IEOR 6711: Stochastic Models I

Professor Whitt, Tuesday, November 12, 2013

regenerative processes and semi-Markov processes

1. Nested Models

We here discuss regenerative processes (§3.7) and semi-Markov processes (§4.8). An SMP is a special regenerative process. The successive transitions into any fixed state of the SMP consitutes the embedded renewal times for the SMP. Then a DTMC and a CTMC are both special cases of an SMP. At transition times, an SMP evolves as a DTMC determined by a transition matrix P. But the times between transitions are random. If the transition times are all exactly 1, then the SMP is a DTMC. If the transtion times are exponential, depending only on the initial state, then the SMP is a CTMC; see the CTMC notes.

2. Regenerative Processes, §3.7 in Ross

Key idea: We start with a stochastic process $\{X(t) : t \geq 0\}$. There is a sequence of random times $\{T_n : n \geq 1\}$ such that they form the successive event times of a renewal process, with $T_0 = 0$ and $T_n > T_{n-1}$. and that the stochastic process $X(T_n + s) : s \geq 0\}$ has a distribution (as a stochastic process) that is independent of n. The process "renews" at the times T_n. We say that there is "an embedded renewal process." The random variables $T_n - T_{n-1}$ form the intervals between renewals and are IID. Ross assumes that the stochastic process $\{X(t) : t \geq 0\}$ is integer-valued, but that is really not necessary. A nice and thorough account is given in Asmussen (2003).

Example: Consider the queue length process in the $GI/GI/s/\infty$ model, i.e., let $X(t)$ be the number of customers in the system at time t. The interarrival times and service times come from independent sequences of IID random variables. The embedded renewal process can be the successive epochs at which an arrival comes to find an empty system, and himself enters service right away. To have a proper steady-state limiting distribution, we will need to assume that the rate in (the reciprocal of the mean interarrival time) is less than the maximum rate out ($s\times$ the reciprocal of the mean service time). That is needed to have the mean time between renewals be finite. We will need the busy cycle (the time between renewals) to be nonlattice. We need some regularity condition. In this example, we have the sample paths of the stochastic process X in the space D (being right-continuous with left limits); see below.

In these notes we discuss the theorems. In particular, we do not write down the conditions here. See the book for precise statements!

Theorem 3.7.1. This theorem establishes a limit for the probability $P(X(t) = j)$ as $t \to \infty$. The proof applies the key renewal theorem. We have the renewal equation

$$g(t) = h(t) + \int_0^t g(t-s)\, f(s)\, ds,$$

where

$$g(t) = P(X(t) = j) \quad \text{and} \quad h(t) = P(X(t) = j, T_1 > t).$$

Main thing to observe: **There is a technical problem in getting this $h(t)$ to be directly Riemann integrable (d.R.i.).** We use the condition that $h(t)$ is less than or equal another function that is d.R.i. Here the other function is $P(T_1 > t)$, which itself is d.R.i. because it is non-increasing and Lebesgue integrable. However, there is an additional condition that the original function $h(t)$ must be bounded and continuous almost everywhere with respect to Lebesgue measure; see p. 154 of Asmussen (2003), handed out. We need to assume more than the distribution F of the time between renewals T is nonlattice. It suffices to assume that it has a density or that the stochastic process X has sample paths in the function space D; see Miller (1972). Ross makes the extra assumption about the density, but he does not explain. There is a tricky technical point here lurking beneath the surface.

Given that h is d.R.i., the key renewal theorem gives the limit as

$$\lim_{t\to\infty} P(X(t) = j) = \frac{\int_0^\infty h(t)\,dt}{E[T]} = \frac{E[\int_0^T 1_{\{X(s)=j\}}\,ds]}{E[T]}.$$

We get the limit above because

$$\begin{aligned}\int_0^\infty h(t)\,dt &= \int_0^\infty P(X(t) = j, T_1 > t)\,dt \\ &= \int_0^\infty E[1_{\{X(t)=j,T_1>t\}}]\,dt \\ &= E[\int_0^\infty 1_{\{X(t)=j,T_1>t\}}\,dt] \\ &= E[\int_0^{T_1} 1_{\{X(t)=j\}}\,dt]\end{aligned}$$

We need to exchange the order of expectation and the integral. That is justified by Tonelli's theorem.

Proposition 3.7.2. We get the associated limit for the proportion of time spent in j by applying the renewal reward theorem, Theorem 3.6.1. We give the system a reward at rate 1 whenever the process X is in the state j. That is the easy part of renewal theory.

3. semi-Markov Processes, §4.8 in Ross

Key idea: In a DTMC time is discrete: We have times 0, 1, 2, etc. Suppose that we now make the transition times random. We have the original DTMC that describes the successive transitions. The DTMC is governed by a transition matrix P. Now we let the transition time be random, of course depending on the origin state i but perhaps also depending on the destination state j. Given that the transition is from i to j, the transition time is governed by the cdf $F_{i,j}(t)$. The DTMC is the special case in which all transition times are identically 1. A CTMC is the special case in which all the transition times are exponential, depending only on the origin state i; i.e., we have, for all j, that $F_{i,j}(t) = H_i(t)$ for the exponential cdf $H_i(t) = 1 - e^{-t/m_i}$. In general, for the time spent in state i upon each visit, we have the cdf

$$H_i(t) \equiv \sum_j P_{i,j} F_{i,j}(t).$$

Let m_i be the expected time spent in state i during each visit to state i.

Examples:

Example 1. A first example is Example 3.1 in the CTMC notes, involving Pooh Bear and the three honey trees. If the transition times are exponentially distributed as stated there, then we have a CTMC. If, instead, the transition times are mutually independent, but are non-exponentially distributed, then we have a SMP. Here we assume that the transition times are negligible; i.e., Pooh is at one tree for a random length of time and then he goes instantaneously to another tree, where he stays for a random length of time.

Example 2. A second example is the taxi example, Exercise 4.50 in Ross. Unlike Example 1 above, the taxi is only at the state instantaneously. We get a SMP if we say that the taxi is in state 1 at time t if the last location visited was state 1. This is part of homework 11.

There are two issues: (i) What is the answer (**form of the limit**) in a question about the long-run behavior? And (ii) How do we justify the **existence of the limit**?

The existence of the limit is contained in Proposition 4.8.1. The form of the limit is in Corollary 4.8.2, Theorem 4.8.3 and Theorem 4.8.4.

Let T_j be the time spent in state j upon each visit; let $T_{j,j}$ be time between successive transitions into j. Let the corresponding mean values be

$$m_j \equiv E[T_j] \quad \text{and} \quad m_{j,j} \equiv E[T_{j,j}].$$

Proposition 4.8.1. The limit of $P(X(t) = j|X(0) = i)$ exists and has the form

$$\lim_{t\to\infty} P(X(t) = j|X(0) = i) = \frac{m_j}{m_{j,j}}.$$

The proof is by applying the limit theorem for alternating renewal processes, Theorem 3.4.4, which in turn follows from the key renewal theorem. This proposition establishes the hard part, the existence of the limit. All the hard work has been done in Chapter 3.

Corollary 4.8.2. This corollary identifies the limit in Proposition 4.8.1 with the long-run proportion of time spent in state j. That follows from Proposition 3.7.2, because an SMP is a regenerative process. In particular, we just apply the SLLN for renewal reward processes.

Theorem 4.8.3. If, in addition to the conditions of Proposition 4.8.1, the underlying discrete-time chain is irreducible and positive recurrent, then the limit has the appealing form

$$\lim_{t\to\infty} P(X(t) = j|X(0) = i) = \frac{\pi_j m_j}{\sum_i \pi_i m_i},$$

where π is the limiting steady-state distribution of the DTMC at transition epochs, with π found by solving $\pi = \pi P$.

Theorem 4.8.3 is proved by applying a LLN argument, like the LLN for renewal reward processes. We also exploit Proposition 4.8.1 to tell us that the limit exists. We are now only determining an alternative expression for that limit.

We now describe the limiting behavior in more detail. This too can be cast as an application of the alternating renewal process. So this is actually not difficult.

Theorem 4.8.4. Theorem 4.8.4 describe the limiting probability of the next state and the remaining time until the transition takes place. Let $S(t)$ denote the next state visited and let $Y(t)$ the time from t until the next transition. If the SMP is irreducible and non-lattice, then

$$\lim_{t\to\infty} P(X(t) = j, Y(t) > x), S(t) = k|X(0) = i) = \frac{P_{j,k}\int_x^\infty (1 - F_{j,k}(y)\, dy}{m_{j,j}}$$

Corollary 4.8.5. In addition,

$$\lim_{t\to\infty} P(X(t) = j, Y(t) > x)|X(0) = i) = \left(\frac{m_j}{m_{j,j}}\right)(1 - H_{j,e}(x)),$$

where $H_{j,e}$ is the stationary-excess cdf associated with the cdf H_j, i.e.,

$$H_{j,e}(x) \equiv \frac{1}{m_j} \int_0^x (1 - H_j(y))\, dy, \quad x \geq 0.$$

Exercises 4.48-4.50 are related.

References

Asmussen, S. 2003. *Applied Probability and Queues*, second edition, Springer.

Miller, D. R. 1972, Existence of Limits in Regenerative Processes. *Annals of Mathematical Statistics* Vol. 43, No. 4, pp. 1275–1282.