particular, $E[I_iI_{i+1}] = 0$, $E[I_iI_{i+2}] = p^4q^4$, $E[I_iI_{i+3}] = 0$, $E[I_iI_{i+4}] = p^5q^5$, $E[I_iI_{i+5}] = 0$ and $E[I_iI_{i+j}] = E[I_i]^2$ for $j \geq 6$.

We end up scaling by $\sqrt{n\gamma}$, where

$$\gamma \equiv \lim_{n\to\infty} \frac{Var(N_A(n))}{n} = (r_0 - r_0^2) + 2\sum_{j=1}^{5}(r_j - r_0^2)$$

where $r_1 = r_3 = r_5 = 0$ and

$$\begin{aligned} r_0 &\equiv E[I_iI_{i+0}] = E[I_i] = p^3q^3, \\ r_2 &\equiv E[I_iI_{i+2}] = p^4q^4, \\ r_4 &\equiv E[I_iI_{i+4}] = p^5q^5, \end{aligned}$$

(For each i, $6 \leq i \leq n-5$, there are $n-10$ j such that $|I_i - I_j| = 0$, but $2(n-10)$ j such that $|I_i - I_j| = 2$ and $2(n-10)$ j such that $|I_i - I_j| = 4$.)

Hence, we have determined the limit of $Var(N_A(n))/n$. Thus, indirectly, we can identify $Var(T_A)$, because we have two CLT's, which must agree.

See Section 4.4 of my book for further discussion of the CLT applied here. See Billingsley (1968, 1969) for more on such CLT's.

ADDITIONAL DETAILS on (b) For patterns A and B in part (a), let N_A and N_B be the numbers of flips until the pattern first occurs. Is $E[N_A] = E[N_B]$?

No, we do *not* have $E[N_A] = E[N_B]$. See below. To understand this, it is useful to reconsider the mean of T_A. When we consider $E[T_A]$, the time between occurrences of $A \equiv HHH$, we do not start with nothing, but we start already having had three $H's$ in a row. Let $N_{C\to D}$ be the number of flips to get pattern D after observing pattern C. (Our notation $N_{C\to D}$ corresponds to $N_{D|C}$ in Ross; we use the arrow to emphasize which pattern comes first.) The number of flips after $A \equiv HHH$ until $A \equiv HHH$ next occurs equals the number of flips to reach A starting from the pattern HH; i.e.,

$$T_A \equiv N_{HHH\to HHH} \stackrel{\mathrm{d}}{=} N_{HH\to HHH} \ .$$

To get to HHH, we must first get to HH. Hence we have

$$E[N_A] = E[N_{HHH}] = E[N_{HH}] + E[N_{HH\to HHH}] = E[N_{HH}] + E[T_{HHH}] \ .$$

Similarly,

$$T_{HH} \equiv N_{HH\to HH} \stackrel{\mathrm{d}}{=} N_{H\to HH} \ ,$$

so that

$$E[N_{HH}] = E[N_H] + E[N_{H\to HH}] = E[N_H] + E[T_{HH}] \ .$$

But, initially, we have

$$E[N_H] = E[T_H] \ .$$

By that reasoning,

$$E[N_{HHH}] = E[T_H] + E[T_{HH}] + E[T_{HHH}] = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \ .$$

But the story is different for the pattern $B \equiv HTH$: First, we have

$$T_B \equiv N_{HTH\to HTH} \stackrel{\mathrm{d}}{=} N_{H\to HTH} \ ,$$

so that

$$E[N_B] = E[N_{HTH}] = E[N_H] + E[N_{H\to HTH}] = E[T_H] + E[T_{HTH}] = \frac{1}{p} + \frac{1}{p^2 q} \ .$$

Hence, we have

$$E[N_A] = E[N_{HHH}] = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \neq \frac{1}{p} + \frac{1}{p^2 q} = E[N_B] \ .$$

Summary of the notation defined above:

$$\text{pattern } A, \quad A(n), \quad P(A(n)), \quad N_A, \quad T_A, \quad N_A(n),$$
$$N_{A\to B}, \quad M_{A,B} \equiv \min\{N_A, N_B\}, \quad P_A, \quad I_j$$