IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt, November 7

Reversibility

1. Four Problems

(1) The Knight Errant (The Random Knight)

A knight is placed alone on one of the corner squares of a chessboard (having $8 \times 8 = 64$ squares). What is the expected total number of moves required for the knight to first return to its initial position, if we assume that the knight moves randomly, taking each of its legal moves in each step with equal probability?

(2) A Big Closed Maze for Markov Mouse

Consider Markov mouse moving from room to room in a closed maze. Suppose that the closed maze for Markov mouse is 10×20, so that it has $10 \times 20 = 400$ rooms, arranged in a rectangular fashion, with doors connecting neighboring rooms. Now there are 10 rows of rooms, with 20 rooms in each row. There are doors connecting neighboring rooms on each row. And there are doors connecting neighboring rooms on each column.

Suppose that the mouse moves randomly according to a Markov chain, moving to one of the available neighboring rooms on each move, with each of the available alternatives chosen with equal probability.

Suppose that the mouse starts in Room 1 (in the upper lefthand corner). What is the expected total number of moves required for the mouse to first return to this initial room?

(3) Random Walk on a Finite Graph

A finite graph consists of a finite set of **vertices** V (or nodes) plus a set of **arcs**. Some pairs of vertices are connected by arcs and some pairs of vertices are not. If there are n vertices, then the vertices can be labelled by integers i with $1 \leq i \leq n$. Then the set A of arcs can be identified by a subset of all subsets of two vertices, i.e., of subsets $\{i, j\}$, where i and j are vertices with $i \neq j$. There is an arc connecting vertices i and j if and only if the subset $\{i, j\}$ belongs to the set A. The **degree** of a vertex is the number of different arcs connected to that vertex.

One vertex is said to be a neighbor of another vertex if there is an arc connecting the two vertices. The graph is said to be **connected** if any two vertices are connected by a collection of arcs. That is, vertices i and j are connected by a collection of arcs if there is an integer k with $1 \leq k \leq n-1$ and k arcs $\{i, i_1\}$, $\{i_1, i_2\}, \ldots, \{i_{k-2}, i_{k-1}\}$, $\{i_{k-1}, j\}$.

Consider a random walk on a connected graph (a Markov chain) that moves from vertex to neighboring vertex, with each neighbor being equally likely at each move.

Suppose that the random walk starts at vertex 1. What is the expected total number of steps taken by the random walk until it first returns to this initial vertex?

(4) Random Walk on a Finite Weighted Graph

See pages 205-6 in Ross.

Consider the graph in Example 3, but let there be a weight assigned to each arc. Specifically, let there be a positive weight $w_{i,j}$ $(0 < w_{i,j} < \infty)$ assigned to the arc $\{i, j\}$ for all arcs $\{i, j\}$ in A. Again consider a random walk on a connected graph (a Markov chain) that moves from vertex to neighboring vertex, but now let the probabilities of moving to each neighboring vertex on each step be proportional to the weight on the arc connecting to that vertex.

Again suppose that the random walk starts at vertex 1. What is the expected total number of steps taken by the random walk until it first returns to this initial vertex?

2. Key facts

(1) These examples are all irreducible finite-state Markov chains.

(2) As a consequence, there is a unique stationary probability vector π, satisfying $\pi = \pi P$.

(3) Examples 1 and 2 can be regarded as a special case of Example 3, which in turn is a special case of Example 4.

(4) In Example 4 (and thus all the examples), the stationary probability vector has a very simple form:

$$\pi_i = \frac{\sum_j w_{i,j}}{\sum_i \sum_j w_{i,j}} \quad .$$

See Proposition 4.7.1.

(5) The simple form of the answer can be verified by just checking.

(6) The simple form of the answer can be explained by time reversibility.

(7) All these examples are time reversible, so that it suffices to solve the detailed-balance equations instead of $\pi = \pi P$, namely,

$$\pi_i P_{i,j} = \pi_j P_{j,i} \quad \textbf{for all} \quad i \quad \textbf{and} \quad j \ .$$

(8) It is easy to check that the claimed solution in (4) satisfies the detailed-balance equations in (7).

(9) It is easy to check that the the detailed-balance equations in (7) imply $\pi = \pi P$, but $\pi = \pi P$ is more general. For the implication, just sum both sides over j.

(10) In an irreducible finite-state Markov chain, the expected number of steps to first return to state i, starting in state i, is $1/\pi_i$. That property is covered by renewal theory. For us, the interarrival times are integer-valued, but that is OK.

3. Detailed Solutions

Example 1

By (10) above, the answer is $1/\pi_1$, where state 1 represents the square in the upper lefthand corner. By (4) above, π_1 has the special form given there, where all the weights are 1. Thus π_i

is proportional to the number of moves the knight has out of square i. Hence π_1 is the number of moves from square 1, divided by the sum of the numbers of moves from all of the squares.

Here is the calculation: There are 4 corner squares, with 2 possible moves out of each; there are $4 \times 2 = 8$ squares with 3 possible moves; there are 20 squares with 4 possible moves; there are $4 \times 4 = 16$ squares with 6 possible moves; and there are $4 \times 4 = 16$ squares with 8 possible moves. Hence

$$\pi_1 = \frac{2}{(4 \times 2) + (8 \times 3) + (20 \times 4) + (16 \times 6) + (16 \times 8)} = \frac{2}{336} = \frac{1}{168} \ .$$

Thus, the expected number of moves until the knight returns to its original corner square is 168.

Example 2

By (10) above, the answer is $1/\pi_1$, where state 1 represents the room in the upper lefthand corner. By (4) above, π_1 has the special form given there, where all the weights are 1. Thus π_i is proportional to the number of doors out of room i. Hence π_1 is the number of doors our of room 1, divided by the sum of the numbers of doors out of all of the rooms.

Here is the calculation: There are 4 corner rooms, each with 2 doors; there are $2 \times (18+8) = 52$ side rooms with 3 doors; and the remaining $200 - 56 = 144$ rooms have 4 doors. Hence, Hence

$$\pi_1 = \frac{2}{(4 \times 2) + (52 \times 3) + (144 \times 4)} = \frac{2}{740} = \frac{1}{370} \ .$$

Thus, the expected number of moves until the mouse returns to its original corner room is 370.

4. Working with the Reverse Chain

We often can work with the reverse chain, defined on page 203, without having to have reversibility. We discussed Theorem 4.7.3 and Example 4.7 (e), which involves the age and remaining life processes, but associated with a renewal process having integer-valued times between renewals.