ON ARRIVALS THAT SEE TIME AVERAGES

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We investigate when *Arrivals See Time Averages* (ASTA) in a stochastic model; i.e., when the stationary distribution of an embedded sequence, obtained by observing a continuous-time stochastic process just prior to the points (arrivals) of an associated point process, coincides with the stationary distribution of the observed process. We also characterize the relation between the two distributions when ASTA does not hold. We introduce a *Lack of Bias Assumption* (LBA) which stipulates that, at any time, the conditional intensity of the point process, given the present state of the observed process, be independent of the state of the observed process. We show that LBA, without the Poisson assumption, is necessary and sufficient for ASTA in a stationary process framework. Consequently, LBA covers known examples of non-Poisson ASTA, such as certain flows in open Jackson queueing networks, as well as the familiar Poisson case (PASTA). We also establish results to cover the case in which the process is observed just after the points, e.g., when departures see time averages. Finally, we obtain a new proof of the Arrival Theorem for product-form queueing networks.

For the M/G/1 queue, it is well known that both the limiting distribution of the queue length just prior to an arrival (the arrival-stationary distribution) and the limiting distribution of the queue length just after a departure (the departure-stationary distribution) coincide with the limiting distribution of the queue length at an arbitrary time (the time-stationary distribution). We want to determine when such useful relations hold in a general stochastic model.

The ASTA Property

Consider a continuous-time stochastic process $X \equiv \{X(t): t \geq 0\}$ that takes values in a general space, and a stochastic point process on the interval $[0, \infty)$, characterized by the counting process $N \equiv \{N(t): t \geq 0\}$ or, equivalently, the sequence of successive points $\{T_n: n \geq 0\}$; i.e., $N(t) = \sup\{n \geq 0: T_n \leq t\}$, $t \geq 0\}$ where $T_0 = 0$ without there being a 0th point. We think of X as representing the state or a partial description of the state of some system. Often X will be Markov, but it need not be. The point process N might represent outside observers totally independent of X; it might represent external arrivals to an open system with state X, or it might represent something else, such as the flow from one queue to another in a queueing network partially described by X.

Assuming that

$$X(t) \Rightarrow X(\infty) \quad \text{as } t \to \infty$$

and (1)

$$X(T_n) \Rightarrow \tilde{X}(\infty) \quad \text{as } n \to \infty$$

where $\Rightarrow$ denotes convergence in distribution or weak convergence, as in Billingsley (1968), we want to know when

$$X(\infty) \stackrel{d}{=} \tilde{X}(\infty) \tag{2}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Alternatively, assuming that $X(\infty)$ is the unique stationary distribution of $\{X(t): t \geq 0\}$ and $\tilde{X}(\infty)$ is the unique stationary distribution of $\{X(T_n): n \geq 1\}$, we want to know when (2) holds. (Then the distribution of $\tilde{X}(\infty)$ is called the Palm distribution.) More generally, we want to relate the two distributions, so that if we know one, then we can calculate the other.

When (2) holds in either of these situations, we say that *Arrivals See Time Averages* (ASTA). Averages have not been considered yet, but they will be in (3)–(6). The ASTA problem has a long history in queueing theory; see Descloux (1967), Franken, König, Arndt and Schmidt (1981), Cooper (1990) and Wolff (1982, 1989). Hence, even though the points T_n need not be

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interpreted as arrivals, we still use the suggestive ASTA terminology.

We assume that the sample paths of X are *left-continuous*, so that $X(T_n)$ coincides with the state *just prior to the nth arrival.* (When we later assume that the sample paths of X have limits from the left, this is just a convention to interpret $X(T_n)$ as the left limit $X(T_n-)$.) For external arrivals, the left continuity ensures that *an arrival does not see itself.* For flows from one queue to another in a queueing network, X may represent all queues *except the originating queue,* so that again the left continuity ensures that the observing customer does not see itself. (We also show how to treat the full network, that is, we give a new proof of the Arrival Theorem for product-form queueing networks.) When we consider departures (Section 4), we assume that the sample paths of X are right-continuous instead of left-continuous, again to ensure that a departing customer does not see itself.

Wolff's PASTA Property

Our approach is based on limiting averages, as in Wolff (1982, 1989). This approach has the advantage of not directly requiring a Markov, stationary or martingale structure. Under a *Lack of Anticipation Assumption* (LAA), that is, $\{N(t+u) - N(t): u \geq 0\}$ is independent of $\{X(s): 0 \leq s \leq t\}$ for all t, Wolff showed that *Poisson Arrivals See Time Averages* (PASTA), in the sense of limiting averages. (As acknowledged by Wolff, LAA and part of the theorem were contributed to Wolff's original work by Whitt 1979.)

Wolff considered the w.p.1 (with probability one) limits of averages of real-valued functions of X, that is

$$V(t) \equiv t^{-1} \int_0^t f[X(s)]\, ds, \quad t>0$$

and (3)

$$W(t) \equiv [N(t)]^{-1} \sum_{k=1}^{N(t)} f[X(T_k)], \quad t>0$$

provided that $N(t) > 0$, with $W(t) = 0$ when $N(t) = 0$, where f is a bounded measurable real-valued function on the state space of X. In particular, Wolff assumed that f is an indicator function of some measurable subset B; i.e., $f(x) = 1_B(x)$, where $1_B(x) = 1$ if $x \in B$ and 0 otherwise. The notation is chosen to suggest a queueing application in which $V(t)$ is the average virtual waiting time over the interval $[0, t]$, while $W(t)$ is the average actual waiting time per customer among the arrivals in $[0, t]$.

Leaving out minor technical regularity conditions, here is Wolff's main result (proved using martingale theory).

Theorem 1. (Wolff's PASTA Theorem). *Suppose that N is Poisson, LAA holds, $f = 1_B$ for some B and the sample paths of $f(X)$ are left-continuous. Then $V(t) \to V(\infty)$ w.p.1 as $t \to \infty$ if and only if $W(t) \to W(\infty)$ w.p.1 as $t \to \infty$, in which case $V(\infty) = W(\infty)$.*

For the arrival averages, it may seem more natural to consider

$$W_n = n^{-1} \sum_{k=1}^{n} f[X(T_k)], \quad n \geq 1 \tag{4}$$

instead of $W(t)$ in (3), but it is easy to see that these are equivalent. Since N is Poisson with finite intensity, say λ, $t^{-1}N(t) \to \lambda$ w.p.1 as $t \to \infty$, so that $W_n \to W(\infty)$ w.p.1 as $n \to \infty$ if and only if $W(t) \to W(\infty)$ w.p.1 as $t \to \infty$. Hence, limits for $W(t)$ as $t \to \infty$ and W_n as $n \to \infty$ are equivalent.

A discrete-time analog of Theorem 1, in which intervals between points have a geometric distribution instead of an exponential distribution, also follows by the same argument, as observed by Halfin (1983). (For new work on discrete time, see Georgiadis (1987) and Makowski, Melamed and Whitt (1989).)

An Elementary Proof

Our first purpose in this paper is to obtain a more elementary proof of Theorem 1, which does not require relatively complicated stochastic process theory (e.g., martingales). We obtain an elementary proof of PASTA in Section 2 by focusing on what we consider to be *the issue of primary practical interest, the equality of $V(\infty)$ and $W(\infty)$*, while freely assuming the existence of all desired limits. Instead of $V(t)$ and $W(t)$ in (3), we focus on related quantities involving *expectations.* In particular we consider

$$\bar{V}(t) \equiv E\left(t^{-1} \int_0^t f[X(s)]\, ds\right), \quad t>0$$

and (5)

$$\bar{W}(t) \equiv (E[N(t)])^{-1} E\left(\sum_{k=1}^{N(t)} f[X(T_k)]\right), \quad t>0.$$

We establish conditions under which $\bar{V}(t) \to \bar{V}(\infty)$ as $t \to \infty$ if and only if $\bar{W}(t) \to \bar{W}(\infty)$ as $t \to \infty$, in which case $\bar{V}(\infty) = \bar{W}(\infty)$. This is related to Theorem 1 when we also have $V(t) \to V(\infty)$ and $W(t) \to W(\infty)$ w.p.1 as $t \to \infty$, where $V(\infty)$ and $W(\infty)$ are constants, as can be established by regenerative structure (for example, Chapters V, VI and XI of Asmussen 1987 and Glynn and Whitt 1987) or stationarity and ergodicity (e.g., Theorem 1.5.5 of Franken et al.). Under extra regularity conditions (e.g., uniform integrability, p. 32 of Billingsley), these w.p.1 limits plus $t^{-1}N(t) \to \lambda$ w.p.1 *imply* that $\bar{V}(t) \to \bar{V}(\infty) = V(\infty)$ and $\bar{W}(t) \to \bar{W}(\infty) = W(\infty)$ as $t \to \infty$. Hence, we will have established what we regard as the most interesting part of Theorem 1 by showing that $\bar{V}(\infty) = \bar{W}(\infty)$.

To obtain the initial desired ASTA result (2), we freely assume the two weak convergence limits in (1) plus

$$\bar{V}(t) \to \bar{V}(\infty) = E(f[X(\infty)])$$

and (6)

$$\bar{W}(t) \to \bar{W}(\infty) = E(f[\tilde{X}(\infty)]) \quad \text{as } t \to \infty$$

for a large class of functions f. Hence, when we establish $\bar{V}(\infty) = \bar{W}(\infty)$ for sufficiently many measurable functions f, we also establish (2); in particular, we work with all bounded continuous real-valued f, because these functions determine the distributions (see p. 9 of Billingsley).

Eliminating the Poisson Assumption

Wolff starts by assuming that the point process N is a Poisson process. However, it is known that certain non-Poisson arrivals also see time averages. For example, consider a stationary open Jackson Markovian queueing network with queue set $\mathcal{N}$ in which queues 1 and 2 belong to a cycle; that is, with positive probability customers who visit queue 1 will also visit queue 2 and vice versa. The process of arrivals to queue 2 from queue 1 is not Poisson, but these arrivals see time averages in the subnetwork $\mathcal{N} - \{1\}$ if they observe just before transit (see, Melamed 1979a, b, 1982 and Walrand 1988). Similarly, the same customers see time averages in the subnetwork $\mathcal{N} - \{2\}$ if they observe immediately after transit. An example of non-Poisson arrivals seeing time averages cited by Wolff (1982, p. 228) is the M/M/1 queue with feedback, where the arrival does not see itself (see Burke 1976 and Disney, König and Schmidt 1984). It can be regarded as the limit of the example in which $\mathcal{N} = \{1, 2\}$, queue 2 is an infinite-server with individual service rate μ and all departures from queue 2 are routed to queue 1; then let $\mu \to \infty$. By the same device (adding the extra queue), customers that move from queue 1 to queue 2 see time averages in the full open Jackson network if they do not count themselves, which is an example of the *Arrival Theorem* (see Theorem 3.12 of Kelly 1979, Lavenberg and Reiser 1980, Sevcik and Mitrani 1981 and Sections 1.7, 2.10 and 4.4 of Walrand).

A second purpose of this paper is to determine more precisely when ASTA holds. First, we show that Theorem 1 is valid without assuming that N is Poisson (Section 2). This does not mean that something close to Poisson is not needed, but that what is needed is already embodied in LAA. In fact, in a stationary process framework, König and Schmidt (1980a, Section 7) had previously shown that the Poisson property did not need to be assumed directly. However, these generalizations of Theorem 1 do not cover the queueing network example above. We go further and give sufficient conditions for ASTA that are weaker than LAA, and contain the queueing network example as a special case (Examples 3 and 4 in Section 5). Indeed, we provide *necessary and sufficient conditions* for ASTA (Theorems 3 and 4). In Section 5 we also show how the Arrival Theorem for product-form queueing networks can be established in this framework.

The Left-Continuity Assumption

We also point out and eliminate a slightly restrictive assumption in Wolff's analysis. As indicated in Theorem 1, Wolff assumes that the stochastic process

$$U(t) \equiv f[X(t)] \qquad (7)$$

with f the indicator function of an arbitrary measurable B in the state space of X is left-continuous w.p.1. Since the indicator function is not continuous, this assumption implicitly puts requirements on the sample paths of X. Of course, there is no problem if X is a pure-jump process, but there could be otherwise. For example, the conditions of Theorem 1 are *not* satisfied when X is reflected Brownian motion on the positive real line with negative drift (which has continuous sample paths and an exponential stationary distribution), N is a Poisson process independent of X, and B is *any* nontrivial measurable subset, because the sample paths of $U(t)$ cannot be made left-continuous. The reason is that the sample paths will not have left limits at every t. Yet, the conclusion of Theorem 1 is obviously true in this case.

We eliminate this difficulty by assuming that the sample paths of X are left-continuous, which requires a topology on the state space of X, and that the function f in (3)–(7) and Theorem 1 is *continuous*; then the sample paths of U are indeed left-continuous. Even though we typically do not work directly with indicator functions, we are able to establish the desired results because expectations of bounded continuous real-valued functions determine distributions. Hence, we obtain $\bar{V}(\infty) = \bar{W}(\infty)$ for *all* measurable f for which the expectations are well defined, and thus (2).

If we want to consider only a single function f, then it suffices for U in (7) to have left-continuous sample paths, which could hold without f being continuous or X being a pure-jump process with left-continuous sample paths.

The Covariance Formula

The ASTA problem has a remarkably simple resolution, as a consequence of a fundamental relation between $X(\infty)$ and $\tilde{X}(\infty)$, which we call the *covariance formula.* It turns out that in a stationary framework

$$E[f(\tilde{X}(\infty))] = \frac{E[\mu(t)f(X(t))]}{E[\mu(t)]} = E[f(X(t))] + \frac{\operatorname{cov}[\mu(t), f(X(t))]}{E[\mu(t)]} \quad (8)$$

where f is an arbitrary measurable real-valued function such that the expectations in (8) are well defined, $\mu(t)$ is the conditional intensity of a jump in (point from) the counting process N at time t given the current state $X(t)$, and $\operatorname{cov}[Y_1, Y_2]$ is the covariance between Y_1 and Y_2, i.e., $\operatorname{cov}[Y_1, Y_2] = E(Y_1, Y_2) - (EY_1)(EY_2)$ (see (19) and Section 3). Given (8), it is immediate that $E[f(\tilde{X}(\infty))] = E[f(X(\infty))]$ if and only if the random variables $\mu(t)$ and $f[X(t)]$ are uncorrelated. In turn, this lack of correlation holds for all bounded measurable f if and only if (2) holds. This motivated us to call the lack of correlation property the *Lack of Bias Assumption* (LBA); see Definitions 2 and 3.

Given that (8) is so important and simple (except for the meaning of μ), it is surprising that it was not noticed before. In fact, (8) is just an expression for the one-dimensional distribution associated with the Palm measure, as in Franken et al. (1981), but this expression has evidently not been noticed before. However, at the same time that we discovered (8), variants of (8) were discovered independently by Brémaud (1989) and Stidham and El-Taha (1989). In fact, Brémaud notes that in the standard stationary framework, which is stronger than we require, a variant of (8) is a consequence of a result by Papangelou (1972) and thus he calls his variant of (8) *Papangelou's formula.* We discuss this further in Section 7. We point out that Papangelou did not actually state this consequence of his theorem. Moreover, (8) is not identical to the corresponding statement in Brémaud (1989).

The Rest of this Paper

In Section 1, we define the model precisely. In Section 2, we present sufficient conditions for ASTA in terms of a *Weak Lack of Anticipation Assumption* (WLAA). In Section 3, we present necessary and sufficient conditions for ASTA in a stationary process framework, using the conditional intensity notion and LBA. In Section 4, we investigate when departures see time averages, by exploiting time reversal. In Section 5, we give several examples from queueing, including open and closed Jackson networks, for which we give a new proof of the Arrival Theorem. In Section 6, we extend our ASTA results to conditional probabilities, paralleling the recent conditional PASTA results of van Doorn and Regterschot (1988) and Georgiadis (1987). We conclude in Section 7 with an additional discussion of the literature.

1. PRELIMINARIES

1.1. Properties of the Stochastic Processes

Consider two stochastic processes X and N defined on a common underlying probability space $(\Omega, \mathcal{F}, P)$. The process X, which is intended to partially describe the state of some system, is assumed to take values in some complete separable metric space E, endowed with the Borel σ-field (generated by the open subsets; the specific metric is not important). The process N, which is intended to represent an arrival process of some sort, is a stochastic point process on $[0, \infty)$, and so, has nondecreasing sample paths with values in the nonnegative integers. We assume that there exists t_0 such that

$$0 < E[N(t)] < \infty \quad \text{for all } t > t_0. \quad (9)$$

We further assume that the sample paths of X and N have left and right limits at all t (all $t > 0$ for left limits). Of particular importance, *we assume that the sample paths of X are left-continuous, while the sample paths of N are right-continuous.* Thus, N can be regarded as a random element of the function space $D \equiv D_{\mathrm{R}}[0, \infty)$, where R indicates that the functions are real-valued, while X can be regarded as a random element of $D \equiv D_E[0, \infty)$ after stipulating that the functions be left-continuous instead of right-continuous. This stipulation does not significantly alter the theory (see Chapter 3 of Billingsley 1968, Section 2 of Whitt 1980 and Chapter 3 of Ethier and Kurtz 1986). It is not essential that processes X and N have sample paths in D; it is a convenient regularity condition to avoid pathologies.

Let T_n be the epoch of the nth point (arrival) in N, with the convention that $T_0 = 0$, and let $N(s, t) = N(t) - N(s)$ for $0 \leq s \leq t$. Since the sample paths of N are integer-valued and in D, the sample paths of N have only finitely many jumps in finite time; that is, N is nonexplosive.

As in Wolff, we will actually work, not directly with X, but with an associated real-valued stochastic process $U \equiv \{U(t): t \geq 0\}$. Let f be a bounded measurable real-valued function on E and let $U(t) = f[X(t)]$. Since the sample paths of X are in D, U is jointly measurable in t and $\omega \in \Omega$, so that it can be integrated without losing measurability. As in Wolff, *we require that U be left-continuous with right limits*, which holds by virtue of our assumptions about X *provided that f is continuous.* (For pure-jump processes, f need not be continuous.) As noted in the Introduction, we can establish the desired (2) by considering only bounded continuous real-valued f; see the corollary to Theorem 2.

1.2. Stochastic and Riemann–Stieltjes Integrals

Much of our analysis depends on the representation of the random sum in the numerator of $\bar{W}(t)$ in (5) as a stochastic integral. However, since N is a point process with nondecreasing sample paths, we do not need an elaborate theory of stochastic integrals as in Chapter 2 of Chung and Williams (1983). Instead, we can regard the stochastic integral as a Riemann–Stieltjes integral for each sample path, that is

$$\sum_{k=1}^{N(t)} U(T_k) = \int_0^t U(s)dN(s) = \lim_{n\to\infty} \sum_{k=0}^{n-1} U\left(\frac{kt}{n}\right) N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right) \quad \text{w.p.1.} \tag{10}$$

We use the sample-path properties of U and N, in particular, the left-continuity of U and the right-continuity of N, to interpret this stochastic integral as a proper Riemann-Stieltjes integral for each sample path (see Chapter 9 of Apostol 1957, especially p. 200). In (10) the subintervals in the partition of $[0, t]$ do not need to be evenly spaced, so long as the width of the largest interval goes to zero in the limit. Moreover, for the Riemann–Stieltjes integral to be well defined, in the kth term of the nth sum in (10), we can evaluate U anywhere in the closed interval $[kt/n, (k + 1)t/n]$. However, for our probabilistic analysis, it is important to use the left endpoint (or the right endpoint when we reverse the left and right continuity properties of X and N to treat departures).

Remark 1. For the Riemann–Stieltjes integral to be well defined, it suffices for U to have w.p.1 no discontinuities from the right (left) where N has its right (left) discontinuities; that is, limits from the left and right are not actually needed. If, with positive probability, U and N have common discontinuities from the left or if they have common discontinuities from the right, then there can be difficulties (see p. 212 of Apostol).

Since U is bounded (because f is bounded), $E[N(t)] < \infty$ by T9), and

$$\sum_{k=0}^{n-1} U\left(\frac{kt}{n}\right) N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right) \leq \left(\sup_{0\leq s\leq t} |U(s)|\right) N(t) < \infty \quad \text{w.p.1} \tag{11}$$

we can apply the Lebesgue dominated convergence theorem to conclude that the expectation of the limit is the limit of the expectations, that is

$$E\left(\sum_{k=1}^{N(t)} U(T_k)\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right) N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right]. \tag{12}$$

2. SUFFICIENT CONDITIONS FOR ASTA

In this section we present a sufficient condition for ASTA. We weaken LAA, but not as much as we do later, so that this section primarily constitutes an elementary proof of PASTA.

Definition 1. The Weak Lack of Anticipation Assumption (WLAA) holds (for U and N) if there exists $u_0 > 0$ such that $U(t)$ and $N(t, t + u)$ are uncorrelated for all $t \geq 0$ and $0 \leq u \leq u_0$.

Note that WLAA is weaker than LAA in three ways: First, we replace the entire past $\{U(s): 0 \leq s \leq t\}$ by the present $U(t)$; second, we replace the entire future $\{N(t + u) - N(t): u \geq 0\}$ by a small increment $N(t + u) - N(t)$; and third, we replace independence by lack of correlation. Obviously LAA implies WLAA, but not conversely. The WLAA condition is closely related to an independence condition introduced by König and Schmidt (1980a); see Theorem 1.6.6 of Franken et al. We discuss this further in Section 7.

We will show that WLAA is sufficient for ASTA under additional weak stationary moment assumptions. Note that, not only is WLAA weaker than LAA, but also N is *not* directly assumed to be Poisson. We use the definitions of $\bar{V}(t)$ and $\bar{W}(t)$ in (5) and t_0 in (9).

Theorem 2. *Suppose that WLAA holds and at least one of a or b holds:*

a. $E[U(t)]$ is independent of t;

b. there exists positive u_0 such that $E[N(t, t+u)] = \lambda u$ for $t \geq 0$ and $0 \leq u \leq u_0$.

Then $\bar{V}(t) = \bar{W}(t)$ for all $t > t_0$, so that $\bar{V}(t) \to \bar{V}(\infty)$ as $t \to \infty$ if and only if $\bar{W}(t) \to \bar{W}(\infty)$ as $t \to \infty$, in which case $\bar{V}(\infty) = \bar{W}(\infty)$.

Proof. By (12) and WLAA

$$E\left(\sum_{k=1}^{N(t)} U(T_k)\right) = \lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right)\right] E\left[N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right].$$

If a holds, so that $E[U(t)] = E[U(0)]$ for all t, then

$$E\left(\sum_{k=1}^{N(t)} U(T_k)\right) = E[U(0)] \lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right] = E[U(0)]E[N(t)]$$

so that

$$\bar{W}(t) = E[U(0)] = t^{-1} \int_0^t E[U(s)]\, ds = \bar{V}(t) \quad \text{for } t > t_0.$$

If, instead, b holds, so that $E[N(t, t+u)] = \lambda u$ for $t \geq 0$ and $0 \leq u \leq u_0$, then

$$E\left(\sum_{k=1}^{N(t)} U(T_k)\right) = \lim_{n\to\infty} \frac{E[N(0, t/n)]}{(t/n)} \sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right)\right]\left(\frac{t}{n}\right) = \lambda \int_0^t E[U(s)]\, ds$$

so that again $\bar{W}(t) = \bar{V}(t)$ for $t > t_0$.

Remark 2. The natural sufficient condition to obtain condition a in Theorem 2 for all bounded measurable f is for X to be stationary, while the natural sufficient condition to obtain condition b is for N to be stationary (see below). Of course, N Poisson implies b, but any stationary point process will do. The need for something like a Poisson process is already contained in WLAA.

We say that U (and similarly for X) is (*strictly*) *stationary* if the finite-dimensional distributions are independent of time, i.e., if

$$[U(t_1), \ldots, U(t_k)] \stackrel{d}{=} [U(t_1 + h), \ldots, U(t_k + h)] \tag{13}$$

for all positive integers k, all k-tuples of positive time points $(t_1, \ldots, t_k)$ and all positive h. We say that the point process N is *stationary* if its *increments* $N(s, t)$ are stationary, i.e., if

$$[N(s_1, t_1), \ldots, N(s_k, t_k)] \stackrel{d}{=} [N(s_1 + h, t_1 + h), \ldots, N(s_k + h, t_k + h)] \tag{14}$$

for all positive integers k, all pairs of positive k-tuples $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ with $s_i \leq t_i$ for all i, and for all positive h. Obviously, for Theorem 2, stationarity of U implies that $E[U(t)]$ is independent of t. Of course, $E[U(t)]$ being independent of t for sufficiently many f (for example, all bounded continuous f) implies that the distribution of $X(t)$ is independent of t, but that is still weaker than stationarity of the full process X, as defined by (13). Similarly, stationarity of N implies that $E[N(t, t+u)]$ is independent of t for all u. Under (9), if N is stationary, then necessarily $E[N(t, t+u)] = \lambda u$ for some λ for all t and u.

Remark 3. In applications, WLAA is not substantially easier to verify than LAA because it is not much easier to work with a small future increment $N(t, t+u)$ than the entire future; see the examples in Section 5. We obtain a more useful condition in Section 3 by replacing the small increment $N(t, t+u)$ with an intensity at time t. In this direction, note that Theorem 2 is still valid if we replace WLAA by an *asymptotic lack of correlation condition*, in particular

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right) N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right] = \lim_{n\to\infty} \sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right)\right] E\left[N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right].$$

The following corollary shows how Theorem 2 enables us to treat the ASTA problem we formulated at the outset.

Corollary. *Suppose that* (1) *holds;* (6) *holds for all bounded continuous real-valued functions f, and at least one of a or b holds:*

a. the distribution of $X(t)$ is independent of t;

b. there exists a positive u_0 such that $E[N(t, t+u)] = \lambda u$ for $t \geq 0$ and $0 \leq u \leq u_0$.

If WLAA holds for all bounded continuous real-valued f, then (2) *holds, i.e.,* $X(\infty) \overset{d}{=} \tilde{X}(\infty)$.

Proof. The conditions of Theorem 2 are satisfied for every bounded continuous real-valued f. (We need the continuity to make the sample paths of U left-continuous.) Hence, $E(f[X(\infty)]) = E(f[\tilde{X}(\infty)])$ for each bounded continuous real-valued f, which implies the conclusion because expectations of these functions determine the distributions; see p. 9 of Billingsley.

Remark 4. To establish WLAA for all bounded continuous f we typically show that $X(t)$ and $N(t, t + u)$ are independent, but note that we can have two *dependent* random variables Y and Z such that $f(Y)$ and Z are uncorrelated for all measurable f. For example, let $(Y, Z) = (1, 0), (1, 2), (0, 1)$ and $(2,1)$ each with probability ¼, so that $E[f(Y)Z] = E[f(Y)]E[Z] = E[f(Y)]$ for all measurable f, but Y and Z are dependent. To convert this into an example in our setting, let $X(t) = Y$ and let N be a doubly stochastic Poisson process with intensity Z for all $t \geq 0$. Then, for any bounded measurable real-valued f

$$E[f(X(t))N(t, t + u)] = E[f(Y)]u = E[f(X(t))]E[N(t, t + u)]$$

so that WLAA holds, but $N(t, t + u)$ is *not* independent of $X(t)$.

Remark 5. Note that the proof of Theorem 2 and the corollary do *not* require that N be a point process. The key Riemann–Stieltjes integral representation (10) is valid if the sample paths of N are of bounded variation on every bounded interval. We thus have an immediate extension of PASTA to LASTA: Lévy Processes (that is, processes with stationary independent increments) See Time Averages, provided that the sample paths are locally of bounded variation and WLAA holds. Examples of such Lévy processes are compound Poisson processes and the gamma process (see pp. 69–72 of Prabhu 1980). As with PASTA, it is not necessary to directly assume the Lévy property; it is typically what is needed to satisfy WLAA.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR ASTA IN A STATIONARY FRAMEWORK

In this section, we obtain necessary and sufficient conditions for ASTA, primarily by replacing the future increment $N(t, t + u)$ in WLAA by a conditional intensity. We assume that X and N are *jointly stationary* in the sense that

$$[X(t), N(t, t + u)] \overset{d}{=} [X(t + h), N(t + h, t + h + u)] \tag{15}$$

for all positive t, u and h, which of course is weaker than joint stationarity in the sense of (13) and (14). As a consequence, $\bar{V}(t)$ in (5) is independent of t, so that trivially $\bar{V}(t) \to \bar{V}(\infty)$ as $t \to \infty$. We will provide conditions under which $\bar{W}(t)$ in (5) is also independent of t (so that trivially $\bar{W}(t) \to \bar{W}(\infty)$ as $t \to \infty$) and $\bar{V}(\infty) = \bar{W}(\infty)$.

Mathematically, the stationarity condition (15) is quite strong as a sufficient condition, but in many applications there is asymptotic stationarity, so that $\bar{V}(\infty)$ and $\bar{W}(\infty)$ are the same as if (15) held. Thus, showing $\bar{V}(\infty) = \bar{W}(\infty)$ under (15) is the key to more general statements. It is not difficult to extend the results in this section to a nonstationary setting in the spirit of Section 2, so that even (15) is not essential. Indeed, this is done in Melamed and Whitt (1990), but (15) makes the discussion easier to follow.

Let λ be the *intensity* of N, $\lambda = E[N(0, 1)]$, which is finite by (9), and let $\mu(t)$ be the *conditional intensity* of N at t given $X(t)$, defined by

$$\mu(t) = \lim_{u \downarrow 0} u^{-1} E[N(t, t + u) \mid X(t)]. \tag{16}$$

Note that our stationarity assumption (15) implies that $\mu(t) \overset{d}{=} \mu(0)$ for all $t > 0$. The mode of convergence in (16) is w.p.1 with respect to the distribution of $X(t)$.

We assume that the limit in (16) exists as a proper random variable and that the limit and expectation are interchangeable

$$E[\mu(t)] = \lim_{u \downarrow 0} E[u^{-1} E[N(t, t + u) \mid X(t)]] = \lambda. \tag{17}$$

Even though (16) and (17) are natural, they constitute restrictions. By Theorem 5.4 of Billingsley, given (16), (17) is equivalent to the family of random variables $\{Y_u\} \equiv \{u^{-1}E[N(t, t + u) \mid X(t)]\}$ indexed by u being *uniformly integrable* as $u \to 0$, for which many sufficient conditions are known. For example, it suffices to have Y_u less than or equal to, or stochastically less than or equal to, a random variable Y with $E[Y] < \infty$ for $0 < u \leq u_0$. The conditions (16) and (17) are fairly easy to verify when $X(t) = g(Z(t))$ for a pure-jump Markov process Z and the point process N counts designated jumps in Z, as can be seen from the queueing examples in Section 5. The more technical approaches in Melamed and Whitt (1990) and Brémaud (1989) avoid the need to justify (16) and (17) by incorporating the conditions in the model framework. At any rate, (16) and (17) are technical conditions that we are prepared to assume, in the same spirit as (1) and (6).

As in (8), we characterize the difference $\bar{W}(\infty) - \bar{V}(\infty)$ in terms of the *covariance* of $U(t)$ and $\mu(t)$

$$\operatorname{Cov}[U(t), \mu(t)] = E[U(t)\mu(t)] - E[U(t)]E[\mu(t)]. \tag{18}$$

Definition 2. The Lack of Bias Assumption (LBA) holds for U and N if $U(t)$ and $\mu(t)$ are uncorrelated, i.e., if $\operatorname{Cov}[U(t), \mu(t)] = 0$.

Theorem 3. *Suppose that X and N are jointly stationary in the sense of* (15) *and that the conditional intensity $\mu(t)$ is well defined, satisfying* (16) *and* (17). *Then both $\bar{V}(t)$ and $\bar{W}(t)$ are independent of t and*

$$\bar{W}(t) - \bar{V}(t) = \lambda^{-1}\operatorname{Cov}[U(0), \mu(0)], \quad t \geq 0 \tag{19}$$

so that $\bar{W}(\infty) = \bar{V}(\infty)$ if and only if LBA holds for U and N. Moreover, (19) *remains valid for all measurable f provided that the expectations are well defined.*

Proof. First, assuming that f is continuous, we have

$$\begin{aligned}
&E\left[t^{-1}\int_0^t U(s)\,dN(s)\right] \\
&= E\left[\lim_{n\to\infty} t^{-1}\sum_{k=0}^{n-1} U\left(\frac{kt}{n}\right)N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right] \quad \text{by (10)} \\
&= \lim_{n\to\infty} t^{-1}\sum_{k=0}^{n-1} E\left[U\left(\frac{kt}{n}\right)N\left(\frac{kt}{n}, \frac{(k+1)t}{n}\right)\right] \quad \text{by (12)} \\
&= \lim_{n\to\infty}\left(\frac{n}{t}\right)E\left[U(0)N\left(0, \frac{t}{n}\right)\right] \quad \text{by (15)} \\
&= \lim_{n\to\infty}\left(\frac{n}{t}\right)E\left[U(0)E\left[N\left(0, \frac{t}{n}\right)\middle| X(0)\right]\right] \quad \text{by conditioning on } X(0) \\
&= E\left[U(0)\lim_{n\to\infty}\left(\frac{n}{t}\right)E\left[N\left(0, \frac{t}{n}\right)\middle| X(0)\right]\right] \quad \text{by (16) and (17)} \\
&= E[U(0)\mu(0)] \quad \text{by (16).}
\end{aligned}$$

(In the second to last step, the convergence and uniform integrability in (16) and (17) are not altered by multiplying by the bounded random variable $U(0)$.) Hence, by (5)

$$\bar{W}(t) = (t/E[N(t)])E[U(t)\mu(t)]$$

and (20)

$$\bar{V}(t) = E[U(t)].$$

Since $E[N(t)] = \lambda t = E[\mu(t)]t$, (19) follows from (20). Finally, to obtain (19) for all measurable f for which the expectations are well defined, extend by taking limits; see pp. 7–9 of Billingsley.

Remark 6. Under (6) the expressions for $\bar{W}(t)$ in (19) and (20) are equivalent to the covariance formula (8). As noted before (6), a sufficient condition for (6) is stationarity and ergodicity.

Remark 7. ASTA says that sampling at the points of N is the same as sampling uniformly over time, i.e., $\bar{W}(\infty) = \bar{V}(\infty)$. The LBA condition identifies the critical lack of bias in the sampling procedure for ASTA to be valid, hence the name. More generally, for $f = 1_B$, if the likelihood of a point in N at time t is positively correlated with $X(t)$ being in the set B, then it is intuitively reasonable that arrivals will find the process X in the set B excessively often, i.e., $\bar{W}(\infty) \geq \bar{V}(\infty)$; the negatively correlated case is analogous. This reasoning makes the zero-correlation condition (LBA) intuitively obvious. Feldman et al. (1981) also express this issue in terms of measurement.

We will apply Theorem 3 to treat the ASTA problem formulated at the outset. To do this, we use a modification of Definition 2.

Definition 3. The Lack of Bias Assumption (LBA) holds for X and N if LBA holds for U and N (Definition 2) for all bounded continuous real-valued f.

Corollary 1. *Suppose that* (1) *holds;* (6) *holds for all bounded continuous real-valued functions f, X and N are jointly stationary in the sense of* (15); *and the conditional intensity $\mu(t)$ is well defined, satisfying* (16) *and* (17). *Then* (2) *holds, i.e., $X(\infty) \stackrel{d}{=} \tilde{X}(\infty)$ if and only if LBA holds for X and N.*

Proof. The proof of Theorem 3 applies for every bounded continuous real-valued f. As in the corollary to Theorem 2, expectations of these functions determine the distributions.

As noted in Remark 6, (19) usefully characterizes the difference $\bar{W}(\infty) - \bar{V}(\infty)$ even when LBA does not hold. For example, (19) is useful for making stochastic comparisons between the two stationary random elements $X(\infty)$ and $\tilde{X}(\infty)$. We say that $X(\infty)$ is *stochastically less than or equal to* $\tilde{X}(\infty)$, and write $X(\infty) \stackrel{d}{\leq} \tilde{X}(\infty)$, if $E(f[X(\infty)]) \leq E(f[\tilde{X}(\infty)])$ for all nondecreasing real-valued functions f, where the state space E is endowed with a closed partial order (see Kamae, Krengel and O'Brien 1977). (A partial order $\leq$ is closed if $x \leq y$ when $x_n \leq y_n$ for all n, $x_n \to x$ and $y_n \to y$ as $n \to \infty$.)

To make stochastic comparisons, we assume that the conditional intensity $\mu(t)$ can be represented as a measurable function of $X(0)$.

Remark 8. The conditional expectation $E[N(t, t + u) | X(t)]$ can always be regarded as a measurable function of $X(t)$ (see p. 299 of Chung 1974), and thus, so can the limit in (16) through any sequence $u_n \downarrow 0$. Given that the limit in (16) is the same for any such sequence, we then have $\mu(t) = g[X(t)]$ for some measurable g.

We say that $\mu(t)$ is a nondecreasing (nonincreasing) function of $X(t)$ w.p.1 if there exists a nondecreasing (nonincreasing) function g such that $\mu(t) = g[X(t)]$ w.p.1. We say that a random vector $Z \equiv (Z_1, \ldots, Z_n)$ in R^n is *associated* if $\mathrm{Cov}[f[Z], g[Z]] \geq 0$ for all nondecreasing bounded measurable real-valued functions f and g on R^n (see p. 29 of Barlow and Proschan 1975).

Corollary 2. *Suppose, in addition to the conditions of Corollary* 1, *that the state-space E has a closed partial order. Then* $X(\infty) \overset{d}{\leq} \overset{d}{\geq} \tilde{X}(\infty)$ *if and only if*

$$\mathrm{Cov}[f[X(t)], \mu(t)] \geq (\leq)\ 0 \tag{21}$$

for all nondecreasing bounded measurable real-valued functions f. In particular, (21) *holds with* $\geq (\leq)\ 0$ *when the conditional intensity* $\mu(t)$ *is a nondecreasing (nonincreasing) function of* $X(t)$ *w.p.1 and* $X(t)$ *is either a real-valued random variable or an associated random vector in* R^n.

Proof. Let f in the definition of U be nondecreasing. Then $\bar{W}(\infty) - \bar{V}(\infty) \geq 0$ on the left side of (19) expresses the stochastic order conclusion.

Remark 9. To see how Corollary 2 can be applied when $X(t)$ is a random vector, consider a k-class queueing system ($k \geq 1$) in which class 1 arrives according to a renewal process. Let $X(t) = [Z(t), -A(t)]$, where $Z(t)$ is the number of customers (or just class-1 customers) in the system and $A(t)$ is the age of the interarrival time for class 1 at time t, that is, the elapsed time since the last class-1 arrival. We expect that $A(t)$ becomes smaller as $Z(t)$ increases; for example, $A(t)$ might be stochastically decreasing in $Z(t)$, that is, $P(A(t) > u \mid Z(t) = s)$ is decreasing in s for all u, which implies that X is associated (see p. 143 of Barlow and Proschan). Assuming that the class-1 interarrival-time distribution has a cdf $G(t)$ with a density $g(t)$, the conditional intensity is

$$\mu(t) = g(a)/[1 - G(a)]$$

when

$$X(t) = (z, -a). \tag{22}$$

Obviously $\mu(t)$ is a nondecreasing (nonincreasing) function of $X(t)$ if and only if the cdf G is DFR (IFR), i.e., has decreasing (increasing) failure rate. Corollary 2 to Theorem 3 thus provides a basis for new stochastic comparisons in the spirit of König and Schmidt (1980b), Chapter 4 of Franken et al., Whitt (1983) and Niu (1984). Since DFR implies NWUE and IFR implies NBUE, the rough example here is consistent with previous results, for example, (4.3.12) on p. 116 of Franken et al.

We conclude this section by giving alternate characterizations of LBA for X and N (Definition 3).

Theorem 4. *Suppose that X and N are jointly stationary in the sense of* (15), *and the conditional intensity* $\mu(t)$ *in* (16) *is well defined, satisfying* (17). *Then the following are equivalent*:

i. *LBA for X and N*;
ii. *ASTA, i.e.,* $X(\infty) \overset{d}{=} \tilde{X}(\infty)$;
iii. $\mu(t) = E[\mu(t)] \equiv \lambda$ *w.p.1*;
iv. $\mu(t)$ *is independent of* $X(t)$.

Proof. Given Corollary 1 to Theorem 3, it suffices to show that i implies iii. Suppose that LBA holds for X and N, so that

$$E[f[X(t)]\mu(t)] = E[f[X(t)]]\lambda \tag{23}$$

for all bounded continuous real-valued f. Then, by taking limits (pp. 7–9 of Billingsley), (23) is valid for all bounded measurable real-valued f, so that λ is a version of the conditional expectation $E[\mu(t) | X(t)]$ (see p. 297 of Chung). However, $E[\mu(t) | X(t)] = \mu(t)$ so that $\mu(t) = \lambda$ w.p.1.

Remark 10. By Theorem 4, LBA holds for X and N (Definition 3) if and only if $\mu(t)$ in (16) is *independent* of $X(t)$. In contrast, independence of $\mu(t)$ and $U(t) = f[X(t)]$ for one f is strictly stronger than LBA for U and N (Definition 2). Similarly, independence of U and $N(t, t + u)$ is strictly stronger than WLAA (Definition 1), as noted in Remark 4.

4. DEPARTURES AND TIME REVERSAL

In queueing, it is also of interest to consider when departures see time averages. Of course, formally, this is already covered by Sections 1–4 because the point process N can represent departures as well as arrivals, but to obtain useful results for departures from queues, we typically must look at X just *after* the departures, so as not to include the departing customer.

When X is a process on the nonnegative integers that move up and down in unit jumps, we can use an upcrossing and downcrossing argument to relate the stationary distribution after departures to the stationary distribution prior to arrivals, for example, see p. 112, Franken et al. However, we want to treat more general processes.

We obtain useful results for departures by assuming that the sample paths of *X and U are right-continuous, while the sample paths of N are left-continuous.* We refer to these as the *reverse continuity conditions.* Given these reverse continuity conditions, we can obtain results paralleling those of Sections 1–4 simply by *reversing time.* The time-reversed processes have the original continuity properties of the processes in Sections 1–4. The time-reversed conditions turn out to be reasonable in the contexts of reversibility and quasireversibility (see Kelly and Walrand).

To illustrate, we first state the analogs of Definition 1 and Theorem 2 in Section 2, omitting the easy proof.

Definition 4. The Reverse Weak Lack of Anticipation Assumption (Reverse WLAA) holds (for U and N) if these exists $u_0 > 0$ such that $U(t)$ and $N(t - u, t)$ are uncorrelated for all $t \geq 0$ and $0 \leq u \leq u_0$ with $t - u \geq 0$.

As before, let t_0 be defined by (9).

Theorem 5. *Suppose that the reverse continuity conditions hold for U and N, Reverse WLAA holds, and at least one of a or b holds:*

a. $E[U(t)]$ *is independent of* t;

b. there exists positive u_0 *such that* $E[N(t - u, t)] = \lambda u$ *for* $t \geq 0$ *and* $0 \leq u \leq u_0$ *with* $t - u \geq 0$.

Then $\bar{V}(t) = \bar{W}(t)$ *for all* $t > t_0$, *so that* $\bar{V}(t) \to \bar{V}(\infty)$ *as* $t \to \infty$ *if and only if* $\bar{W}(t) \to \bar{W}(\infty)$ *as* $t \to \infty$, *in which case* $\bar{V}(\infty) = \bar{W}(\infty)$.

We next state the analogs of Definition 2 and Theorem 3 in Section 3. Paralleling (15), we assume that X and N are jointly stationary in the sense that

$$[X(t), N(t - u, t)] \stackrel{d}{=} [X(t + h), X(t + h - u, t + h)] \qquad (24)$$

for all positive t, u and h with $t - u \geq 0$. Paralleling (16), let $\tilde{\mu}(t)$ be the *reverse conditional intensity* of N at t given $X(t)$, defined by

$$\tilde{\mu}(t) = \lim_{u \downarrow 0} u^{-1} E[N(t - u, t) \mid X(t)], \quad t > 0. \qquad (25)$$

As in (17), we assume that the expectation of the limit in (25) is the limit of the expectations. Notice that $\tilde{\mu}(t)$ is precisely the conditional intensity of the left-continuous version of N at t given $X(t)$ in *reverse time.* (See Section 1.7 of Kelly for related notions for Markov processes.)

Definition 5. The Reverse Lack of Bias Assumption (Reverse LBA) holds for U and N if $U(t)$ and $\tilde{\mu}(t)$ are uncorrelated.

Theorem 6. *Suppose that the reverse continuity conditions hold for U and N; X and N are jointly stationary in the sense of* (24); *and the conditional intensity* $\tilde{\mu}(t)$ *is well defined, satisfying* (25) *and the analog of* (17). *Then* $\bar{W}(t)$ *is independent of t and*

$$\bar{W}(\infty) - \bar{V}(\infty) = \lambda^{-1}\mathrm{Cov}[U(t), \tilde{\mu}(t)] \qquad (26)$$

so that $\bar{W}(\infty) = \bar{V}(\infty)$ *if and only if Reverse LBA holds for U and N.*

Again, the proof is immediate by time reversal.

5. EXAMPLES IN QUEUEING

We will illustrate the concepts in the previous sections through examples in queueing.

Example 1. To show that there is a significant difference between WLAA and LBA, and that we can have LBA and ASTA without N being Poisson, consider a standard M/M/1 queue with arrival rate λ and traffic intensity ρ, where $0 < \rho < 1$. Let $X_1(t)$ represent the number of customers in the system at time t, assuming that an arrival inaugurates a busy cycle at $t = 0$. Modify each sample path of $X_1(t)$ by repeating each busy cycle exactly once more. Formally, let B_n be the epoch of the beginning of the nth busy cycle associated with X_1 (when there is an arrival to an empty system), with $B_0 = 0$, and let $X(t)$ be defined by

$$X(2B_n + t) = X(B_n + B_{n+1} + t) = X_1(B_n + t) \quad \text{for } 0 \leq t < B_{n+1} - B_n \quad \text{for } n \geq 0. \qquad (27)$$

Finally, take the stationary version of X. Let N be the arrival process associated with X; i.e., arrivals occur when there are jumps up in X. For this example, it is obvious that N is not Poisson and that LAA does not hold. Furthermore, it is not difficult to show that WLAA does not hold either (see an unpublished appendix), but LBA and ASTA do hold. In particular, $\mu(t)$ is well defined via (16) and $\mu(t) = \lambda$ w.p.1, where λ is the arrival rate. Apply Theorems 3 and 4.

Example 2. An elementary queueing example in which ASTA holds when the arrival process is non-Poisson was given on p. 863 of König, Miyazawa and

Schmidt (1983). Let $X(\infty)$ be the time-stationary number of customers in the standard GI/M/1 queue with traffic intensity ρ and interarrival-time cdf $A(x)$ having mean λ^{-1} and the property that the smallest positive root of the equation

$$\sigma = \int_0^\infty \exp(-\lambda\rho^{-1}x(1-\sigma))\, dA(x) \tag{28}$$

is equal to $\rho < 1$. As a consequence, $X(\infty)$ has the geometric stationary distribution of an M/M/1 queue, even through $A(x)$ need not be exponential. Moreover, the stationary variable $\tilde{X}(\infty)$ obtained by considering the embedded process at arrival epochs (or departure epochs) has the same geometric distribution. For this example, LBA for the arrivals and reverse LBA for the departures necessarily hold because ASTA holds, but LBA for the arrivals does not seem easy to establish directly. This illustrates that LBA and the covariance formula (8) are not panaceas. However, we show that it is not difficult to establish reverse LBA for the departures, using the exponential service times. For any $n \geqslant 0$

$$\begin{aligned}&\lim_{u\downarrow 0} u^{-1}E[N(t-u,t)\,|\,X(t)=n]\\ &\quad= \lim_{u\downarrow 0} u^{-1}P(N(t-u,t)=1\,|\,X(t)=n)\end{aligned} \tag{29}$$

where

$$\begin{aligned}&P(N(t-u,t)=1\,|\,X(t)=n)\\ &\quad=\frac{P(N(t-u,t)=1, X(t)=n)}{P(X(t)=n)}\\ &\quad=\frac{P(N(t-u,t)=1, X(t-u)=n+1)}{P(X(t)=n)}+o(u)\end{aligned}$$

with

$$\begin{aligned}&\frac{P(N(t-u,t)=1,\ X(t-u)=n+1)}{P(X(t)=n)}\\ &\quad=\frac{P(X(t-u)=n+1)P(N(t-u,t)=1\,|\,X(t-u)=n+1)}{P(X(t)=n)}\\ &\quad=\rho P(N(t-u,t)=1\,|\,X(t-u)=n+1)\end{aligned} \tag{30}$$

so that

$$\begin{aligned}&\lim_{u\downarrow 0} u^{-1}E[N(t-u,t)\,|\,X(t)=n]\\ &\quad=\rho\lim_{u\downarrow 0} u^{-1}E[N(t-u,t)\,|\,X(t-u)=n+1]\\ &\quad=\lambda.\end{aligned} \tag{31}$$

Alternatively, proceeding forward in time, we can apply (8) or, equivalently, (19) to calculate $E[f(\tilde{X}(\infty))]$ when the point process N is the departure process. Note that the state embedded at departures but excluding the departing customer is $X(T_n) - 1$. In particular, since $\mu(t) = \lambda\rho^{-1}$ on $\{X(t) > 0\}$ and 0 otherwise

$$\begin{aligned}&E[f(\tilde{X}(\infty)-1)]\\ &\quad=\frac{E[\mu(t)f(X(t)-1)]}{E[\mu(t)]}\quad\text{by (8)}\\ &\quad=E[f(X(t)-1)\,|\,X(t)>0]\quad\text{by form of }\mu(t)\\ &\quad=E[f(X(t))]\quad\text{by geometric distribution.}\end{aligned} \tag{32}$$

Example 3. We will discuss the example of non-Poisson ASTA mentioned in the Introduction. Consider a *stationary* open Jackson network of multiserver queues with queue set $\mathcal{N} = \{1, 2, \ldots, m\}$ and two distinguished queues, say 1 and 2, which participate in a cycle (that is, with positive probability customers can go from queue 1 to queue 2 and vice versa). Let $Z \equiv (Z_1, \ldots, Z_m)$ be the full state of the (Markovian) network, i.e., the vector representing the number of customers at each queue, and let $\hat{Z}_i$ be the state of $\mathcal{N} - \{i\}$. Let T_n be the epochs of the traffic stream on arc (1, 2), that is, the epochs when customers move from queue 1 to queue 2. It is known that the point process N with epochs T_n is *not* Poisson (see Melamed 1979a, b or Section 4.11 of Walrand). Nevertheless, customers observing states $\hat{Z}_1(T_n-)$ (that is, customers recording the state of $\mathcal{N} - \{1\}$ just prior to entering queue 2) do see time averages in $\mathcal{N} - \{1\}$.

To fit this into our framework, we let $X(t) = \hat{Z}_1(t-)$, so that X is a *partial description* of the state of the network and X has left-continuous paths. If we were considering external arrivals, then we could let $X(t) = Z(t-)$ and X would represent the full state of the network, but now we want to exclude queue 1 from the observed process.

Let $\eta(t)$ be the intensity of N at t given the full state $Z(t-)$, defined as in (16) by

$$\eta(t) = \lim_{u\downarrow 0} u^{-1}E[N(t, t+u)\,|\,Z(t-)]. \tag{33}$$

It is elementary that

$$\begin{aligned}\eta(t) &= q(Z(t-), Z(t-) - e_1 + e_2)\\ &= \mu_1 r_{12}\min\{s_1, Z_1(t-)\}\end{aligned} \tag{34}$$

where $q(\cdot, \cdot)$ is the transition rate function of the Markov process Z, e_i is the ith unit vector, μ_i is the service rate of an individual server at queue i, s_i is the number of servers at queue i, and r_{ij} is the probability of routing from queue i to queue j.

Notice that $\eta(t)$ in (34) is a function of only $Z_1(t-)$. Since the stationary distribution of $Z(t)$ in this network is product form, $Z_1(t-)$ is independent of $\hat{Z}_1(t-) = X(t)$. Hence

$$\begin{aligned}\mu(t) &\equiv \lim_{u\downarrow 0} u^{-1}E[N(t,t+u) \mid X(t)]\\ &= \lim_{u\downarrow 0} u^{-1}E[E[N(t,t+u) \mid Z(t-)] \mid X(t)]\\ &= E[\eta(t) \mid X(t)] = E[\eta(t)] \qquad (35)\end{aligned}$$

so that $\mu(t)$ is constant. Thus, ASTA and LBA for X and N hold by virtue of Theorem 4.

We will consider what the customers flowing from queue 1 to queue 2 see in the subnetwork $\mathcal{N} - \{2\}$ just after entering queue 2. To fit this into our framework, let $X(t) = \hat{Z}_2(t+)$ and let N have left-continuous paths. Instead of (33), we let

$$\tilde{\eta}(t) = \lim_{u\downarrow 0} u^{-1}E[N(t-u,\ t) \mid Z(t+)]. \qquad (36)$$

To compute $\tilde{\eta}(t)$, we use the fact that the transition function $\tilde{q}(\cdot,\ \cdot)$ for the reversed process associated with Z satisfies

$$\pi(x)q(x,\ y) = \pi(y)\tilde{q}(y,\ x) \qquad (37)$$

where $x,\ y \in R^m$ and π is the equilibrium distribution of Z (see p. 28 of Kelly or p. 63 of Walrand). Then

$$\tilde{\eta}(t) = \tilde{q}(Z(t+) - e_1 + e_2,\ Z(t+)) \qquad (38)$$

and

$$\begin{aligned}&\tilde{q}(x - e_1 + e_2,\ x)\\ &= \frac{\pi(x)}{\pi(x-e_1+e_2)}q(x, x-e_1+e_2)\\ &= \frac{\pi(x)}{\pi(x-e_1)}\frac{\pi(x-e_1)}{\pi(x-e_1+e_2)}q(x, x-e_1+e_2)\\ &= \frac{\theta_1}{\mu_1(x_1)}\frac{\mu_2(x_2+1)}{\theta_2}\mu_1(x_1)r_{12} = \frac{\theta_1}{\theta_2}\mu_2(x_2+1)r_{12} \qquad (39)\end{aligned}$$

where $\mu_i(x_i) = \min\{s_i, x_i\}\mu_i$ is the service rate at queue i when there are x_i customers at queue i, and $\theta = (\theta_1, \ldots, \theta_m)$ is the unique solution to the traffic rate equations

$$\theta = \lambda + \theta R \qquad (40)$$

with $\lambda = (\lambda_1, \ldots, \lambda_m)$ the vector of external arrival rates and $R \equiv (r_{ij})$ the routing matrix. From (38) and (39), we conclude that

$$\tilde{\eta}(t) = \begin{cases}(\theta_1/\theta_2)(Z_2(t+)+1)\mu_2 r_{12} & \text{if } Z_2(t+)+1 < s_2\\ (\theta_1/\theta_2)s_2\mu_2 r_{12} & \text{if } Z_2(t+)+1 \geq s_2.\end{cases} \qquad (41)$$

Since $\tilde{\eta}(t)$ depends on $Z(t+)$ only through $Z_2(t+)$, $\tilde{\eta}(t)$ is independent of $\hat{Z}_2(t) = X(t)$, so that reverse LBA holds and departures see time averages by virtue of Theorem 6.

By a minor modification of the arguments above, *we can also treat the entire network to obtain the Arrival Theorem*, but it is important to remember that the Arrival Theorem is not quite ASTA as we have defined it, because in the Arrival Theorem the customer going from queue 1 to queue 2 should not see itself, whereas our convention is to regard the process X as having left-continuous paths. The first way to treat the entire network is to augment the network by inserting an infinite-server queue with high service rate on the path from queue 1 to queue 2. If we consider arrivals, then we consider the customers flowing from the new queue to queue 2 and in the analysis delete the new queue. First, by the previous argument, the arrivals to queue 2 from the new queue see time averages in the original network (we delete the extra queue), independent of the service rate at the new queue. Second, as we increase the service rate at the new queue, the arrival times at queue 2 from the new queue approach the departure times from queue 1 of those customers going to queue 2. This argument is intuitively clear, but leaves a continuity proof, which can be made rigorous.

The second way to treat the entire network is to directly apply (8). For this purpose, we consider functions f of the form $f(x_1, \hat{x}_1) = f_1(x_1)f_2(\hat{x}_1)$. Let $\tilde{Z}(\infty)$ have the stationary distribution of $Z(T_n-)$. Then

$$\begin{aligned}&E[f(\tilde{Z}(\infty)]\\ &= \frac{E[\eta(t)f(Z(t-))]}{E[\eta(t)]}\\ &= \frac{E[\mu_1 r_{12}\min\{s_1, Z_1(t-)\}f_1(Z_1(t-))]E[f_2(\hat{Z}_1(t-))]}{\theta_1 r_{12}}\\ &= \sum_{n=1}^{\infty} \mu_1\theta_1^{-1}\min\{s_1, n\}f_1(n)\\ &\quad \cdot P(Z_1(t-) = n)E[f_2(\hat{Z}_1(t-))]\\ &= \sum_{n=1}^{\infty} f_1(n)P(Z_1(t-) = n-1)E[f_2(\hat{Z}_1(t-))]\\ &= E[f_1(Z_1(t-)+1)]E[f_2(\hat{Z}_1(t-))] \qquad (42)\end{aligned}$$

using the known form for the stationary distribution of $Z_1(t-)$, as in (39), which in turn implies the Arrival Theorem. (Expectations of functions of this form determine the distribution of the vector $Z(t-)$.)

Example 4. Consider a stationary *closed* Jackson network that has a fixed number K of customers circulating in it. Let $Z^c(t)$ represent the full state of the closed network and let $\tilde{Z}^c(\infty)$ have the stationary distribution of $Z^c(T_n-)$ where T_n are the transition epochs from queue 1 to queue 2. Let $Z_1^c(t)$ and $\hat{Z}_1^c(t)$ be defined for $Z^c(t)$ just as for $Z(t)$ in Example 3. It is known that arrivals at queue 2 do not see time averages in $\mathcal{N} - \{1\}$ just prior to arrival (nor in $\mathcal{N} - \{2\}$ just after arrival). Notice that $Z_1^c(t-)$ is no longer independent of $\hat{Z}_1^c(t-)$ because the sum of all the components is K. Thus, conditioning on $\hat{Z}_1^c(t-)$ gives some information about $Z_1^c(t-)$; evidently $\mu(t)$ and $U(t)$ are correlated.

However, as in Example 3, it is easy to obtain the Arrival Theorem for closed networks from (8). The argument is a minor modification of (42). To see this, let $Z^o(t)$ denote the process $Z(t)$ in the open network of Example 3 corresponding to the closed network with population K. The process Z^o can be constructed from Z^c by treating one queue in the closed model as an entrance-exit queue, i.e., by inserting an external Poisson process to this queue and letting all internal arrivals leave the network (see pp. 1914, 1932 of Whitt 1984). The key fact is that the (stationary) distribution of $Z^c(t-)$ for the closed network can be expressed as the conditional distribution of $Z^o(t-)$ for the open network given that the sum of the components of $Z^o(t-)$ is K.

For notational simplicity, we assume that $m = 2$. For the closed network, let $E[\cdot]_K$ denote the expectation operator given a total population of K. Then

$$E[f(\tilde{Z}^c(\infty))]_K = \frac{E[\eta(t)f(Z^c(t-))]_K}{E[\eta(t)]_K} = \frac{E[\mu_1 r_{12}\min\{s_1, Z_1^c(t-)\}f_1(Z_1^c(t-))f_2(\hat{Z}_1^c(t-))]_K}{E[\mu_1 r_{12}\min\{s_1, Z_1^c(t-)\}]_K}$$

$$= \frac{\sum_{n=1}^K \mu_1\theta_1^{-1}\min\{s_1, n\}f_1(n)f_2(K-n)P(Z_1^o(t-) = n)P(\hat{Z}_1^o(t-) = K-n)}{\sum_{n=1}^K \mu_1\theta_1^{-1}\min\{s_1, n\}P(Z_1^o(t-) = n)P(\hat{Z}_1^o(t-) = K-n)}$$

$$= \frac{\sum_{n=1}^K f_1(n)f_2(K-n)P(Z_1^o(t-) = n-1)P(\hat{Z}_1^o(t-) = K-n)}{\sum_{n=1}^K P(Z_1^o(t-) = n-1)P(\hat{Z}_1^o(t-) = K-n)}$$

$$= \frac{\sum_{n=0}^{K-1} f_1(n+1)f_2(K-1-n)P(Z_1^o(t-) = n)P(Z_1^o(t-) = K-1-n)}{P(Z_1^o(t-) + \hat{Z}_1^o(t-) = K-1)}$$

$$= E[f_1(Z_1^c(t-) + 1)f_2(\hat{Z}_1^c(t-))]_{K-1}. \quad (43)$$

Consequently, $\tilde{Z}^c(\infty)$ with population K is distributed as $Z^c(t) + e_1$ with population $K - 1$, which is the Arrival Theorem for closed Jackson networks. (Similar results hold for more general product-form networks, e.g., see Sections 1.7, 1.9, 1.10, 4.3, 4.4 and 4.11 of Walrand.)

6. CONDITIONAL ASTA

PASTA has been generalized to conditional probabilities by Van Doorn and Regterschot (1988), which is useful for treating a stochastic process in a random environment. (Related results were obtained by Georgiadis 1987.) In this section, we indicate how to generalize our ASTA results along the same lines. To a large extent, nothing new is required.

Augment the framework in Section 1 with a new process $Z \equiv \{Z(t): t \geq 0\}$ on the underlying probability space $(\Omega, \mathcal{F}, P)$ having the same sample path properties as X. We are interested in conditional probabilities of the form $P(X(t) \in A \mid Z(t) \in B)$, where B typically contains a single point. Paralleling (3), the two averages are

$$V^*(t) \equiv \frac{\int_0^t f[X(s), Z(s)]\, ds}{\int_0^t g[Z(s)]\, ds}$$

and (44)

$$W^*(t) \equiv \frac{\int_0^t f[X(s), Z(s)]\, dN(s)}{\int_0^t g[Z(s)]\, dN(s)}$$

where f and g are bounded measurable real-valued functions. However, paralleling (5), we actually treat

$$\bar{V}^*(t) \equiv \frac{E[\int_0^t f[X(s), Z(s)]\, ds]}{E[\int_0^t g[Z(s)]\, ds]}$$

and (45)

$$\bar{W}^*(t) \equiv \frac{E[\int_0^t f[X(s), Z(s)]\, dN(s)]}{E[\int_0^t g[Z(s)]\, dN(s)]}.$$

Paralleling (7), let

$$U_1(t) = f[X(t), Z(t)], \quad t \geqslant 0$$

and (46)

$$U_2(t) = g[Z(t)], \quad t \geqslant 0.$$

As before, we require that $U_1(t)$ and $U_2(t)$ have left-continuous sample paths, which can be obtained in general by restricting attention to continuous f and g. Recall that t_0 is defined in (9).

Theorem 7. *Suppose that WLAA holds for both pairs* (U_1, N) *and* (U_2, N) *and that at least one of a or b holds*:

a. $E[U_1(t)]$ *and* $E[U_2(t)]$ *are independent of* t;

b. there exists a positive u_0 *such that* $E[N(t, t + u)] = \lambda u$ *for* $t \geqslant 0$ *and* $0 \leqslant u \leqslant u_0$.

Then $\bar{V}^*(t) = \bar{W}^*(t)$ *for all* $t > t_0$, *so that* $\bar{V}^*(t) \to \bar{V}^*(\infty)$ *as* $t \to \infty$ *if and only if* $\bar{W}^*(t) \to \bar{W}^*(\infty)$ *as* $t \to \infty$, *in which case* $\bar{V}^*(\infty) = \bar{W}^*(\infty)$.

Proof. It suffices to apply the proof of Theorem 2 to the numerators and denominators of $\bar{V}^*(t)$ and $\bar{W}^*(t)$ in (45) separately, after dividing both by t in $\bar{V}^*(t)$ and $E[N(t)]$ in $\bar{W}^*(t)$.

The following is the generalization of the corollary to Theorem 2.

Corollary. *Suppose that*

a. $(X(t), Z(t)) \Rightarrow (X(\infty), Z(\infty))$ *as* $t \to \infty$ *and* $(X(T_n), Z(T_n)) \Rightarrow (\tilde{X}(\infty), \tilde{Z}(\infty))$ *as* $n \to \infty$;

b. $\bar{V}^*(t) \to \bar{V}^*(\infty) = E[f[X(\infty), Z(\infty)]]/E[g[Z(\infty)]]$ *and* $\bar{W}^*(t) \to \bar{W}^*(\infty) = E[f[\tilde{X}(\infty), \tilde{Z}(\infty)]]/E[g[\tilde{Z}(\infty)]]$ *as* $t \to \infty$ *for all bounded continuous real-valued f and g with* $E[g[Z(\infty)]] > 0$ *and* $E[g[\tilde{Z}(\infty)]] > 0$.

Suppose that at least one of c or d holds:

c. the distribution of $(X(t), Z(t))$ *is independent of* t;

d. there exists a positive u_0 *such that* $E[N(t, t + u)] = \lambda u$ *for* $t \geqslant 0$ *and* $0 \leqslant u \leqslant u_0$.

If WLAA holds for both pairs (U_1, N) *and* (U_2, N) *for all bounded continuous real-valued f and g, then*

$$P(X(\infty) \in A \mid Z(\infty) \in B) = P(\tilde{X}(\infty) \in A \mid \tilde{Z}(\infty) \in B)$$

for all B such that $P(Z(\infty) \in B) > 0$ *and* $P(\tilde{Z}(\infty) \in B) > 0$.

In fact, the corollary to Theorem 7 can be obtained directly from the corollary to Theorem 2 by interpreting the new process (X, Z) as the old process X. Then $f[X(t), Z(t)]$ and $g[Z(t)]$ are just two candidate bounded continuous real-valued functions of $(X(t), Z(t))$. Given $(X(\infty), Z(\infty)) \stackrel{d}{=} (\tilde{X}(\infty), \tilde{Z}(\infty))$, it is immediate that the conditional distributions are equal. Similarly, Theorem 3 can be extended by simply replacing X by (X, Z) in (15)–(18).

7. DISCUSSION OF THE LITERATURE

There are at least three approaches to the ASTA problem besides the limiting average approach used here, depending on the stochastic process theory to be applied.

7.1. Markov Processes and the Arrival Theorem

The first approach applies when X is a continuous-time Markov chain or a pure-jump Markov process and N is a point process generated by a subset of the jumps, as occurs with the flows in product-form queueing networks (see Kelly 1979, Melamed 1979a, b, 1982, Lavenberg and Reiser 1980, Sevcik and Mitrani 1981, Whittle 1986, Disney and Kiessler 1987, Walrand 1988 and Serfozo 1989a, b). An advantage of this approach is that the quantities of interest usually can be calculated explicitly. A major result within this Markov framework is the Arrival Theorem for product-form Markov queueing networks. It is significant that the Arrival Theorem can be regarded as a special case of the ASTA results, as illustrated in Examples 3 and 4 of Section 5. Indeed, the original motivation for this work was to find a common framework for PASTA and the Arrival Theorem. When we remove the queue from which a customer comes, the Arrival Theorem is a direct consequence of ASTA (but not PASTA, as we noted at the outset). When we consider the entire network, the Arrival Theorem is a consequence of (8). In Section 5 we only considered open and closed Jackson networks, but essentially the same analysis applies to more general product-form networks (Section 4.4 of Walrand). Our results provide another useful perspective on the Arrival Theorem, but we do not claim that they simplify the proof; for example, the Markov chain proofs in Serfozo and Walrand are already quite short.

7.2. Stationary Processes and the König-Schmidt Independence Condition

The second alternative approach applies when $\{X(t): t \geq 0\}$ and $\{X(T_n): n \geq 1\}$ are stationary processes related by the Palm theory (see Miyazawa 1977, König and Schmidt 1980a, b, 1989, Franken et al. 1981, Baccelli and Brémaud 1987, König, Schmidt and Van Doorn 1989, and Brémaud 1989, 1990). In particular, in the stationary process framework, the model considered here is a process with embedded marked point process (PMP) (p. 44 of Franken et al.). In the full stationary process framework, stronger results than Theorem 1 were established. First, it was established that if N is an exogenous Poisson arrival process, or any Poisson process such that future arrivals (the process $\{N(t + u) - N(t): u \geq 0\}$) are independent of the state $X(t)$ for all t, then in great generality the arrival-stationary distribution of $X(T_n)$ coincides with the time-stationary distribution of $X(t)$; see Sections 1.4–1.6 and 4.1 of Franken et al. In fact, in Theorem 1.6.6 there, which is based on König and Schmidt (1980a), it is shown that the Poisson property is *not* needed. They showed that for ASTA it suffices to have T_1 (the forward residual time until the next point in N) be independent of the current state $X(0)$.

This König–Schmidt independence condition is closed related to the WLAA introduced in Section 2. However, even in a stationary framework, neither implies the other and both suffer from the defect of considering the future behavior of N over an interval instead of at a point. If we strengthened WLAA to require *independence* of $U(t)$ and $N(t, t + u)$ for *all* t, $u \geq 0$, then WLAA would imply that $U(t)$ is independent of the interval to the next point in N after t. However, lack of correlation is strictly weaker than independence (see Remark 4). König, Schmidt and Van Doorn give several examples showing that ASTA can hold without the König–Schmidt independence condition. Further extensions have been provided after our paper: Brémaud (1990) presents a necessary and sufficient condition for ASTA which can be thought of as the infinitesimal limit of the König–Schmidt independence conditions. It is also similar to Section 3. König and Schmidt (1989) present a closely related necessary and sufficient condition for ASTA, which exploits certain families of regular subsets of the state space of $X(t)$, as on p. 48 of Franken et al. (1981).

A PASTA result in a stationary framework with an instantaneous proof was communicated much earlier by Strauch (1970), but there are two difficulties with Strauch's result. First, his condition that the conditional distribution of the current state $X(t)$ given the history $\{N(s): s \leq t\}$ of the Poisson process be independent of whether a point occurs at time t is not entirely obvious; it might be considered no less obvious than the desired conclusion itself. Second, Strauch does not justify working with conditional probabilities in which the conditioning event has probability zero; in fact, the Palm theory associated with stationary processes justifies this analysis, as is shown in Franken et al.

7.3. Martingales and the Stochastic Intensity

The third approach is based on the Martingale theory of point processes, as contained in Brémaud (1981) and Varaiya and Walrand (1981); this is the approach of Wolff (1982). It has the disadvantage of being rather technical, but the basic concepts get immediately to the heart of the issue. In particular, the conditional intensity μ in (8) is easily made precise using the stochastic intensity, say ν, associated with the Martingale theory; just let $\mu(t) = E[\nu(t) | X(t)]$. We give alternative proofs (of not entirely equivalent theorems) in this framework in Melamed and Whitt (1990). We also establish an anti-PASTA result there; that is, we show that ASTA implies that N must be Poisson in a certain Markov setting (see Miyazawa 1982, König, Miyazawa and Schmidt 1983 and Green and Melamed 1990). (Obviously the non-Poisson ASTA examples do not satisfy these conditions.)

Finally, martingales provide a natural way to relate the w.p.1 limits of the averages in (3) as well as the limits of the expectations in (6); see Georgiadis (1987) and Makowski et al. (1989).

7.4. Variants of the Covariance Formula

Brémaud (1989) derived a variant of the covariance formula (8) by applying a theorem of Papangelou (1972). Papangelou's theorem relates the stochastic intensity in the martingale theory of point processes as in Brémaud (1981) to the Palm measure in the stationary theory of point processes as in Franken et al. (1981); see pp. 24–26 of Baccelli and Brémaud (1987) and Brémaud (1989). Given a simple stationary point process N with respect to a measure P, Papangelou's theorem states that the associated Palm measure P^0 is absolutely continuous with respect to P if and only if N has a stochastic intensity ν, in which case the Radon–Nikodym derivative dP^0/dP is just $(\nu(t)/E[\nu(t)])dt$. Formula (8), modified by having ν

instead of μ, is a consequence of this result, although this evidently was not appreciated before Brémaud (1989). In this stationary-Martingale framework we can define μ by $\mu(t) = E[\nu(t) \mid X(t)]$ so that (8) is equivalent to the same statement with $\nu(t)$ in place of $\mu(t)$. (This step is discussed further in Melamed and Whitt 1990.)

Note that the strong equivalence $P = P^0$ holds if and only if $\nu(t) = E[\nu(t)]$ w.p.1, which in turn holds if and only if P is Poisson, by Watanabe's (1964) theorem (p. 25 of Brémaud 1981). The reason we get interesting results for non-Poisson processes is that we consider the special functions $f(x(0))$ where $\{x(t): t \geq 0\}$ is the sample path. We have shown that $\mu(t) = E[\mu(t)]$ w.p.1 is equivalent to LBA and ASTA (Theorem 4).

It is significant that (8) also can be obtained by a sample-path argument. Paralleling the elegant treatment of $L = \lambda W$ by sample-path methods in Stidham (1974), Stidham and El-Taha independently derived a version of (8) as well as other conservation laws. The approaches in Brémaud (1989) and Stidham and El-Taha are both appealing because they enable us to treat $L = \lambda W$ and ASTA in a common framework.

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