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AN LIL VERSION OF $L = \lambda W$*†

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This paper establishes a law-of-the-iterated-logarithm (LIL) version of the fundamental queueing formula $L = \lambda W$: Under regularity conditions, the continuous-time arrival counting process and queue-length process jointly obey an LIL when the discrete-time sequence of interarrival times and waiting times jointly obey an LIL, and the limit sets are related. The standard relation $L = \lambda W$ appears as a corollary. LILs for inverse processes and random sums are also established, which are of general probabilistic interest because the usual independence, identical-distribution and moment assumptions are not made. Moreover, an LIL for regenerative processes is established, which can be used to obtain the other LILs.

1. Introduction and summary. The fundamental queueing formula $L = \lambda W$ established mathematically by Little [18] and Stidham [24] represents a relation among strong laws of large numbers (SLLNs): For a large class of queueing systems, the time-average of the queue-length process converges w.p.1 (with probability one) to a limit L (obeys an SLLN) if the customer-averages of the interarrival times and the waiting times converge w.p.1 to limits λ^{-1} and W, respectively, and the limits are related by $L = \lambda W$. Underlying $L = \lambda W$ is a relation among cumulative processes in continuous time (the integral of the queue-length process) and in discrete-time (the sum of the waiting times). Except for remainder terms that usually are asymptotically negligible, the two cumulative processes are random time-transformations of each other. This relation among the stochastic processes is the basis for corresponding relations among other classical limit theorems besides SLLNs, such as central limit theorems (CLTs), weak laws of large numbers (WLLNs) and laws of the iterated logarithm (LILs). In [6] we established relations among all these classical limit theorems in the setting of $L = \lambda W$ by exploiting functional limit theorems, i.e., weak convergence of probability measures on the function space $D[0, \infty)$ and arguments related to the continuous mapping theorem, as in Billingsley [3]. In [9] we discussed applications of the CLTs to statistical estimation of queueing parameters.

It is also of interest to know whether it is possible to establish similar relations among the associated ordinary (nonfunctional) limit theorems. Since the ordinary limit theorems are weaker (except for the SLLN; see Theorem 4 of [8]), both the condition and the conclusion are weaker, so that neither result contains the other. In [8] we showed that it is indeed possible to establish similar relations among the corresponding ordinary CLTs and WLLNs. The purpose of this paper is to establish similar relations among the ordinary LILs. As in [8], we find that the ordinary limit theorems are

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harder, requiring extra conditions and very different arguments. Comparing the LIL results here with the stronger functional laws of the iterated logarithm (FLILs) in §7 of [6] also illustrates the advantage of the functional limit theorems when you can get them. The ordinary limit theorems are projections of the functional limit theorems; to consider only ordinary limit theorems is to live in flatland [1]; see Remark 1.3 and Example 1 below. However, the statements and proofs here have appeal because of their simplicity. We use elementary methods; e.g., no knowledge of functional limit theorems is required.

As in [6] and [8], we use the standard $L = \lambda W$ framework involving the sequence of ordered pairs of random variables $\{(A_k, D_k)\colon k \geqslant 1\}$ where $0 \leqslant A_k \leqslant A_{k+1}$ and $A_k \leqslant D_k$ for all k. This framework is obviously very general, so that there are many applications. (However, an even more general framework, encompassing the extension to $H = \lambda G$, relating more general customer averages and time averages in queueing models, is introduced and analyzed in [10].) In queueing, we interpret A_k and D_k as the arrival and departure epochs of the kth arriving customer, where arrival and departure are understood to be with respect to the system under consideration. For example, if we are interested in the waiting time before beginning service, then the relevant system is the waiting room or queue, not counting the servers, and the departure epochs D_k refer to the instants customers leave the queue and begin service.

Let the associate interarrival times be $U_k = A_k - A_{k-1}$ for $k \geqslant 1$ where $A_0 = 0$ without there being a 0th customer. Let the queue length at time t, $Q(t)$, be the number of k with $A_k \leqslant t \leqslant D_k$ and let the waiting time of the kth customer be $W_k = D_k - A_k$. Let $N(t)$ and $D(t)$ count the number of arrivals and departures, respectively, in the interval $[0, t]$.

The starting point for our ordinary-LIL version of $L = \lambda W$ is an ordinary joint LIL for A_n and the cumulative process associated with W_n; we *assume*

$$(1.1)\qquad \phi(n)\left(A_n - \lambda^{-1}n,\ \sum_{k=1}^{n} W_k - wn\right) \underset{\sqrt{}}{\rightarrow} K_{AW} \quad \text{as } n \to \infty,$$

where $0 < \lambda < \infty$, $w < \infty$,

$$(1.2)\qquad \phi(t) = (2t \log\log t)^{-1/2}, \qquad t \geqslant 3,$$

K_{AW} is a compact subset of R^2 with the usual Euclidean norm $\|\cdot\|$, and the notation $X_n \underset{\sqrt{}}{\rightarrow} K$ as $n \to \infty$ denotes that w.p.1 the sequence $\{X_n\colon n \geqslant 3\}$ is relatively compact with K as the set of limit points for all convergent subsequences. Recall that a set is relatively compact if its closure is compact. A sequence in R^d is relatively compact if every subsequence contains a convergent subsequence; the set of all limit points is thus compact; see Chapter 9 of Royden [22]. The LIL convergence is quite remarkable: It says that except for a set of sample paths of probability zero, each sample path of $\{X_n\colon n \geqslant 1\}$ is arbitrarily close to each point in K infinitely often, and arbitrarily close to each point not in K only finitely often. This kind of LIL is sometimes called a *compact LIL* to distinguish it from other forms, such as statements that only focus on the lim sup or lim inf. The references provide background on LILs.

Throughout this paper we use the standard LIL normalizing function (1.2), but the results also hold for other normalizing functions; see Remark (3.6) of [6]. Since $\phi(t)/t \to 0$, an LIL with the normalization (1.2) is a refinement of an SLLN. For practical purposes, $\phi(t)$ is usually not much greater than $(2t)^{1/2}$ since $\log\log t$ grows so slowly; e.g., for $t = 10^3$, 10^6 and 10^9, $\phi(t)/(2t)^{1/2} = 1.39$, 1.62 and 1.74 respectively.

Given (1.1), we have LILs for the marginals separately by simply applying the projection, e.g.,

$$\phi(n)\left(A_n - n\lambda^{-1}\right) \underset{\sqrt{}}{\to} K_A \quad \text{as } n \to \infty \tag{1.3}$$

where K_A is the projection of K_{AW} on its first coordinate. Since the projections are continuous, the marginal limit sets K_A and K_W are compact subsets of R. Usually the limit sets K are convex as well. (The only compact convex subsets of R are closed bounded intervals.) However, it is not to be expected that $K_{AW} = K_A \times K_W$.

The object now is to obtain LILs for the arrival counting process $N(t)$ and the cumulative process $\int_0^t Q(s)\,ds$, *assuming* the initial LIL (1.1). We first state two results that are of general probabilistic interest outside of queueing theory. Our first result is for the "inverse" processes $\{A_n: n \geqslant 1\}$ and $\{N(t): t \geqslant 0\}$ alone; our second result is for the family of random sums $\{\sum_{k=1}^{N(t)} W_k: t \geqslant 0\}$. Our proofs begin in §2.

THEOREM 1. *Assume that* $0 < \lambda < \infty$. *The LIL* (1.3) *holds if and only if*

$$\phi(t)\left(N(t) - \lambda t\right) \underset{\sqrt{}}{\to} K_N = -\lambda^{3/2} K_A \quad \text{as } t \to \infty. \tag{1.4}$$

in which case the limit sets K_A *and* K_N *are convex.*

REMARKS. (1.1) It is significant that there are no independence, identical-distribution or moment conditions in Theorem 1. Of course, such conditions play an important role in establishing (1.3) or (1.4), but they are not needed to go from one to the other. The SLLN, WLLN, and CLT analogs of Theorem 1 are Theorem 2(a) of [6] (well known), Theorem 3 of [8] and Theorem 6 of [8], respectively. Functional-limit-theorem analogs appear in §7 of [26] and references cited there. By Theorem 4 of [8], a SLLN is equivalent to the corresponding functional strong law.

(1.2) For the special case of a renewal process in which the renewal interval has finite second moments, (1.3) and its FLIL generalization are well known; see Strassen [25] and Gut [11]. In the renewal case, the LIL (1.4) follows immediately from the FLIL in Theorem 2.3 of Iglehart [15]. For the special case of a renewal process, $E(U_n^2) < \infty$, the LIL (1.3) and the CLT for A_n are all equivalent; see p. 507 of Kuelbs and Zinn [17]. By Theorem 1 above and Theorem 6 of [8], these are also equivalent to the LIL (1.4) and the CLT for $N(t)$. A recent strong approximation for renewal processes that yields an LIL is contained in Horvâth [13].

(1.3) The situation is easier when we can work with FLILs; we can then simply apply §7 of [26] to get the equivalence of FLILs, from which (1.3) and (1.4) follow. However, with the Skorohod J_1 topology on $D[0, \infty)$, see [3] and [26], the FLILs must have limit sets containing only functions with continuous paths. No such restriction is required for Theorem 1. Other FLILs can be obtained with the M_1 topology though; see Wichura [27] and §7 of [26]. The FLILs yield a stronger conclusion than the ordinary LILs. In particular, if either FLIL version of (1.3) or (1.4) holds, then as in Corollary 7.1 in [6] we can characterize the joint limiting behavior. In particular,

$$\phi(t)\left(A_{[\lambda t]} - t, N(t) - \lambda t\right) \underset{\sqrt{}}{\to} K_{AN} \quad \text{in } R^2 \quad \text{as } t \to \infty \quad \text{where} \tag{1.5}$$

$$K_{AN} = \{(x, -\lambda x): x \in K_A\}. \tag{1.6}$$

In this paper we establish an FLIL as well as an LIL for (1.1), and thus also (1.3), in a regenerative context; see Theorems 8 and 9 and Remark 1.9. Thus, we establish (1.5) as well as (1.3) and (1.4). However, our main focus is on ordinary LILs.

EXAMPLE 1. To see that the stronger joint limit (1.5) does not hold under (1.3) alone, let $\lambda = 1$ and

$$A_n = \begin{cases} 2^k + \phi(2^k)^{-1}, & 2^k \leqslant n \leqslant 2^k + \phi(2^k)^{-1}, \quad k \leqslant 2, \\ n, & \text{otherwise.} \end{cases} \tag{1.7}$$

Then $\phi(n)(A_n - n) \underset{\sqrt{}}{\to} [0,1]$ as $n \to \infty$ and $\phi(t)(N(t) - t)) \underset{\sqrt{}}{\to} [-1,0]$ as $t \to \infty$, but $\phi(t)(A_{[t]} - t, N(t) - t) \underset{\sqrt{}}{\to} K_{AN}$ as $t \to \infty$ where $K_{AN} = \{(0,0), (1-x, -x)\colon 0 \leqslant x \leqslant 1\}$. It is also not difficult to see that the corresponding FLIL for $\{A_n\colon n \geqslant 1\}$ in (1.7) does not hold with continuous limit functions. Finally, also note that, even though the limit sets K_A and K_N are convex, the limit set K_{AN} in this example is not even connected. ∎

Even when U_k and W_k are both nonnegative, the limit set K_{AW} in (1.1) need not be a convex subset of R^2.

EXAMPLE 2. To see that K_{AW} need not be convex, modify Example 1 above as follows. Let A_n and W_n be defined by

$$A_n = \begin{cases} 2^{2k} + \phi(2^{2k})^{-1}, & 2^{2k} \leqslant n \leqslant 2^{2k} + \phi(2^{2k})^{-1}, \quad k \geqslant 1, \\ n, & \text{otherwise,} \end{cases}$$

$$W_n = \begin{cases} 2^{2k+1} + \phi(2^{2k+1})^{-1}, & 2^{2k+1} \leqslant n \leqslant 2^{2k+1} + \phi(2^{2k+1})^{-1}, \quad k \geqslant 1, \\ n, & \text{otherwise.} \end{cases}$$

Then $K_{AW} = \{(0, x), (x, 0)\colon 0 \leqslant x \leqslant 1\}$. ∎

Our next result is for $N(t)$ and the random sum $\sum_{k=1}^{N(t)} W_k$. (A limit theorem similar to (1.10) below appears in Gut [11].) For this purpose, let Γ be the matrix

$$\Gamma = \begin{pmatrix} -\lambda & -\lambda w \\ 0 & 1 \end{pmatrix} \tag{1.8}$$

and note that Γ is invertible because $0 < \lambda < \infty$. We do not require that W_k be nonnegative here; in particular, the results of Theorem 2(b)–(e) are applicable not just to queues, but more generally as well. Elements of R^2 are regarded as row vectors.

THEOREM 2. *Assume that* $0 < \lambda < \infty$ *and* $w < \infty$.

(a) *If* W_k *is nonnegative, then the limit set of* $\{\phi(t)(\sum_{k=1}^{N(t)} W_k - \lambda w t)\colon t \geqslant 3\}$ *is convex.*

(b) *If* (1.1) *holds, then the limit set of*

$$\left\{\phi(t)\left(N(t) - \lambda A_{N(t)}, \sum_{k=1}^{N(t)} W_k - \lambda w A_{N(t)}\right)\colon t \geqslant 3\right\} \tag{1.9}$$

is a subset of $\lambda^{1/2} K_{AW} \Gamma$.

(c) *If* (1.1) *holds and* $U_n > 0$ *for all n w.p.*1, *then*

$$\phi(t)\left(N(t) - \lambda A_{N(t)}, \sum_{k=1}^{N(t)} W_k - \lambda w A_{N(t)}\right) \underset{\sqrt{}}{\to} \lambda^{1/2} K_{AW} \Gamma \quad \text{as } t \to \infty. \tag{1.10}$$

(d) *If* $\phi(n) U_n \to 0$ *w.p.*1 *as* $n \to \infty$, *then*

$$\phi(t)\left(t - A_{N(t)}\right) \to 0 \quad \text{w.p.1 as } t \to \infty. \tag{1.11}$$

(e) *Under the assumptions of* (c) *and* (d),

$$\phi(t)\left(N(t) - \lambda t, \sum_{k=1}^{N(t)} W_k - \lambda wt\right) \underset{\sqrt{}}{\to} \lambda^{1/2} K_{AW}\Gamma \quad \text{as } t \to \infty. \tag{1.12}$$

The nonnegativity of U_n and W_n is crucial for the convexity of the limit sets K_A and K_W.

EXAMPLE 3. To see that the nonnegativity is necessary for K_W to be convex, let W_n be defined so that $\sum_{k=1}^{n} W_k = nw + (-1)^n/\phi(n)$; then $K_W = \{-1, +1\}$. In contrast, convexity is easy in the classical case of partial sums of i.i.d. real-valued random variables with finite second moments because the individual terms are $o(\phi(n))$. ■

The extra conditions in Theorem 2(c)–(e) are needed for the conclusion (1.12).

EXAMPLE 4. To see that (1.12) need not hold under (1.1) alone, return to Example 2 and note that the limit set for $\phi(t)\{N(t) - \lambda t, \sum_{k=1}^{N(t)} W_k - \lambda wt)$ is $\{(-x, -x), (0, x)\colon 0 \leqslant x \leqslant 1\}$, which is not equal to $\lambda^{1/2} K_{AW}\Gamma = \{(0, x), (-x, 0)\colon 0 \leqslant x \leqslant 1\}$. ■

It is possible to remove the positivity condition in Theorem 2(c) by controlling the growth rate of both $N(t) - N(t-)$ and W_k, but something is needed.

EXAMPLE 5. We show that (1.10) need not hold without the assumed positivity for U_n. In the setting of Example 3 let $U_{2n} = 0$ for all n. The $\sum_{k=1}^{N(t)} W_k = wN(t) + \phi(N(t))^{-1}$, so that

$$\phi(t)\left[N(t) - \lambda t, \sum_{k=1}^{N(t)} W_k - \lambda wt\right] \underset{\sqrt{}}{\to} \left\{(x, wx + \lambda^{1/2})\colon x \in -\lambda^{3/2} K_A\right\} \quad \text{as } t \to \infty$$

by Theorem 1. ■

Unfortunately, we have difficulty in showing that $\sum_{k=1}^{N(t)} W_k$ and $\int_0^t Q(s)\,ds$ can be appropriately related, despite the strong foundation in §2 of [6]. We need to impose extra conditions on the fluctuations of (A_n, W_n). Our conditions are not too difficult to verify because they are for A_n and W_n separately.

THEOREM 3. *If $n^{-1}A_n \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, and there exist positive constants α and β with $\alpha \leqslant \beta \leqslant 1/2$ such that*

$$\text{(i) } n^{-\alpha} W_n \to 0 \quad w.p.1 \quad \text{as } n \to \infty \quad \text{and} \tag{1.13}$$

(ii) *for any $\epsilon > 0$,*

$$\left(A_{[n+\epsilon n^\beta]} - A_n\right)/\epsilon n^\beta \to \lambda^{-1} \quad w.p.1 \quad \text{as } n \to \infty, \tag{1.14}$$

then

$$t^{-\beta} Q(t) = t^{-\beta}(N(t) - D(t)) \to 0 \quad w.p.1 \quad \text{as } t \to \infty \quad \text{and} \tag{1.15}$$

$$t^{-(\alpha+\beta)}\left|\sum_{k=1}^{N(t)} W_k - \int_0^t Q(s)\,ds\right| \to 0 \quad w.p.1 \text{ as } t \to \infty. \tag{1.16}$$

We now combine Theorems 2 and 3 to obtain the LIL version of $L = \lambda W$.

THEOREM 4. *If* (1.1) *holds with $0 < \lambda < \infty$ and $w < \infty$, $U_n > 0$ for all n w.p.1, and* (1.13) *and* (1.14) *hold for positive constants α and β such that $\alpha \leqslant \beta$ and $\alpha + \beta < 1/2$, then*

$$\phi(t)\left(N(t) - \lambda t, \int_0^t Q(s)\,ds - \lambda wt\right) \underset{\sqrt{}}{\to} K_{NQ} \equiv \lambda^{1/2} K_{AW}\Gamma \quad \text{as } t \to \infty. \tag{1.17}$$

REMARK. (1.4) The fluctuation condition (1.14) may be interpreted as an a.s. version of the Anscombe [2] condition that is frequently used in random time-change weak convergence arguments. Condition (1.14) also implies that $\phi(n)U_n \to 0$ w.p.1 as required in Theorem 2(d). To see this, note that (1.14) implies that $\phi(n)(A_n - A_{[n-\epsilon n^\beta]}) \to 0$ w.p.1, while $0 \leqslant U_n \leqslant A_n - A_{[n-\epsilon n^\beta]}$.

It is obviously not trivial to verify fluctuation condition (1.14), but (1.14) is convenient to apply Borel-Cantelli arguments. Furthermore, (1.14) is a standard conclusion of a strong approximation theorem for the process $\{U_n\colon n \geqslant 1\}$; see Csörgő and Révesz [4] and Philipp and Stout [21]. For example, we apply Theorem 3.2.1 of Csörgő and Révesz [4] to obtain a cleaner sufficient condition for the case in which the interarrival times are i.i.d.

THEOREM 5. *If* $\{U_n\colon n \geqslant 1\}$ *is i.i.d. with* $E(U_n^{2/\beta}) < \infty$, $\beta < 1$, *then* (1.14) *is satisfied.*

REMARKS. (1.5) Our proof of Theorem 5 follows directly from a strong approximation result for partial sums of i.i.d. random variables, and a known result on the increments of Brownian motion. With more work, it should be possible to improve this result significantly. In fact, a referee has proved that (1.14) holds with $\beta = 1/2$ under the condition of Theorem 5 with $\beta = 1$. This improvement carries over to Theorem 6 (Remark 1.9 below), but *not* Theorems 4, 8 and 9 because Theorem 8 does not use (1.14) and Theorems 4 and 9 needs $\alpha + \beta < 1/2$.

(1.6) For $\beta = 1/3$, a weaker result than Theorem 5 (the same conclusion under a slightly stronger moment condition) can be established by a relatively elementary argument based on the Borel-Cantelli theorem. In particular, it is not difficult to show that if $\{U_n\colon n \geqslant 1\}$ is i.i.d. and $EU_n^{6(1+\delta)} < \infty$ for some $\delta > 0$, then (1.14) holds for $\beta = 1/3$.

(1.7) The real power of (1.14) is for treating *dependent* sequences $\{U_n\colon n \geqslant 1\}$ such as regenerative, ϕ-mixing and martingales. One way to do this is to apply strong approximation theorems; see Philipp and Stout [21]. The strong approximation theorem gives us $A_n - B(n) = O(n^{1/2-\delta})$ w.p.1 for some $\delta > 0$ where $\{B(t)\colon t \geqslant 0\}$ is a nice stochastic process such as Brownian motion. Then

$$\left(A_{[n+\epsilon n^\beta]} - A_n\right)/\epsilon n^\beta = \left(B([n+\epsilon n^\beta]) - B(n)\right)/\epsilon n^\beta = O(n^{1/2-\delta-\beta}),$$

so that in order to invoke known properties about $B(t)$ we need to take $\beta > 1/2 - \delta$. The quality of the strong approximation for A_n decreases as δ decreases. In turn, the requirement for W_n in (1.13) increases as δ decreases. (The permissible α decreases as β increases and δ decreases.)

(1.8) A major goal in this paper is to see what can be done relating the various LILs without resorting to FLILs. However, the strong approximation theorems used to establish (1.14) are intimately connected to FLILs. On the other hand, it is easy to see that the conditions of Theorem 4 do *not* also imply the FLIL generalization of (1.1). In particular, it is easy to see that (1.1) and (1.13) do not imply an FLIL for partial sums of the waiting times W_n.

EXAMPLE 6. To see that the conditions of Theorem 4 do not directly imply the FLIL analog of (1.1), let $A_n = n$ for all n. Trivially $\phi(n)(A_n - n) \underset{f}{\to} \{0\}$ as $n \to \infty$, so that it suffices to consider W_n alone. In particular, it suffices to show that an ordinary LIL for the partial sums of W_n plus (1.13) does not imply an associated FLIL for the partial sums of W_n. Suppose that $0 < \epsilon < \alpha < 1/2$ and let $\{n_k\colon k \geqslant 1\}$ be a rapidly increasing subsequence of positive integers such as $n_k = 2^{2^k}$. Let $W_j = 0$ except for special j, namely, $W_j = n_k^{\alpha-\epsilon}$, $m_k = n_k - (\phi(n_k)n_k^{\alpha-\epsilon})^{-1} < j \leqslant n_k$. Note that

$\phi(n_k)\sum_{j=m_k}^{j=n_k} W_j \approx 1$ for each k. It is easy to see that (1.13) holds and $\phi(n)\sum_{j=1}^{n} W_j \underset{\sqrt{}}{\to} [0,1]$ as $n \to \infty$, so that the LIL (1.1) holds. However, there is no FLIL for the partial sums of W_n. To see this, consider the subsequence $\{n_k\}$. As $n_k \to \infty$, $\{\phi(n_k)\sum_{j=1}^{[n_k t]} W_j\colon n_k \geqslant 1\}$ converges w.p.1 to 1 for $t = 1$ and converges w.p.1 to 0 for each other t in the neighborhood of 1. Since the prospective limit function is not in $D[0, \infty)$, the FLIL cannot hold. ■

To further illustrate the use of (1.14) in conjunction with the strong approximation theorems, we apply Theorem 3.2.1 of Csörgő and Révesz [4] again to establish another sufficient condition for (1.14) that will be part of a general result for regenerative processes.

THEOREM 6. *Let $\{A_n\colon n \geqslant 0\}$ be a (possibly) delayed regenerative process with regeneration times $T_{-1} = 0 \leqslant T_0 < T_1 < \cdots$. If, for $0 < \gamma < 1$,*

$$E\left[(T_1 - T_0)^{2/\gamma}\right] < \infty \quad \text{and} \quad E\left[\left(\sum_{k=T_0}^{T_1-1} U_k\right)^{2/\gamma}\right] < \infty, \tag{1.18}$$

then (1.14) *holds with $\lambda^{-1} = E(\sum_{k=T_0}^{T_1-1} U_k)/E(T_1 - T_0)$ and $\beta = \gamma$.*

REMARKS. (1.9) By applying Remark 1.5 (rather than Theorem 5) in the proof of Theorem 6, one can show that if (1.18) holds with $\gamma = 1$, then (1.14) is valid for regenerative processes with $\beta = 1/2$.

(1.10) We give a direct proof of Theorem 6 by applying Theorem 3.2.1 of Csörgő and Révesz [4] again. An alternate proof, following Remark (1.7) above, might be to directly apply a strong approximation theorem for regenerative processes. For example, the strong approximation theorem for Markov chains in §10 of Philipp and Stout [21] extends easily to regenerative processes. However, it does not yield (1.14). The condition $\delta < 2$ of p. 118 there prevents it from applying. We conjecture that Theorem 10.1 of Philipp and Stout [21] is valid for all $\delta > 0$, but their proof is only valid for $\delta < 2$. In particular, Lemma 10.2.1 there requires $\delta < 2$. The difficulty in obtaining strong approximation theorems for regenerative processes is illustrated by Horváth's [13] treatment of the special case of renewal processes.

We now develop a simple sufficient condition for (1.13). We use the following basic result.

PROPOSITION 7. *If $\{X_n\colon n \geqslant 1\}$ is a sequence of random variables for which $n^{-1}\sum_{k=1}^{n} X_k^p \to X$ w.p.1 as $n \to \infty$, where $p > 0$ and $|X| < \infty$ w.p.1, then*

$$n^{-1/p} \max_{1 \leqslant k \leqslant n} |X_k| \to 0 \quad w.p.1 \quad \text{as } n \to \infty.$$

We then can apply Birkhoff's ergodic theorem to obtain the following consequence. Note that there is no independence condition.

COROLLARY 1. *If $\{X_n\colon n \geqslant 1\}$ is stationary with $E(|X_n|^p) < \infty$ for $p > 0$, then $n^{-1/p}\max_{1 \leqslant k \leqslant n}|X_k| \to 0$ w.p.1.*

The next result combines Theorems 4 and 5 and Corollary 1 to obtain simple widely applicable sufficient conditions for the fluctuation conditions (1.13) and (1.14) and the LIL (1.17).

COROLLARY 2. (a) *If $\{U_n\colon n \geqslant 1\}$ is i.i.d. and $\{W_n\colon n \geqslant 1\}$ is stationary with $E(U_n^6) < \infty$ and $E(W_n^7) < \infty$, then both* (1.13) *and* (1.14) *hold with $\alpha = 1/7$ and $\beta = 1/3$.*

(b) *If, in addition* (1.1) *holds with* $0 < \lambda < \infty$, $w < \infty$ *and* $U_n > 0$ *w.p.*1, *then the LIL* (1.17) *holds.*

In order for Theorems 1–4 to have applied value, of course we need to have an initial LIL. To apply Theorem 4 in cases of interest, we need (1.1) as well as the fluctuation conditions (1.13) and (1.14). As a basis for a large class of applications, we provide sufficient conditions in the regenerative context. The rest of this section thus parallels [7].

We suppose that the sequence $\{(U_n, W_n): n \geq 0\}$ is regenerative (possibly delayed) with regeneration times $\{T_n: n \geq 0\}$ such that

$$(1.19)\quad E(T_1 - T_0)^2 < \infty, \quad E\left(\sum_{k=T_0}^{T_1-1} |U_k|\right)^2 < \infty \quad \text{and} \quad E\left(\sum_{k=T_0}^{T_1-1} |W_k|\right)^2 < \infty,$$

but here we do not need to require that U_k or W_k be nonnegative. (If nonnegativity does not hold, then we need the absolute values in (1.19).) Let λ and w be defined by

$$(1.20)\quad \lambda^{-1} = E\left(\sum_{k=T_0}^{T_1-1} U_k\right)\Bigg/E(T_1 - T_0) \quad \text{and} \quad w = E\left(\sum_{k=T_0}^{T_1-1} W_k\right)\Bigg/E(T_1 - T_0),$$

and let

$$(1.21)\qquad Z_j = \sum_{k=T_{j-1}}^{T_j-1} \left(U_k - \lambda^{-1}, W_k - w\right).$$

Let C be the covariance matrix of Z_k and assume that it is positive-definite, so that it possesses an invertible square root, i.e.,

$$(1.22)\qquad C = BB' = B'B.$$

Let $\|x\|$ be the usual Euclidean norm in R^2, i.e., $\|x\| = (x_1^2 + x_2^2)^{1/2}$ for $x = (x_1, x_2)$.

The key to proving the following results is the LIL independent random vectors in R^2 with uncorrelated marginals; see Lemma 2 of Finkelstein [5].

THEOREM 8. *If the basic sequence* $\{(U_n, W_n): n \geq 0\}$ *is regenerative satisfying* (1.19)–(1.22), *then the LIL* (1.1) *holds with*

$$(1.23)\qquad K_{AW} = \left\{xB\left(E[T_1 - T_0]\right)^{1/2}: \|x\| \leq 1\right\}.$$

REMARKS. (1.11) If the covariance matrix C of Z_k in (1.21) is singular, then the LIL holds with the limit set K_{AW} being a line segment.

(1.12) It turns out that the conditions of Theorem 8 also imply the FLIL analog of (1.1); cf. Remark 1.8 above. This is a consequence of FLIL for partial sums of i.i.d. random vectors; see Theorem 1 and Corollary 1 of Philipp [20]. (We provide the additional supporting details in §9.) In this special case, we thus can get (1.5) and (1.17) as well as (1.1) without checking the extra conditions in Theorem 4 or Theorem 6; see [7].

Finally, we combine Theorems 6 and 8 to obtain conditions to have both (1.17) and (1.1) in the regenerative context. In view of Remark (1.12) above, we are primarily checking that conditions (1.13) and (1.14) are reasonable. At the expense of some extra moment conditions, we obtain all the LILs directly via the fluctuation conditions (1.13) and (1.14), without reference to FLILs.

THEOREM 9. *If the basic sequence $\{(U_n, W_n): n \geqslant 0\}$ in the $L = \lambda W$ framework (with $U_n > 0$ and $W_n \geqslant 0$ w.p.1) is regenerative (possibly delayed) as above with*
(i) $E(T_1 - T_0)^6 < \infty$,
(ii) $E[(\sum_{k=T_0}^{T_1-1} U_k)^6] < \infty$,
(iii) $E(\sum_{k=T_0}^{T_1-1} W_k^7) < \infty$,
(iv) $C = E(Z_1^t Z_1)$ *for* Z_1 *in* (1.21) *is positive definite,*
then (1.1) *and* (1.10)–(1.17) *hold with* K_{AW} *in* (1.23), K_{NQ} *in* (1.17), $\alpha = 1/7$ *and* $\beta = 1/3$.

The rest of this paper is devoted to proving Theorems 1–8. We remark in closing that LILs for specific queues have been proved by Iglehart [14], [15]. Of course, there are many LILs in the literature that could be applied to obtain (1.1) besides Theorem 8. For example, a LIL for martingales is on p. 126 of Hall and Heyde [12].

2. Proof of Theorem 1. We first give a revealing proof under the assumption that $U_k > 0$ for all k, and then a quite different one without this condition. Throughout we assume that $0 < \lambda < \infty$. We use the following lemmas in this first proof and later.

LEMMA 0. *If $\{x_n: n \geqslant 1\}$ is a sequence of real numbers such that $x_n/n \to c > 0$ as $n \to \infty$, then $\phi(x_n)/\phi(n) \to c^{-1/2}$ as $n \to \infty$.*

LEMMA 1. *If* (1.3) *or* (1.4) *holds with* $0 < \lambda < \infty$, *then*
(a) $n^{-1}A_n \to \lambda^{-1}$ *as* $n \to \infty$ *and* $t^{-1}N(t) \to \lambda$ *as* $t \to \infty$ *w.p.*1, *and*
(b) $\phi(A_n)/\phi(n) \to \lambda^{1/2}$ *as* $n \to \infty$ *and* $\phi(N(t))/\phi(t) \to \lambda^{-1/2}$ *as* $t \to \infty$ *w.p.*1.

PROOF OF LEMMA 0. Since $x_n/n \to c$, $\log x_n - \log n \to \log c$, so that $\log x_n/\log n \to 1$. This in turn implies that $\log\log x_n - \log\log n \to 0$, so that $\log\log x_n/\log\log n \to 1$ and $\phi(x_n)/\phi(n) \to c^{-1/2}$.

PROOF OF LEMMA 1. (a) Suppose that (1.3) holds. Since $n\phi(n) \to \infty$, $n^{-1}A_n \to \lambda^{-1}$ directly from (1.3). Then $t^{-1}N(t) \to \lambda$ by Theorem 2(a) of [6]. A similar argument applies starting with (1.4).

(b) This is an immediate consequence of Lemma 0 and Lemma 1(a). ∎

PROOF WITH POSITIVE INTERARRIVAL TIMES. Fix a sample path and let K_A and K_N be the limit sets for (1.3) and (1.4). The positivity assumption implies that the set $\{\phi(A_n)(n - \lambda A_n): A_n \geqslant 3\}$ is a subset of $\{\phi(t)(N(t) - \lambda t): t \geqslant 3\}$. Since $\phi(A_n)/\phi(n) \to \lambda^{1/2}$ by Lemma 1(b), this implies that $-\lambda^{3/2}K_A \subseteq K_N$. On the other hand, the set $\{\phi(N(t))(A_{N(t)} - N(t)\lambda^{-1}): N(t) \geqslant 3\}$ coincides with the set $\{\phi(n)(A_n - \lambda^{-1}n): n \geqslant 3\}$. Since $\phi(N(t))/\phi(t) \to \lambda^{-1/2}$ by Lemma 1(b), $\phi(t)(N(t) - \lambda A_{N(t)}) \mathrel{\sqrt{\to}} -\lambda^{3/2}K_A$ as $t \to \infty$. A similar argument proves that $\phi(t)(N(t) - \lambda A_{N(t)+1}) \mathrel{\sqrt{\to}} -\lambda^{3/2}K_A$ as $t \to \infty$. Note, however, that

$$N(t) - \lambda A_{N(t)+1} \leqslant N(t) - \lambda t \leqslant N(t) - \lambda A_{N(t)}. \tag{2.1}$$

It follows, by the convexity of K_A (Lemma 3 below), that $K_N \subseteq -\lambda^{3/2}K_A$, which completes the proof. ∎

REMARK. (2.1) A somewhat shorter proof could be obtained from (2.1) under the assumption that $U_n = o(\phi(n))$; then $A_{N(t)+1} - A_{N(t)} = o(\phi(t))$. However, we did not assume that $U_n = o(\phi(n))$. The LIL only implies that $U_n = O(\phi(n))$.

PROOF IN THE GENERAL CASE. We no longer assume positivity of U_k. The new proof is obtained by combining the following three lemmas.

LEMMA 2. *For every sample path, the set of limit points of $\{\phi(t)(N(t) - \lambda t): t \geqslant 3\}$ as $t \to \infty$ is a closed interval.*

PROOF. Since the set of limit points is necessarily compact, we need only show that it is convex to complete the proof. Suppose that $a < b$ are two limit points of $\{f(t): t \geq 0\}$ as $t \to \infty$ where $f(t) = \phi(t)(N(t) - \lambda t)$. Then, for any $\epsilon > 0$, there exists a sequence $t_n \to \infty$ such that $f(t_{2n}) \geq b - \epsilon$ and $f(t_{2n+1}) \leq a + \epsilon$. Since the only discontinuities of f are positive jumps, it follows that for any $c \in [a + \epsilon, b - \epsilon]$, there exists $t_n' \in [t_{2n}, t_{2n+1}]$ such that $f(t_n') = c$. Hence, c is a limit point of $\{f(t): t \geq 0\}$ as $t \to \infty$. Since ϵ was arbitrary, any point c lying between a and b is a limit point, so the set of limit points is convex. ∎

LEMMA 3. *For every sample path, the set of limit points of* $\{\phi(n)(A_n - \lambda^{-1}n): n \geq 3\}$ *as* $n \to \infty$ *is a closed interval.*

PROOF. The reasoning is essentially the same as in Lemma 2, because the sequence $\{A_n: n \geq 1\}$ is nondecreasing. Of course, here there is discreteness, but it is easy to see that it is asymptotically negligible as $n \to \infty$: Any jump down at n is bounded in absolute value by $\lambda^{-1}\phi(n) \to 0$. Let $g(n) = \phi(n)(A_n - \lambda^{-1}n)$. If $g(n_{2k}) > b > a > g(n_{2k+1})$ for some increasing subsequence $\{n_k\}$, then for any c with $a < c < b$ and for any $\epsilon > 0$, $|g(n) - c| < \epsilon$ for infinitely many n. ∎

LEMMA 4. *For each sample path,*

$$\varliminf_{t\to\infty} \phi(t)(N(t) - \lambda t) = -\lambda^{3/2} \varlimsup_{n\to\infty} \phi(n)(A_n - \lambda^{-1}n) \quad \text{and}$$

$$\varlimsup_{t\to\infty} \phi(t)(N(t) - \lambda t) = -\lambda^{3/2} \varliminf_{n\to\infty} \phi(n)(A_n - \lambda^{-1}n).$$

PROOF. We use the basic inverse relation

$$A_n \leq t \quad \text{if and only if} \quad N(t) \geq n. \tag{2.2}$$

Suppose that $\phi(n)(A_n - \lambda^{-1}n) > c$, then

$$A_n > \lambda^{-1}n + c/\phi(n) \equiv t_n,$$

so that $N(t_n) < n$ and

$$\phi(t_n)(N(t_n) - \lambda t_n) < \phi(t_n)(n - \lambda t_n) = -\lambda c\phi(t_n)/\phi(n).$$

Since $t_n \to \infty$ and $\phi(t_n)/\phi(n) \to \lambda^{1/2}$ as $n \to \infty$ by Lemma 1(b), if $\phi(n)(A_n - \lambda^{-1}n) > c$ infinitely often (only finitely often), then $\phi(t_n)(N(t_n) - \lambda t_n) < -\lambda^{3/2}c + \epsilon$ infinitely often ($< -\lambda^{3/2}c - \epsilon$ only finitely often). A similar result holds starting with $\phi(t)(N(t) - \lambda t) < c$. ∎

REMARK. (2.2) Note that we never used the fact that the limit sets K_A and K_N are deterministic.

3. Proof of Theorem 2. First, part (a) is covered by the same proof as in Lemma 2: The process $\{\phi(t)(\sum_{k=1}^{N(t)} W_k - \lambda w t): t \geq 3]$ has only positive jump discontinuities.

To prove (b), we first note that

$$\left\{\phi(N(t))\left(A_{N(t)} - \lambda^{-1}N(t), \sum_{k=1}^{N(t)} W_k - N(t)w\right): N(t) \geq 3\right\} \tag{3.1}$$

is a random subset of

$$\left\{\phi(n)\left(A_n - \lambda^{-1}n, \sum_{k=1}^{n} W_k - nw\right): n \geq 3\right] \tag{3.2}$$

where $N(t) \to \infty$ as $t \to \infty$. Since $\phi(N(t))/\phi(t) \to \lambda^{-1/2}$ by Lemma 0,

$$\{X(t)\} \equiv \left\{\phi(t)\left(A_{N(t)} - \lambda^{-1}N(t), \sum_{k=1}^{N(t)} W_k - N(t)w\right): N(t) \geq 3\right\} \tag{3.3}$$

has limit set contained in $\lambda^{1/2}K_{AW}$. But the process in (1.9) is $X(t)\Gamma$ for $X(t)$ in (3.3) and Γ in (1.8), so that the limit set of (1.9) is indeed contained in $\lambda^{1/2}K_{AW}\Gamma$.

Part (c) follows because if $U_n > 0$ for all n, then $N(t)$ increases in jumps of size 1, so that the limit points of (3.1) and (3.2) coincide (as in the first proof of Theorem 1 under the positivity condition).

For part (d), if $\phi(n)U_n \to 0$ w.p.1, then $\phi(N(t))U_{N(t)-1} \to 0$ w.p.1, so that $\phi(t)U_{N(t)+1} \to 0$ w.p.1, which in turn implies (1.11).

Part (e) is obtained by combining (c) and (d). ∎

4. **Proof of Theorem 3.** Let $I(B)$ be the indicator function of the set B; i.e., $I(B)(\omega) = 1$ if $\omega \in B$ and 0 otherwise. Reasoning as in §5 of [8], recall that, for any $\gamma > 0$,

$$\begin{aligned} Q(t) = N(t) - D(t) &= \sum_{k=1}^{N(t)} I(A_k \leq t, A_k - W_k > t) \\ &\leq \sum_{k=1}^{N(t)} I(A_k \leq t, A_k + \gamma k^{\alpha} > t) + \sum_{k=1}^{\infty} I(W_k > \gamma k^{\alpha}) \\ &\leq \sum_{k=1}^{N(t)} I(A_k + \gamma k^{\alpha} > t) + \sum_{k=1}^{\infty} I(W_k > \gamma k^{\alpha}) \\ &\leq \sum_{k=1}^{N(t)} I(A_k + \gamma k^{\alpha} > A_{N(t)}) + \sum_{k=1}^{\infty} I(W_k > \gamma k^{\alpha}) \end{aligned}$$

where $\sum_{k=1}^{\infty} I(W_k > \gamma k^{\alpha})$ is finite w.p.1 by virtue of (1.13). Hence, to establish (1.15), it suffices to show that

$$t^{-\beta} \sum_{k=1}^{N(t)} I(A_k + \gamma k^{\alpha} > A_{N(t)}) \to 0 \quad \text{w.p.1},$$

which in turn is equivalent to

$$N(t)^{-\beta} \sum_{k=1}^{N(t)} I(A_k + \gamma k^{\alpha} > A_{N(t)}) \to 0 \quad \text{w.p.1}$$

because $t^{-1}N(t) \to \lambda$ by virtue of the assumed convergence $n^{-1}A_n \to \lambda^{-1}$ w.p.1 and Theorem 2(a) of [6]. Hence, it suffices to show that

$$n^{-\beta} \sum_{k=1}^{n} I(A_k + \gamma k^{\alpha} > A_n) \to 0 \quad \text{w.p.1}$$

which is implied by

$$n^{-\beta}\sum_{k=1}^{n} I(A_k+\gamma n^{\alpha}>A_n)\to 0 \quad \text{w.p.1},$$

but

$$(4.1)\qquad n^{-\beta}\sum_{k=1}^{n} I(A_k+\gamma n^{\alpha}>A_n)$$

$$\leqslant n^{-\beta}\sum_{k=1}^{[n-\epsilon n^{\beta}]} I(A_k+\gamma n^{\alpha}>A_n)+n^{-\beta}\sum_{k=[n-\epsilon n^{\beta}]+1} 1$$

$$\leqslant n^{-\beta}[n-\epsilon n^{\beta}+1]I\left(A_{[n-\epsilon n^{\beta}]}+\gamma n^{\beta}>A_n\right)+n^{-\beta}(\epsilon n^{\beta}+1).$$

By (1.14), γ can be chosen so that $A_n - A_{[n-\epsilon n^{\beta}]} < \gamma n^{\beta}$ only finitely often, w.p.1. Hence the first term on the right side of (4.1) is 0 for all sufficiently large n, w.p.1. Since ϵ was arbitrary, the proof of (1.15) is complete.

By Theorem 1 of [6],

$$(4.2)\qquad \left|\sum_{k=1}^{N(t)} W_k-\int_0^t Q(s)\,ds\right|\leqslant \sum_{k=D(t)+1}^{N(t)} W_k$$

$$\leqslant (N(t)-D(t))\max\{W_k\colon 1\leqslant k\leqslant N(t)\}.$$

Under (1.13), $\max\{W_k\colon 1\leqslant k\leqslant n\}/n^{\alpha}\to 0$ w.p.1, so that $\max\{W_k\colon 1\leqslant k\leqslant N(t)\}/N(t)^{\alpha}\to 0$ and $\max\{W_k\colon 1\leqslant k\leqslant N(t)\}/t^{\alpha}\to 0$ w.p.1. Hence, (1.16) follows by combining this with (1.15) and (4.2). ∎

5. Proof of Theorem 5. We apply Theorem 3.2.1 of Csörgő and Révesz [4], letting $a_n=\epsilon n^{\beta}$ and $H(x)=x^{2/\beta}$. By (3.1.6) there, for any $\epsilon>0$,

$$(5.1)\qquad \overline{\lim_{n\to\infty}}\, \psi_1(n)\left|A_{[n+\epsilon n^{\beta}]}-\epsilon n^{\beta}\lambda^{-1}-A_n\right|=\operatorname{Var}(U_1)\quad\text{where}$$

$$(5.2)\qquad \psi_1(n)=\left[2\epsilon n^{\beta}\left(\log(n/\epsilon n^{\beta})+\log\log n\right)\right]^{-1/2}.$$

As a consequence of (5.1), for all $\epsilon>0$,

$$\lim_{n\to\infty} n^{-\beta}\left|A_{[n+\epsilon n^{\beta}]}-A_n\right|=\epsilon\lambda^{-1}\quad\text{w.p.1.} \quad ∎$$

6. Proof of Theorem 6. By Theorem 5 (Theorem 3.2.1 of Csörgő and Révesz [4]) twice,

$$(6.1)\qquad \left(A_{T_{[n+\delta n^{\beta}]}}-A_{T_n}\right)\Big/\delta n^{\beta}\to E\left(\sum_{k=T_0}^{T_1-1} U_k\right)\quad\text{w.p.1}\quad\text{as } n\to\infty\quad\text{and}$$

$$(6.2)\qquad \left(T_{[n+\gamma n^{\beta}]}-T_n\right)\Big/\gamma n^{\beta}\to E(T_1-T_0)\quad\text{w.p.1}\quad\text{as } n\to\infty$$

for any $\delta, \gamma > 0$. Let $L(x) = \max\{k \geqslant -1: T_k \leqslant x\}$. (Note that $L(x)$ is well-defined.) From renewal theory, it follows that $L(n)/n \to 1/E(T_1 - T_0)$ w.p.1 as $n \to \infty$, so that $L(n) \to \infty$ as $n \to \infty$ w.p.1. Hence,

$$\left[A_{T_{[L(n)+\delta L(n)^\beta]}} - A_{T_{L(n)}}\right]\Big/\delta L(n)^\beta \to E\left(\sum_{k=T_0}^{T_1-1} U_k\right) \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

which implies that

$$\text{(6.3)} \qquad \left[A_{T_{[L(n)+\delta L(n)^\beta]}} - A_{T_{L(n)}}\right]\Big/n^\beta \to \frac{\delta E\left(\sum_{k=T_0}^{T_1-1} U_k\right)}{\left[E(T_1 - T_0)\right]^\beta} \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Since $L(n) \leqslant n$,

$$\left|A_n - A_{T_{L(n)}}\right| \leqslant \max_{0 \leqslant k \leqslant n+1}\left(\sum_{j=T_{k-1}}^{T_k} U_j\right),$$

so that, by our moment hypothesis and Corollary 1 to Proposition 7,

$$\text{(6.4)} \qquad \left(A_{T_{L(n)}} - A_n\right)\Big/n^{\beta/2} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Combining (6.3) and (6.4), we obtain

$$\text{(6.5)} \qquad \left[A_{T_{[L(n)+\delta L(n)^\beta]}} - A_n\right]\Big/n^\beta \to \frac{\delta E\left(\sum_{k=T_0}^{T_1-1} U_k\right)}{\left[E(T_1 - T_0)\right]^\beta} \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

By the same reasoning applied to (6.2) instead of (6.1),

$$\left[T_{[L(n)+\gamma L(n)^\beta]} - T_{L(n)}\right]\Big/\gamma L(n)^\beta \to E(T_1 - T_0) \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

so that

$$\text{(6.6)} \quad \left[T_{[L(n)+\gamma L(n)^\beta]} - T_{L(n)}\right]\Big/n^\beta \to \gamma\left[E(T_1 - T_0)\right]^{1-\beta} \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Since

$$|T_{L(n)} - n| \leqslant \max_{0 \leqslant k \leqslant n+1} |T_k - T_{k-1}|,$$

our moment of hypothesis and Corollary 1 to Proposition 7 yield

$$\text{(6.7)} \qquad |T_{L(n)} - n|\big/n^{\beta/2} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Combining (6.6) and (6.7), we obtain

$$\left[T_{[L(n)+\gamma L(n)^\beta]} - n\right]\Big/n^\beta \to \gamma\left[E(T_1 - T_0)\right]^{1-\beta} \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

By (6.7),

$$\left|T_{L(n+\epsilon n^\beta)} - (n + \epsilon n^\beta)\right|\Big/n^\beta \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

so that, choosing $\gamma = \epsilon/[E(T_1 - T_0)]^{1-\beta}$, we find that

$$\left[T_{[L(n)+\gamma L(n)^{\beta}]} - T_{L(n+\epsilon n^{\beta})}\right]/n^{\beta} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty. \tag{6.8}$$

We claim that (6.8) implies the limit

$$\left[\left[L(n) + \gamma L(n)^{\beta}\right] - L(n + \epsilon n^{\beta})\right]/n^{\beta} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

To see this, suppose that there exists $\eta > 0$ such that $[L(n) + \gamma L(n)^{\beta}] - L(n + \epsilon n^{\beta}) > \eta n^{\beta}$ *i.o.* Then (6.2) implies that

$$\left[T_{[L(n)+\gamma L(n)^{\beta}]} - T_{L(n+\epsilon n^{\beta})}\right]/n^{\beta} > (\eta/2)E(T_1 - T_0) \quad i.o.,$$

which contradicts (6.8); a similar proof works for $\eta < 0$.

Since $[L(n) + \gamma L(n)^{\beta}] = L(n + \epsilon n^{\beta}) + o(n^{\beta})$, it follows from (6.1) that

$$\left[A_{T_{[L(n)+\gamma L(n)^{\beta}]}} - A_{T_{L(n+\epsilon n^{\beta})}}\right]/n^{\beta} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

We conclude from (6.5) that

$$\left[A_{T_{L(n+\epsilon n^{\beta})}} - A_n\right]/n^{\beta} \to \frac{\gamma E\left(\sum_{k=T_0}^{T_1-1} U_k\right)}{[E(T_1 - T_0)]^{\beta}} = \epsilon\lambda^{-1} \quad \text{w.p.1} \quad \text{as } n \to \infty$$

for λ^{-1} defined in Theorem 6. It follows from (6.4) that

$$\left|A_{T_{L(n+\epsilon n^{\beta})}} - A_{[n+\epsilon n^{\beta}]}\right| = o(n^{\beta}),$$

so that

$$\left(A_{[n+\epsilon n^{\beta}]} - A_n\right)/\epsilon n^{\beta} \to \lambda^{-1} \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

as stated in (1.14). ■

7. Proof of Proposition 7. Under the condition, $n^{-1}X_n^p = n^{-1}\sum_{k=1}^{n} X_k^p - n^{-1}\sum_{k=1}^{(n-1)} X_k^p \to 0$ w.p.1, so that $n^{-1/p}X_n \to 0$ w.p.1. For $m \geq n$,

$$\begin{aligned} n^{-1/p} \max_{1 \leq k \leq n} |X_k| &\leq n^{-1/p} \max_{1 \leq k \leq m} |X_k| + \max_{m \leq k \leq n} k^{-1/p}|X_k| \\ &\leq n^{-1/p} \max_{1 \leq k \leq m} |X_k| + \max_{k \geq m} k^{-1/p}|X_k|. \end{aligned}$$

FIrst let $n \to \infty$, then $m \to \infty$. ■

8. Proof of Theorem 8. We first apply (1.22) to obtain i.i.d. vectors $Z_j B^{-1}$ with uncorrelated marginals; i.e.,

$$\begin{aligned} \text{Cov}(Z_1 B^{-1}) &= E\left[(Z_1 B^{-1})^t (Z_1 B^{-1})\right] = E\left[(B^{-1})^t Z_1^t Z_1 B^{-1}\right] \\ &= (B^{-1})^t B^t B B^{-1} = (B^t)^{-1} B^t = I. \end{aligned}$$

Next apply the LIL for i.i.d. random vectors in R^2 with uncorrelated marginals, see Lemma 2 of Finkelstein [5], to obtain

$$\phi(n)\left(\sum_{k=1}^{n} Z_k B^{-1}\right) \sqrt{\to} K_2 \quad \text{as } n \to \infty, \tag{8.1}$$

where K_2 is the unit disc in R^2, i.e., $K_2 = \{x \in R^2\colon \|x\| \leqslant 1\}$. As a consequence,

$$\phi(n) \sum_{k=1}^{n} Z_k \sqrt{\to} K_2 B \quad \text{as } n \to \infty. \tag{8.2}$$

From (8.2), we have

$$\phi(L(n)) \sum_{k=1}^{L(n)} Z_k \sqrt{\to} K_2 B \quad \text{as } n \to \infty. \tag{8.3}$$

where $L(n) = \max\{k\colon T_k \leqslant n\}$, $n \geqslant 0$, so that by Lemma 0

$$\phi(n) \sum_{k=1}^{L(n)} Z_k \sqrt{\to} K_{AW} \quad \text{as } n \to \infty. \tag{8.4}$$

Finally,

$$\left\|\left(A_n - n\lambda^{-1}, \sum_{k=1}^{n} W_k - nw\right) - \sum_{k=1}^{L(n)-1} Z_k\right\|$$

$$\leqslant \left\| \sum_{k=T_{L(n)}}^{n} \left(U_k - \lambda^{-1}, W_k - w\right)\right\|$$

$$\leqslant \sum_{k=T_{L(n)}}^{n} |U_k| + \sum_{k=T_{L(n)}}^{n} |W_k| + (\lambda^{-1} + w)\left(n + 1 - T_{L(n)}\right)$$

$$\leqslant \max_{1 \leqslant j \leqslant n}\left\{\sum_{k=T_j}^{T_{j+1}-1} |U_k|\right\} + \max_{1 \leqslant j \leqslant n}\left\{\sum_{k=T_j}^{T_{j+1}-1} |W_k|\right\} + \max_{1 \leqslant j \leqslant n}\left\{(T_{j+1} - T_j)(\lambda^{-1} + w)\right\},$$

which is $o(n^{-1/2})$ w.p.1 by our moment condition (1.19) and Corollary 1 to Proposition 7. ∎

9. Remark (1.10): The FLIL. In this final section we provide additional details to show that the conditions of Theorem 8 indeed imply the FLIL generalization of (1.1), i.e., the sequential compactness of

$$\left\{\phi(n)\left(A_{[n\cdot]} - [n\cdot]\lambda^{-1}, \sum_{k=0}^{[n\cdot]} W_k - [n\cdot]w\right) : n \geqslant 3\right\} \tag{9.1}$$

in the function space $D[0,\infty)$ w.p.1. (See [3], [6] and [26] for additional background on $D[0,\infty)$ and FLILs.) Thus (1.17) can also be obtained from §6 of [6] in this regenerative case under the weaker conditions of Theorem 8 instead of Theorem 9.

To establish the sequential compactness of (9.1), it suffices to identify for any subsequence $\{n_k\colon k \geqslant 1\}$ a further subsequence $\{n_i'\colon i \geqslant 1\}$ and a limit function y such that

$$(9.2)\qquad \sup_{0\leqslant t\leqslant \tau}\left\|\phi(n_i')\left(A_{[n_i't]}-[n_i't]\lambda^{-1},\ \sum_{k=0}^{[n_i't]}W_k-[n_i't]w\right)-y(t)\right\|\to 0$$

$$\text{w.p.1}\quad \text{as } n_i'\to\infty$$

for each $\tau > 0$. We use the topology of uniform convergence on compact subsets because the limit functions will all be continuous. We use a previously established FLIL to identify $\{n_i'\}$ and y given $\{n_k\}$. The stated conditions directly imply the FLIL for the i.i.d. random vectors at regeneration points (Theorem 1 and Corollary 1 of Philipp [20]); i.e., given any subsequence $\{n_k\}$ there exist a further subsequence n_i' and a continuous limit function x such that, for each $\tau > 0$,

$$(9.3)\qquad \sup_{0\leqslant t\leqslant \tau}\left\|\phi(n_i')\left(A_{T_{[n_i't]}}-T_{[n_i't]}\lambda^{-1},\ \sum_{k=0}^{T_{[n_i't]}}W_k-T_{[n_i't]}w\right)-x(t)\right\|\to 0$$

$$\text{w.p.1}\quad \text{as } n_i'\to\infty,$$

which clearly implies that for each τ

$$(9.4)\qquad \sup_{0\leqslant t\leqslant \tau}\left\|\phi(n_i')\left(A_{T_{L([n_i't])}}-T_{L([n_i't])}\lambda^{-1},\ \sum_{k=0}^{T_{L([n_i't])}}W_k-wT_{L([n_i't])}\right.\right.$$
$$\left.\left.-x\left(n_i'^{-1}L([n_i't])\right)\right)\right\|\to 0\quad \text{w.p.1}$$

Now,

$$\sup_{0\leqslant t\leqslant \tau}\left|A_{T_{L([nt])}}-A_{[nt]}\right|\leqslant \sup_{0\leqslant k\leqslant [n\tau]}\sum_{j=T_k}^{T_{k+1}-1}|U_j|=o(n^{1/2})\quad \text{w.p.1}$$

by the moment condition (1.9) and Corollary 1 to Proposition 7. Similarly,

$$\sup_{0\leqslant t\leqslant \tau}\left|\sum_{k=0}^{T_{L([nt])}}W_k-\sum_{k=0}^{[nt]}W_k\right|=o(n^{1/2})\quad \text{w.p.1}\quad \text{and}$$

$$\sup_{0\leqslant t\leqslant \tau}\left|T_{L([nt])}-[nt]\right|=o(n^{1/2})\quad \text{w.p.1},$$

so that by (9.4)

$$\sup_{0\leqslant t\leqslant \tau}\left\|\phi(n_i')\left(A_{[n_i't]}-[n_i't]\lambda^{-1},\ \sum_{k=0}^{[n_i't]}W_k-[n_i't]w\right)-x\left(n_i'^{-1}L([n_i't])\right)\right\|$$
$$\to 0\quad \text{w.p.1}.$$

By the FLLN, for each τ,

$$\sup_{0 \leqslant t \leqslant \tau} \left| n^{-1} T_{[nt]} - tE(T_1 - T_0) \right| \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

so that

$$\sup_{0 \leqslant t \leqslant \tau} \left| n^{-1} T_{L[nt]} - n^{-1} L([nt]) E(T_1 - T_0) \right| \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty.$$

Hence, since

$$\sup_{0 \leqslant t \leqslant \tau} \left| T_{L([nt])} - [nt] \right| = o(n^{1/2}) \quad \text{w.p.1},$$

$$\sup_{0 \leqslant t \leqslant \tau} \left| \frac{t}{E(T_1 - T_0)} - n^{-1} L([nt]) \right| \to 0 \quad \text{w.p.1}.$$

Using the uniform continuity of x on compacts, it follows that for each τ,

$$\sup_{0 \leqslant t \leqslant \tau} \left| x\left(\frac{L([n_i t])}{n_i} \right) - x\left(\frac{t}{E(T_1 - T_0)} \right) \right| \to 0 \quad \text{w.p.1} \quad \text{as } n_i \to \infty.$$

What we have shown so far is that the sequence (9.1) is indeed relatively compact and that the set of limit points L^* includes the set

$$\hat{L} = \{ y \colon y(t) = x(t/E(T_1 - T_0)), x \in L \}. \tag{9.5}$$

when L is the set of limit points of (9.3).

To show that $L^* = \hat{L}$, let $y \in L^*$. Then there exists a subsequence n_i such that for each τ,

$$\sup_{0 \leqslant t \leqslant \tau} \left\| \phi(n_i) \left(A_{[n_i t]} - [n_i t] \lambda^{-1}, \sum_{k=0}^{[n_i t]} W_k - [n_i t] w \right) - y(t) \right\| \to 0$$

$$\text{w.p.1} \quad \text{as } n_i \to \infty.$$

This implies that, for each τ,

$$\sup_{0 \leqslant t \leqslant \tau} \left\| \phi(n_i) \left(A_{T_{[n_i t]}} - T_{[n_i t]} \lambda^{-1}, \sum_{k=0}^{T_{[n_i t]}} W_k - T_{[n_i t]} w \right) - y\left(n_i^{-1} T_{[n_i t]} \right) \right\| \to 0$$

$$\text{w.p.1} \quad \text{as } n_i \to \infty.$$

By the same uniform continuity argument as above,

$$\sup_{0 \leqslant t \leqslant \tau} \left\| y\left(n_i^{-1} T_{[n_i t]} \right) - y\left(E(T_1 - T_0) t \right) \right\| \to 0 \quad \text{w.p.1} \quad \text{as } n_i \to \infty.$$

Thus, $y(E(T_1 - T_0)t) \in L$, which implies that $L^* \subseteq \hat{L}$, completing the proof. ∎

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References

[1] Abbott, E. A. (1952). *Flatland.* Sixth ed., Dover, New York.

[2] Anscombe, F. J. (1952). Large-Sample Theory of Sequential Estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[3] Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[4] Csörgő, M. and Révesz, P. (1981). *Strong Approximations in Probability and Statistics.* Academic Press, New York.

[5] Finkelstein, H. (1971). The Law of the Iterated Logarithm for Empirical Distributions. *Ann. Math. Statist.* **42** 607–615.

[6] Glynn, P. W. and Whitt, W. (1986). A Central-Limit-Theorem Version of $L = \lambda W$. *Queueing Systems* **1** 191–215.

[7] ______ and ______ (1987). Sufficient Conditions for Functional-Limit-Theorem versions of $L = \lambda W$. *Queueing Systems* **1** 279–287.

[8] ______ and ______ (1988). Ordinary CLT and WLLN versions of $L = \lambda W$. *Math. Oper. Res.* (to appear).

[9] ______ and ______ (1988). Indirect Estimation via $L = \lambda W$. *Oper. Res.* (to appear).

[10] ______ and ______ (1988). Extensions of the Queueing Relations $L = \lambda W$ and $H = \lambda G$. *Oper. Res.* (to appear).

[11] Gut. A. (1987). On the Law of the Interated Logarithm for Randomly Indexed Partial Sums with Two Applications. *Studia Sci. Math. Hungar.* (to appear).

[12] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and Its Applications.* Academic Press, New York.

[13] Horváth, L. (1984). Strong Approximation of Renewal Processes. *Stochastic Process Appl.* **18** 127–138.

[14] Iglehart, D. L. (1971). Multiple Channel Queues in Heavy Traffic. IV. Law of the Iterated Logarithm. *Z. Wahrsch. Verw. Gebiete* **17** 168–180.

[15] ______ (1971). Functional Limit Theorems for the Queue GI/G/1 in Light Traffic. *Adv. in Appl. Probab.* **3** 269–281.

[16] Kuelbs, J. (1985). The LIL When X Is in the Domain of Attraction of a Gaussian Law. *Ann. Probab.* **13** 825–859.

[17] ______ and Zinn, J. (1983). Some Results on LIL Behavior. *Ann. Probab.* **11** 506–557.

[18] Little, J. D. C. (1961). A Proof of the Queueing Formula: $L = \lambda W$. *Oper. Res.* **9** 383–387.

[19] Marcus, M. B. and Zinn, J. (1984). The Bounded Law of the Iterated Logarithm for the Weighted Empirical Distribution Process in the Non-i.i.d. Case. *Ann. Probab.* **12** 335–360.

[20] Philipp, W. (1979). Almost Sure Invariance Principles for Sums of B-Valued Random Variables. In *Probability in Banach Spaces II*, Lecture Notes in Math. **709** 171–193.

[21] ______ and Stout, W. (1975). *Almost Sure Invariance Principles for Partial Sum of Weakly Dependent Random Variables.* Mem. Amer. Math. Soc. No. **161**, Providence, RI.

[22] Royden, H. L. (1968). *Real Analysis.* Second ed. Macmillan, London.

[23] Sheu, S. S. (1974). Some Iterated Logarithm Results for Sum of Independent Two-Dimensional Random Variables. *Ann. Probab.* **2** 1139–1151.

[24] Stidham, S. Jr. (1974). A Last Word on $L = \lambda W$. *Oper. Res.* **22** 417–421.

[25] Strassen, V. (1964). An Invariance Principle for the Law of the Iterated Logarithm. *Z. Wahrsch. Verw. Gebiete.* **3** 211–226.

[26] Whitt, W. (1980). Some Useful Functions for Functional Limit Theorems. *Math. Oper. Res.* **5** 67–85.

[27] Wichura, M. J. (1974). Functional Laws of the Iterated Logarithm for the Partial Sums of i.i.d. Random Variables in the Domain of Attraction of a Completely Asymmetric Stable Law. *Ann. Probab.* **2** 1108–1138.

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