IEOR 6711: Stochastic Models I

Professor Whitt, Tuesday, October 22, 2013

Renewal Theory: Proof of Blackwell's theorem

1 Delayed Renewal Processes: Exploiting Laplace Transforms

The proof of Blackwell's theorem in Chapter 9 exploits delayed renewal processes, especially the equilibrium renewal process.

We first discussed **delayed renewal processes**, as in Section 3.5 of Ross. The intervals are again independent. The first interval then has distribution G, while all subsequent intervals have cdf F, where we assume F has mean $\mu \equiv E[X] = \int_0^\infty x\, dF(x) < \infty$. We keep the assumption that all intervals between points are mutually independent, but now we give the first interval a different distribution.

Some of the properties of delayed renewal processes can be established by Laplace transforms. Again, we emphasize that Ross uses unconventional notation for Laplace transforms, so that our notation differs. See pages 19-20. For a general function $g(t)$, I would write its Laplace transform as

$$\hat{g}(s) \equiv \int_0^\infty e^{-st} g(t)\, dt \ ,$$

but Ross writes instead

$$\hat{g}(s) \equiv \int_0^\infty e^{-st}\, dg(t) \ .$$

That is really a Laplace-Stieltjes transform of g. In fact, arbitrary functions do not have Laplace-Stieltjes transforms. For a cdf F with a density (pdf) f, we would write

$$\hat{F}(s) \equiv \int_0^\infty e^{-st} F(t)\, dt \ ,$$

and

$$\hat{f}(s) \equiv \int_0^\infty e^{-st}\, dF(t) = \int_0^\infty e^{-st} f(t)\, dt \ ,$$

which makes

$$\hat{F}(s) = \frac{\hat{f}(s)}{s} \ .$$

I too use a Laplace-Stieltjes transform here, but I have different notation for ordinary Laplace transform and for Laplace-Stieltjes transform. Laplace-Stieltjes transforms are well defined if the function is monotone, as with a cdf or the renewal function, or even for functions of bounded variation, which are differences of two monotone functions.

We can use transforms to establish that $m_e(t) = t/\mu$, which implies equation (1). In particular, the Laplace transform is

$$\hat{m}_e(s) \equiv \int_0^\infty e^{-st} m_e(t)\, dt = \frac{1}{s^2 \mu}.$$

For a delayed renewal process, we have

$$\mathbf{m_D(t) \equiv E[N(t)] = G(t) + \int_0^t m(t-u)\, dG(u)} \ ,$$

where F is the cdf between renewals, except for the first interval, which has cdf G, and m is the renewal function of the ordinary renewal process associated with cdf F. Using Laplace transforms, we get

$$\hat{m}_D(s) \equiv \int_0^\infty e^{-st} m_D(t)\, dt = \hat{G}(s) + \hat{m}(s)\hat{g}(s).$$

Since

$$\hat{m}(s) = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} \quad \text{and} \quad \hat{G}(s) = \frac{\hat{g}(s)}{s},$$

we get

$$\hat{m}_D(s) = \frac{\hat{g}(s)}{s(1-\hat{f}(s))}.$$

Since,

$$\hat{f}_e(s) = \frac{1-\hat{f}(s)}{s\mu},$$

we get

$$\hat{m}_e(s) = \frac{1}{s^2\mu},$$

as claimed.

As an aside, we point out that it is also possible to start from the representation

$$\mathbf{m_D(t)} = \sum_{\mathbf{n=1}}^{\infty} \mathbf{G} * \mathbf{F_{n-1}(t)},$$

as a variant of Proposition 3.2.1, where the first F is replaced by G. Then with transforms we get

$$\hat{m}_D(s) = \sum_{n=1}^{\infty} (\hat{g}(s)\hat{f}(s)^{n-1})/s = (\hat{g}(s)/\hat{f}(s))\hat{m}(s),$$

which also gives the formula above.

2 The Equilibrium Renewal Process

We next focus on **equilibrium renewal processes**. A delayed renewal process becomes an equilibrium renewal process when the first exceptional interval is distributed as F_e, the **stationary-excess cdf** or equilibrium-lifetime cdf or equilibrium-residual-lifetime cdf; i.e.,

$$F_e(t) \equiv \frac{1}{E[X]} \int_0^t P(X > s)\, ds \equiv \frac{1}{\mu} \int_0^t F^c(s)\, ds$$

where $\mu \equiv E[X]$ and $F^c(t) \equiv 1 - F(t) \equiv P(X > t)$, $t \geq 0$.

The equilibrium renewal processes are an important reference point for Blackwell's theorem, because for an equilibrium renewal process, the limit we want to prove holds as an equality. Let $m_D(t)$ be the delayed renewal function, i.e., $m_D(t) \equiv E[N_D(t)]$.

Let $m_e(t) \equiv E[N_e(t)]$, where $N_e \equiv \{N_e(t) : t \geq 0\}$ is the equilibrium renewal counting process. If the first interval has distribution F_e instead of G, then we have the equilibrium renewal process, with the property that

$$m_e(t+b) - m_e(t) = \frac{b}{\mu} \quad \text{for all} \quad t\ , \tag{1}$$

where $\mu = E[X]$, where X is a time between renewals, having cdf F. See Theorem 3.5.2 on page 131. We proved this in class. Our proofs of parts (i) and (ii) were different, because we use different notation for Laplace transforms and because we started with the renewal equation for $P(Y(t) > x)$ to then construct the expression for the residual lifetime in a delayed renewal process, $P(Y_D(t) > x)$. In particular, we first considered the ordinary renewal process and wrote down

$$P(Y(t) > x) = P(X_1 > t + x) + \int_0^t P(Y(t - s) > x)\, dF(s)$$

and then wrote down its solution in the usual way as

$$P(Y(t) > x) = F^c(t + x) + \int_0^t F^c(t + x - s)\, dm(s).$$

We then observed that if the first interval has cdf G, then we simply must change the first term on the right, getting

$$\mathbf{P(Y_D(t) > x) = G^c(t + x) + \int_0^t F^c(t + x - s)\, dm(s).}$$

We then do calculations (rearranging an integral) to see that $P(Y_e(t) > x) = F_e^c(x)$ when we make two substitutions above: $G = F_e$ and $m(t) = m_e(t) \equiv t/E[X]$, as shown in Ross.

A further important reference case is a Poisson process. Then $F = F_e$ and $m_e(t) = m(t) = \lambda t$ for all t. (It is important to note that $F = F_e$ if and only if F is exponential.) We then have $m_D(t) = \lambda t$ provided the initial distribution G is the exponential distribution F with mean $1/\lambda$. Otherwise, we even have some trouble proving the limit (3) for a delayed Poisson process (where $G \neq F$). The argument here covers the delayed Poisson process as well as the more general case in (1).

3 Stationary Point Processes

There is an important connection to the **theory of stationary point processes**; see Section 3.8 in Ross. There are many books on stationary point processes; e.g., see K. Sigman (1995), *Stationary Marked Point Processes: An Intuitive Approach*, Chapman and Hall, and F. Baccelli and P. Brémaud (1994) *Elements of Queueing Theory*, Springer. The counting process $N_e(t) : t \geq 0\}$ associated with an equilibrium renewal process is a stationary point process (from the continuous-time counting process perspective); it has stationary increments (but not necessarily independent increments). It turns out that the interval until the first point in a stationary point process always has cdf F_e.

On the other hand, the ordinary renewal process $\{N(t) : t \geq 0\}$ is the associated Palm version, because the associated discrete-time sequence $\{X_n : n \geq 1\}$ is a stationary process. (For a renewal process, this sequence $\{X_n : n \geq 1\}$ is actually IID.) The equilibrium renewal process differs from the ordinary renewal process only by the distribution of the first interval. For more general stationary processes, the Palm transformation is more complicated. As an application of the Palm transformation, I mentioned the fundamental queueing identity $L = \lambda W$; e.g., see W. Whitt, A Review of $L = \lambda W$ and Extensions, *Queueing Systems*, vol. 9, No. 3, 1991, pp. 235-268:

http://www.columbia.edu/ ww2040/ReviewLlamW91.pdf

4 Blackwell's Theorem

Consider a renewal process $\{N(t) : t \geq 0\}$ with times between renewals X_k having cdf F. Let $m(t) \equiv E[N(t)]$ be the renewal function.

This lecture is devoted to a discussion of **Blackwell's theorem** and its proof. We focus on the proof given in Chapter 9 of Ross, in particular, as given on page 422. That proof has an extra condition, as stated in (9.3.4) on p. 423, but it illustrates a clever argument, based on **coupling**. A great reference on coupling is the book by Lindvall (1992); see p. 456 of Ross.

A great reference on renewal theory and a source of further references is Asmussen (2003). The basics are in Chapter V, but important related theory is in Chapters VI and VII, with more related material throughout the book. We discussed Section V.4 giving four equivalent statements of the renewal theorem, one of which is the Blackwell theorem and another is the key renewal theorem. The other two are convergence in distribution as $t \to \infty$ of the residual lifetime (forward excess) $Y(t)$ and convergence in distribution of the age (backward excess) $A(t)$ to limits distributed according to the equilibrium excess F_e.

Of course we need to assume that F is **non-lattice**, but there is a corresponding result for the lattice case. We also assume that the mean $E[X] < \infty$, but there is also a statement for the case $E[X] = \infty$.

We want to prove:

Theorem 4.1 (Blackwell's theorem) *If F is a non-lattice cdf with finite mean, then*

$$m(t+b) - m(t) \to \frac{b}{\mu} \quad as \quad t \to \infty \ , \tag{2}$$

where $\mu = E[X]$, where X is a time between renewals, having cdf F. If, instead, we have a delayed renewal process with initial cdf G, and using F thereafter, then

$$m_D(t+b) - m_D(t) \to \frac{b}{\mu} \quad as \quad t \to \infty \ , \tag{3}$$

This is Proposition 3.5.1 (iii) on page 125. It extends Theorem 3.4.1 (i) on page 110.

5 Stochastic Order Relations

The extra condition imposed in (9.34) on p. 423 of Ross is that the hazard rate (or failure rate) function is well defined and bounded away from 0 and ∞, i.e., there exist constants c_1 and c_2 such that

$$0 < c_1 \leq h(x) \equiv \frac{f(x)}{F^c(x)} \leq c_2 < \infty \quad \text{for all} \quad x. \tag{4}$$

In order to have (4), we need for F to have a pdf f.

The extra condition (4) can be expressed as a stochastic order relation. From that perspective, it says that

$$Y_1 \geq_h X \geq_h Y_2,$$

where $\geq_h$ is "greater than or equal to" in **hazard-rate stochastic ordering** defined in (9.3.1) on p. 420 and Y_i is an exponential random variable with mean $1/c_i$. Useful background is the material on stochastic order in Chapter 9. Hazard rate ordering $X_1 \leq_h X_2$ means that $h_{X_1}(x) \geq h_{X_1}(x)$ for all $x \geq 0$. Hazard rate ordering is equivalent to $X_{1,t} \leq_{st} X_{2,t}$, where $\le_{st}$ is ordinary stochastic order, as in §9.1, and X_t is a random variable with the cdf $P(X_t \le s) \equiv P(X \le s+t | X > t)$ for $s \ge 0$; see p. 421. That is easy to see because

$$P(X \le s+t | X > t) = \exp\{-\int_t^{s+t} h(u)\, du\}.$$

Since the hazard rate function h arises as the derivative of the logarithm of $F^c(t)$. As a consequence, we have the useful representation

$$F^c(t) = \exp\{-\int_0^t h(x)\, dx\}, \quad t \ge 0.$$

6 Proof of Blackwell's theorem

The proof of Blackwell's theorem is given on pages 423-424 of Ross. The following discussion elaborates on the overall argument and points not discussed enough in Ross. We do not repeat many of his (important) detailed steps. This should be read together with Ross.

The following lemma shows that if the cdf F satisfies (4) than so does the stationary-excess cdf F_e. The lemma plays a key role in the proof.

Lemma 6.1 *Consider a nonnegative random variable X with cdf F having probability density function (pdf) f and hazard rate function $h \equiv f/F^c$. Let F_e be the associated stationary-excess cdf, having pdf $f_e \equiv F^c/E[X]$ and hazard rate function $h_e \equiv f_e/F_e^c$. If the inequalities in* (4) *hold, then*

$$c_1 \le h_e(x) \le c_2 \quad \textit{for all} \quad x. \tag{5}$$

Proof of the lemma. The argument is essentially the same for each of the inequalities. Hence, consider the second inequality. Note that the second inequality in (4) is equivalent to

$$f(x) \le c_2 F^c(x) \quad \text{for all} \quad x. \tag{6}$$

Given (6), we can integrate both sides over the interval $[t, \infty)$ to get

$$E[X]f_e(t) = \int_t^\infty f(x)\, dx \le c_2 \int_t^\infty F^c(x)\, dx = c_2 E[X] F_e^c(t), \tag{7}$$

which is equivalent to the second inequality in (5). The same reasoning applies to the first inequalities. ■

The proof of Theorem 4.1 in Chapter 9 uses coupling. We construct the delayed renewal process and the ordinary renewal process on the same space by thinning a common Poisson process with the upper bound rate λ_2. (See §9.3 for this step.) We make the delayed renewal process be the equilibrium renewal process, so it satisfies the limit to be established as an equality. We discussed how this construction is a common way to simulate a nonhomogeneous Poisson process. In that setting, we generate the nonhomogeneous Poisson process by simulating the ordinary Poisson process and then thinning it to construct the nonhomogeneous Poisson process.

With this construction of the two renewal processes, after some random time, these two delayed renewal processes have a common point. We then couple the two processes by redefining the renewal process after this common point to coincide with the equilibrium renewal process. This leaves the distribution of each of the two processes unchanged (as a process by itself), but of course the construction introduces a special form of dependence between the two processes, which will not concern us. Then further calculations yield the limit in (3).

The remaining details are given on pages 420-424. In particular, care is needed to properly establish (9.3.5) on p. 424.

7 Summary

In summary we have the following key concepts:

- Blackwell's renewal theorem, Theorems 3.4.1 and 3.5.1 (iii),
- lattice and non-lattice probability distributions, p. 109,
- stochastic order, failure-rate stochastic order (hazard-rate stochastic order), Chapter 9,
- delayed renewal process, equilibrium renewal process, Section 3.5,
- stationary point processes, Section 3.8,
- age, residual lifetime, stationary-excess cdf or equilibrium residual-lifetime cdf F_e,
- coupling.