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COMPARING COUNTING PROCESSES AND QUEUES

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Abstract

Several partial orderings of counting processes are introduced and applied to compare stochastic processes in queueing models. The conditions for the queueing comparisons involve the counting processes associated with the interarrival and service times. The two queueing processes being compared are constructed on the same probability space so that each sample path of one process lies below the corresponding sample path of the other process. Stochastic comparisons between the processes and monotone functionals of the processes follow immediately from this construction. The stochastic comparisons are useful to obtain bounds for intractable systems. For example, the approach here yields bounds for queues with time-dependent arrival rates.

QUEUEING; STOCHASTIC ORDER; STOCHASTIC COMPARISONS; COUNTING PROCESS; POINT PROCESS ON THE LINE; BOUNDS

1. Introduction and summary

A rather obvious proposition about queueing systems is that the congestion should increase if the customers arrive more quickly or are served more slowly. However, when we try to make this proposition precise, we discover that there are many ways to define what is meant by both the conclusion and the conditions, and without the proper combination the proposition need not be valid. The purpose of this paper is to further clarify and amplify this basic proposition. We determine conditions under which a very strong form of the conclusion is valid. In particular, we find conditions under which it is possible to construct the two queueing stochastic processes being compared on the same underlying probability space so that each sample path of one process lies below the corresponding sample path of the other process. The possibility of such a construction has been shown by Kamae, Krengel and O'Brien (1977) to be equivalent to stochastic order of all the finite-dimensional distributions. The artificial construction of course immediately implies many stochastic comparisons. For example, various monotone functionals such as first-passage times will be stochastically ordered. Stochastic order for the limiting distributions is

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also a consequence of the sample-path orderings, but it is often possible to obtain this conclusion under weaker conditions; e.g., see p. 1628 of Jacobs and Schach (1972). The various stochastic comparisons are useful to obtain bounds for intractable systems. In fact, this research was largely motivated by the desire to obtain bounds for queueing systems in which the arrival rates are non-stationary. For this reason, the counting processes in Section 2 and the arrival processes of the queueing systems in Sections 3 and 4 are often arbitrary. It is helpful to think of non-homogeneous Poisson processes.

The investigation of sample-path comparisons for queues here is a sequel to the recent work of Sonderman (1978), (1979a, b). We obtain new results for queues principally by applying comparisons involving conditional failure rate functions; see Definition 1. These comparisons extend recent comparison results for renewal processes and semi-Markov processes by Miller (1979) and Sonderman (1980). Roughly speaking, the idea is to say that arriving faster means the arrival rate conditional on any history in one system is greater than the arrival rate conditional on any history in the other system. Of course, for two Poisson processes, this notion reduces to a simple comparison of the constant rates. The use of conditional failure rates makes it possible to compare queueing systems with not only non-Poisson arrival streams but also non-renewal arrival streams. The ordered conditional failure rates allow us to construct the slower counting process by thinning the faster counting process, so that the sequence of event epochs associated with the slower process becomes a subsequence of the sequence of event epochs associated with the faster process. This general thinning idea goes back at least to Jensen (1953) and has been used frequently for queueing comparisons in recent years, e.g., p. 1630 of Jacobs and Schach (1972), Miller (1979), Sonderman (1980) and references there. However, we use it in a more general way. Moreover, we believe the approach using conditional failure rates makes the essential properties transparent. However, the strong path-order conclusions we obtain in this paper require quite restrictive conditions. It is also important to study weaker conclusions that hold under less restrictive conditions. An account of related work that does this is contained in Stoyan (1977).

We now indicate how the rest of this paper is organized. We begin in Section 2 by defining several different partial orderings for counting processes, the strongest being the one discussed above involving the conditional failure rates. For the most part, Section 2 is a review, but we believe it provides a useful overview. In Section 3 we apply the partial orderings for counting processes to obtain sample-path comparisons for queueing processes. In Section 4 we extend the comparison results to queueing systems with a series or an acyclic network of stations. We obtain positive comparison results in this setting by introducing an appropriate partial ordering on the m-dimensional state space.

We conclude the paper in Section 5 by observing that the methods here can be used to compare generalized semi-Markov processes.

Since the arguments required here are similar to those displayed by Jacobs and Schach (1972), Miller (1979) and Sonderman (1979a, b), (1980), we frequently omit proofs.

2. Comparing counting processes

Let $A_i \equiv \{A_i(t), t \geqq 0\}$ be a counting process (point process on the positive half line), i.e., a stochastic process with non-decreasing right-continuous non-negative integer-valued sample paths, for $i = 1, 2$. Let $T_i = \{T_i(n), n \geqq 0\}$ be the associated sequence of event epochs, defined by

$$T_i(n) = \inf\{s \geqq 0 : A_i(s) \geqq n\}, \qquad n \geqq 1,$$

with $T_i(0) = 0$ and $T_i(n) = +\infty$ if $A_i(t) < n$ for all t. We shall use several different partial orderings of counting processes. To express them, let $L(A_i)$ represent the probability law (distribution) of the stochastic process A_i on the space of its sample paths. Also recall that a non-negative random variable has a failure rate $r(t)$ if its c.d.f. $F(t)$ is absolutely continuous with respect to Lebesgue measure and has a density $f(t)$; then $r(t) = f(t)/(1 - F(t))$ for all t such that $F(t) < 1$, see p. 53 of Barlow and Proschan (1975). (It is also possible to have discrete failure rates when $F(t)$ is absolutely continuous with respect to counting measure and has a probability mass function. Where failure rates are used in the following discussion, the results can be extended to cover this case as well as mixtures of the two.)

Definition 1. For $j = 1, \cdots, 5$, the ordering $A_1 \leqq_j A_2$ means:

($j = 1$) The conditional distributions $P([T_i(A_i(t)+1) - t] \leqq u \mid A_i(s),\ 0 \leqq s \leqq t)$ have failures rates for each t and i (almost surely with respect to A_i), and for some $\lambda(t)$ the failure rate for $i = 1$ ($i = 2$) is bounded above (below) by $\lambda(t)$, $t \geqq 0$.

($j = 2$) It is possible to construct on a common probability space two new counting processes $\tilde{A}_1$ and $\tilde{A}_2$ with associated event epoch sequences $\tilde{T}_1$ and $\tilde{T}_2$ such that $L(\tilde{A}_i) = L(A_i)$ for each i, the sequence $\{\tilde{T}_1(n), n \geqq 1\}$ is a subsequence of $\{\tilde{T}_2(n), n \geqq 1\}$ and $\tilde{A}_1(t) - \tilde{A}_1(t-) \leqq \tilde{A}_2(t) - \tilde{A}_2(t-)$ for all $t \geqq 0$ and all sample paths.

($j = 3$) It is possible to construct on the same probability space two counting processes $\tilde{A}_1$ and $\tilde{A}_2$ with associated event epoch sequences $\tilde{T}_1$ and $\tilde{T}_2$ such that $L(\tilde{A}_i) = L(A_i)$ for each i and $\tilde{T}_1(n) - \tilde{T}_1(n-1) \geqq \tilde{T}_2(n) - \tilde{T}_2(n-1)$ for all $n \geqq 1$ and all sample paths.

($j = 4$) It is possible to construct on the same probability space two new counting processes $\tilde{A}_1$ and $\tilde{A}_2$ such that $L(\tilde{A}_i)=L(A_i)$ for each i and $\tilde{A}_1(t)\leqq\tilde{A}_2(t)$ for all $t\geqq 0$ and all sample paths.

($j=5$) For all $t\geqq 0$, $A_1(t)$ is stochastically less than or equal to $A_2(t)$, i.e., $P(A_1(t)\geqq x)\leqq P(A_2(t)\geqq x)$ for all x and t.

Remarks. (1) We use the term 'partial ordering' loosely. As binary relations, the ordering $\leqq_5$ is not antisymmetric and the ordering $\leqq_1$ is not reflexive. If $L(A_1)=L(A_2)$, then $A_1\leqq_2 A_2$. However, $L(A_1)=L(A_2)$ and $A_1\leqq_1 A_2$ both hold if and only if A_1 and A_2 are Poisson processes with common intensity.

(2) This list of orderings does not include all the possibilities. For example, with respect to $\leqq_5$, it is possible to define other stochastic orderings between random variables; see Kirstein (1976) and Stoyan (1977) and references there.

(3) For the conditional probability distribution in the ordering $\leqq_1$ to be well defined, we need to work with regular conditional probabilities; see Section 4.3 of Breiman (1968) or Chapter V of Parthasarathy (1967). However, the sample paths of the counting processes belong to the function space $D[0,\infty)$; the function space $D[0,\infty)$ with the usual topology is metrizable as a complete separable metric space; and the Borel σ-field generated by this topology agrees with the standard notions of measurability; see Lindvall (1973) and Section VII of Parthasarathy (1967). Hence, the probability space we are dealing with is a standard Borel space, so that regular conditional probabilities exist.

We now indicate how the orderings are related. Let $i\rightarrow j$ mean that ordering $\leqq_i$ implies ordering $\leqq_j$ but they are not equivalent and let $i\leftrightarrow j$ mean the two orderings are equivalent.

Theorem 1. (a) In general $1\rightarrow 2\rightarrow 4\rightarrow 5$ and $3\rightarrow 4$.

(b) For two renewal processes,

$$1\rightarrow 2\rightarrow 3\leftrightarrow 4\leftrightarrow 5.$$

We briefly discuss Theorem 1 instead of providing a detailed proof because it contains only a minor extension of results in the literature. In particular, these orderings have been considered before for renewal processes. Miller (1979) showed that the ordering $\leqq_1$ implies the ordering $\leqq_2$ for renewal processes. Similar reasoning shows that implication holds in the more general setting here: If $A_1\leqq_1 A_2$, then the process A_1 can be constructed by thinning the process A_2; i.e., if the conditional failure rates at time t are $r_1(t)$ and $r_2(t)$ and an event occurs in process 2 at time t, then let an event also occur at time t in process 1 with probability $r_1(t)/r_2(t)$, and adjust the conditional failure rates to reflect the events which do occur. The orderings $\leqq_1$ and $\leqq_2$ were also applied to the comparison of queuing processes and semi-Markov processes by Sonderman (1979a,b), (1980). Miller (1979) introduced ordering $\leqq_1$ as a condition to get ordering $\leqq_2$. However, for results here and in Sonderman (1980) the stronger properties of ordering $\leqq_1$ are needed; ordering $\leqq_2$ is often not sufficient.

The ordering $\leqq_3$ for renewal processes follows from a well-known construction for real-valued random variables; see Lemma 1 of Sonderman (1979a). The ordering $\leqq_3$ was applied in queueing by Jacobs and Schach (1972). The ordering $\leqq_4$, which we also use for stochastic processes that do not have non-decreasing sample paths, was extensively studied and shown to be equivalent to the usual stochastic order of all finite-dimensional distributions by Kamae, Krengel and O'Brien (1977). Schmidt (1976) showed that the ordering $\leqq_5$ does not imply the ordering $\leqq_3$ in general.

To express the simple conditions for the orderings $\leqq_1$ and $\leqq_3$ for renewal processes, we use the following orderings for non-negative random variables X_1 and X_2.

Definition 2. (a) The ordering $X_1 \leqq_r X_2$ means that the distributions of X_1 and X_2 have failure rates $r_1(t)$ and $r_2(t)$, and $\inf_{t\geqq 0} r_1(t) \geqq \sup_{t\geqq 0} r_2(t)$.

(b) The ordering $X_1 \leqq_{st} X_2$ means that X_1 is stochastically less than or equal to X_2, i.e., $P(X_1 \geqq x) \leqq P(X_2 \geqq x)$ for all x.

It is not difficult to see that for renewal processes two of the orderings of counting processes are characterized by these two orderings for random variables applied to the time between renewals.

Theorem 2. Suppose $A_1(A_2)$ is a renewal counting process for which $X_1(X_2)$ is a time between renewals. Then

(a) $A_1 \leqq_1 A_2$ if and only if $X_1 \geqq_r X_2$;

(b) $A_1 \leqq_3 A_3$ if and only if $X_1 \geqq_{st} X_2$.

We now illustrate the ordering $\leqq_r$ with a few examples.

Examples. (1) Let $E(\lambda)$ be an exponentially distributed random variable with mean λ^{-1}. Obviously, $E(\lambda_1) \leqq_r E(\lambda_2)$, $E(\lambda_1) \leqq_{st} E(\lambda_2)$ and $\lambda_1 \geqq \lambda_2$ are equivalent.

(2) Let $G(\lambda, \alpha)$ be a random variable with a gamma distribution, i.e., with the density $g_{\lambda,\alpha}(t) = \lambda^\alpha t^{\alpha-1} e^{-\lambda t}/\Gamma(\alpha)$, $t \geqq 0$. Then $G(\lambda_1, \alpha_1) \leqq_r G(\lambda_2, \alpha_2)$ if and only if $\lambda_1 \geqq \lambda_2$ and $\alpha_1 \leqq 1 \leqq \alpha_2$; see pp. 73–75 of Barlow and Proschan (1975).

(3) The sum $E(\lambda_1) + \cdots + E(\lambda_n)$ of independent exponential random variables, often called a hypoexponential random variable, is IFR (increasing failure rate), p. 100 of Barlow and Proschan (1975). For $n \geqq 2$, the failure rate increases from 0 to $\min\{\lambda_1, \cdots, \lambda_n\}$. Hence, $E(\lambda_1) + \cdots + E(\lambda_n) \geqq_r E(\lambda)$ if and only if $\lambda \leqq \min\{\lambda_1, \cdots, \lambda_n\}$.

(4) Let $M(p_1, \cdots, p_n; \lambda_1, \cdots, \lambda_n)$ be the mixture of n exponential random variables with parameters $\lambda_1, \cdots, \lambda_n$ and weights $p_1, \cdots, p_n$, having density

$\sum_{i=1}^{n} p_i\lambda_i e^{-\lambda_i t}$, $t \geqq 0$. This mixture, often called a hyperexponential random variable, is DFR (decreasing failure rate), p. 103 of Barlow and Proschan (1975). The failure rate decreases from $\sum_{i=1}^{n} p_i\lambda_i$ to $\min\{\lambda_1, \cdots, \lambda_n\}$. Hence,

$$E(\lambda) \geqq_r M(p_1, \cdots, p_n; \lambda_1, \cdots, \lambda_n) \geqq_r E(\bar{\lambda})$$

if and only if $\lambda \leqq \min\{\lambda_1, \cdots, \lambda_n\}$ and $\sum_{i=1}^{n} p_i\lambda_i \leqq \bar{\lambda}$.

(5) The Weibull distribution with c.d.f. $F(t) = 1 - e^{-(\lambda t)^\alpha}$ has a failure rate $\alpha\lambda(\lambda t)^{\alpha-1}$, $t > 0$, $\alpha \neq 1$. Hence, for $\alpha \neq 1$, this distribution has no upper bound and only the trivial lower bound 0 in the $\leqq_r$ partial ordering.

We close this section by noting that all the partial orderings for counting processes except $\leqq_3$ extend to superpositions.

Theorem 3. For $i = 1, 2$, let A_{ij}, $j = 1, \cdots, n$, be independent counting processes. For $k = 1, 2, 4$ or 5, if $A_{1j} \leqq_k A_{2j}$ for all j, then $A_{11} + \cdots + A_{1n} \leqq_k A_{21} + \cdots + A_{2n}$.

Proof. For $k = 2$, 4 and 5, the result is trivial. For $k = 1$, note that the conditioning events for each superposition process form a sub-sigmafield of the conditioning events for each vector of component processes. Hence, first condition on the vector of component processes and then integrate. The basic principle is that $E(X \mid \mathcal{F}_1) \leqq \lambda$ a.s. if X is an integrable random variable, $E(X \mid \mathcal{F}_2) \leqq \lambda$ a.s. and $\mathcal{F}_1$ is a sub-sigmafield of the sigmafield $\mathcal{F}_2$.

3. Comparing queueing models

In the queueing systems we consider, customers are served in order of their arrival by the first available server without defections after entering the system. If there is a finite waiting room and the system is full when a customer arrives, that customer leaves without receiving service or affecting future arrivals. We also assume that the service-time distributions are independent of the arrival process. We use the notation $A/A/c/k$ to refer to such a system with c servers and a waiting room of size $k - c$, $1 \leqq c \leqq k \leqq \infty$. The A's mean that the arrival process and the service process are arbitrary. We substitute G, M, etc. for A in the usual way when the counting process is renewal, Poisson, etc. By having arbitrary arrival processes instead of renewal arrival processes, we are able to treat systems with stations in series where the arrivals at one station are the departures from the one before; see Theorem 12 and its corollary.

In this section, we first compare discrete-time queueing processes (Theorems 4–7) and then continuous-time queueing processes (Theorems 8–11). The conditions are expressed in terms of the counting processes $A_i \equiv \{A_i(t), t \geqq 0\}$ and $S_i \equiv \{S_i(t), t \geqq 0\}$ associated with the interarrival and service times in the ith system. Let $\{u_i(n), n \geqq 1\}$ and $\{v_i(n), n \geqq 1\}$ be the sequences of interarrival times and service times, where we associate the service times with successive arrivals. Then A_i and S_i are defined as

$$A_i(t)=\max\{n\geqq 0: u_i(1)+\cdots+u_i(n)\leqq t\},\qquad t\geqq 0,$$

and

$$S_i(t)=\max\{n\geqq 0: v_i(1)+\cdots+v_i(n)\leqq t\},\qquad t\geqq 0.$$

Note that $A_i(t)$ represents the number of arrivals in the interval $[0, t]$, but $S_i(t)$ does not represent the number of service completions in $[0, t]$ because there may be several servers and these servers typically will be idle from time to time.

In the following theorems, we use all the orderings in Definition 1. Each time ordering $\leqq_j$ appears as a condition it is possible to show by counterexample that ordering $\leqq_{j+1}$ would not suffice. We also use the orderings $\leqq_4$ and $\leqq_5$ in Definition 1 for stochastic processes without non-decreasing sample paths.

We begin by considering the sequence of waiting time (not counting service times) of successive customers in an $A/A/c/\infty$ system. Let $W_i\equiv\{W_i(n), n\geqq 0\}$ represent this sequence in the ith system. We also use the orderings $\leqq_4$ and $\leqq_5$ for such discrete-time processes. One of the most elementary comparison results is the following.

Theorem 4. If $A_1\leqq_3 A_2$ and $S_1\geqq_3 S_2$ in an $A/A/c/\infty$ system, then $W_1\leqq_4 W_2$.

Theorem 4 is essentially due to Kiefer and Wolfowitz (1955); it was proved by Jacobs and Schach ((1972), Theorem 2.2), but they only stated the following corollary.

Corollary. If $A_1\leqq_3 A_2$ and $S_1\geqq_3 S_2$ in a $G/G/c/\infty$ system, then $W_1\leqq_5 W_2$.

The proof also yields the ordering $\leqq_4$ for the Kiefer–Wolfowitz vector of workloads facing each server at successive arrival epochs, and thus also for the total workload in service time remaining in the system at arrival epochs. Related results when independence is relaxed have been proved by O'Brien (1975).

It is significant that Theorem 4 is not valid if the ordering $\leqq_3$ for either A_i or S_i is replaced by $\leqq_4$. As a contrast, consider the sequence $\{L(n), n\geqq 0\}$ where $L(n)$ represents the total work in service time to enter the system just prior to the arrival of the nth customer. It is trivial that $L_1\leqq_4 L_2$ if $A_1\leqq_4 A_2$ and $S_1\geqq_4 S_2$.

We now consider comparisons of embedded queue-length processes. Let $Q_i^A(n)$ be the number of customers in the ith system at the epoch of the arrival of the nth customer (but not including the nth customer). Jacobs and Schach ((1972), Theorems 3.1 and 3.2) concluded that the ordering $Q_1^A\leqq_5 Q_2^A$ is valid in a $G/G/c/\infty$ system assuming that $A_1 \leqq_3 A_2$ and $S_1 \leqq_3 S_2$. However, Theorem 3.2 there seems to be incorrect. Their proof of Theorem 3.1 yields the following result.

Theorem 5. If $L(A_1) = L(A_2)$ and $S_1 \geqq_3 S_2$ in an $A/A/c/\infty$ system, then $Q_1^A \leqq_4 Q_2^A$.

However, $A_1 \leqq_3 A_2$ and $L(S_1) = L(S_2)$ do not seem to be enough for $Q_1^A \leqq_5 Q_2^A$ in a $G/G/c/\infty$ system.

Counterexample. Consider two $G/D/1/\infty$ systems with $P(v_1(1) = 2) = P(v_2(1) = 2) = 1$, $P(u_1(1) = 0) = P(u_2(1) = 1) = 1 - \varepsilon$ and $P(u_1(1) = 6{\cdot}1) = P(u_2(1) = 6{\cdot}1) = \varepsilon$. Then

$$P(Q_1^A(4) = 0) = (1 - \varepsilon)^2 \varepsilon + O(\varepsilon^2)$$

and

$$P(Q_2^A(4) = 0) = O(\varepsilon^2),$$

while

$$P(Q_1^A(4) \geqq 3) = (1 - \varepsilon)^4 > P(Q_2^A(4) \geqq 3) = 0.$$

Hence, for sufficiently small ε, $Q_1^A(4)$ and $Q_2^A(4)$ are not stochastically comparable, i.e., $Q_1^A \leqq_5 Q_2^A$ fails.

The following positive result and its proof are similar to Theorem 1 of Sonderman (1979b). We now let the queueing systems have finite waiting rooms.

Theorem 6. If $c_1 \geqq c_2$, $c_1 + k_1 \leqq c_2 + k_2$, $A_1 \leqq_3 A_2$ and $S_1 \geqq_1 S_2$ in two $A/A/c/k$ systems, then $Q_1^A \leqq_4 Q_2^A$.

Proof. The argument is similar to the argument for Theorem 1 of Sonderman (1979b). We construct two new queueing systems on the same probability space with embedded sequences $\tilde{Q}_1^A$ and $\tilde{Q}_2^A$ so that $\tilde{Q}_1^A(n) \leqq \tilde{Q}_2^A(n)$ for all n and all sample points and $L(\tilde{Q}_i^A) = L(Q_i^A)$ for each i. To do this, use any two arrival processes $\tilde{A}_1$ and $\tilde{A}_2$ such that $\tilde{T}_1(n+1) - \tilde{T}_1(n) \geqq \tilde{T}_2(n+1) - \tilde{T}_2(n)$ for all n with $L(\tilde{A}_i) = L(A_i)$ for each i, which exist by the assumed ordering $\leqq_3$. Suppose all the arrivals and service completions have been generated for the two systems up to the epochs $\tilde{T}_1(n)$ and $\tilde{T}_2(n)$, respectively, so that $\tilde{Q}_1^A(k) \leqq \tilde{Q}_2^A(k)$, $0 \leqq k \leqq n$. We show how to guarantee that $\tilde{Q}_1^A(n+1) \leqq \tilde{Q}_2^A(n+1)$ while keeping the correct distributions. Let the conditional failure rates of the residual service times be consistent with the histories up to times $\tilde{T}_1(n)$ and $\tilde{T}_2(n)$, respectively. Using any construction after $\tilde{T}_i(n)$, let $U(n)$ be the elapsed time after these epochs, if any, until the numbers of customers in the two systems first agree, i.e., let $\tilde{Q}_i(t)$ be the number of customers in the ith system at time t and let

$$U(n)=\inf\{s\geqq 0:\tilde{Q}_1(\tilde{T}_1(n)+s)=\tilde{Q}_2(\tilde{T}_2(n)+s)\}.$$

If $U(n)\geqq \tilde{T}_2(n+1)-\tilde{T}_2(n)$, there is no problem because $\tilde{Q}_1^A(n+1)\geqq \tilde{Q}_2^A(n+1)$ is guaranteed. If $U(n)<\tilde{T}_2(n+1)-\tilde{T}_2(n)$, then make a special construction in the two systems beginning at times $\tilde{T}_1(n)+U(n)$ and $\tilde{T}_2(n)+U(n)$, respectively. Construct successive departures in the first system consistent with the conditional failure rates. Then use the orderings $S_1\geqq_1 S_2$ and $c_1\geqq c_2$ to construct the departures in the second system after $\tilde{T}_2(n)+U(n)$ by thinning the departure sequence for the first system after $\tilde{T}_1(n)+U(n)$. Perform this careful construction whenever the number of customers in the two systems are the same, but carry out any construction consistent with the conditional failure rates whenever the number of customers in the first system is strictly less than the number in the second system. Since $\tilde{T}_1(n+1)-\tilde{T}_1(n)\geqq \tilde{T}_2(n+1)-\tilde{T}_2(n)$ for each n, this procedure guarantees that $\tilde{Q}_1^A(n+1)\leqq \tilde{Q}_2^A(n+1)$.

A slightly stronger result follows by the same reasoning if the service times are exponentially distributed.

Theorem 7. If $c_1+k_1\leqq c_2+k_2$, $A_1\leqq_3 A_2$, $\mu_1\geqq\mu_2$ and $c_1\mu_1\geqq c_2\mu_2$ in two $A/M/c/k$ systems where μ_i^{-1} is the mean service time in the ith system, then $Q_1^A\leqq_4 Q_2^A$.

We now consider the continuous-time queue-length process. Let $Q_i(t)$ be the number of customers in the ith system at time t. There is plenty of evidence to show that $Q_1\leqq_5 Q_2$ need not hold if $A_1\leqq_3 A_2$ and $S_1\geqq_3 S_2$ in a $G/G/c/k$ system; see p. 1628 of Jacobs and Schach (1972) and the counterexamples in Sonderman (1978). The following comparison result is a generalization of Theorems 4.2 and 6.3 of Jacobs and Schach (1972) which follows by a minor modification of their proofs.

Theorem 8. If $A_1\leqq_2 A_2$ and $S_1\geqq_3 S_2$ in an $A/G/c/\infty$ system, then $Q_1\leqq_4 Q_2$.

Proof. The ordering $S_1\geqq_3 S_2$ means that the service times $\tilde{v}_i(n)$ can be constructed so that $\tilde{v}_1(n)\leqq\tilde{v}_2(n)$ for all n. However, do not assign the service times in order of arrival in system 2. Instead, let the service times $\tilde{v}_1(n)$ and $\tilde{v}_2(n)$ satisfying $\tilde{v}_1(n)\leqq\tilde{v}_2(n)$ be assigned to the customers arriving at $T_1(n)$. Let the extra arrivals in system 2 be assigned service times from an independent copy of S_2. Since S_2 is assumed to be a renewal process independent of A_2, this does not alter the distribution of S_2 or the associated queueing processes. After this construction, all the arrivals in system 1 are matched by arrivals in system 2 with longer service times. Moreover, there may be extra arrivals in system 2. Then apply a slight modification of the proof of Theorems

2.2 and 3.1 in Jacobs and Schach (1972) to show that first the waiting times and then the departure points of the matched customers are ordered, which implies the desired result.

Remark. It is apparent from the proof above that if $L(A_1)=L(A_2)$ instead of $A_1 \leqq_2 A_2$ in Theorem 8, then the result holds for an $A/A/c/\infty$ system.

It is easy to construct examples showing that Theorem 8 is not valid for a system with a finite waiting room, even if $L(S_1)=L(S_2)$. The following positive result requires stronger conditions than both Theorems 6 and 8. It is the natural generalization of the comparison result for $M/M/c/k$ systems that follows immediately from Theorem 5.1 of Kirstein (1976) or Theorem 3.2 of Sonderman (1980).

Theorem 9. If $c_1 \geqq c_2$, $c_1+k_1 \leqq c_2+k_2$, $A_1 \leqq_2 A_2$ and $S_1 \geqq_1 S_2$ in two $A/A/c/k$ systems, then $Q_1 \leqq_4 Q_2$.

Proof. Do a construction similar to the one outlined for Theorem 6. In particular, it is possible to piece together constructions for the subintervals $[T_1(n), T_1(n+1))$. Both systems have an arrival at $T_1(n)$. As long as the number in the first system is strictly less than the number in the second system, use any construction consistent with the conditional failure rates. However, whenever the number of customers in the two systems is equal, construct the successive departures in the second system by thinning the departures in the first stream. The assumption $S_1 \geqq_1 S_2$ implies that departures occur one at a time, so that the number of customers in the second system will never jump below the number in the first. Moreover, the orderings $c_1 \geqq c_2$ and $S_1 \geqq_1 S_2$ imply that the conditional failure rates are ordered as needed for the thinning when the number in each system is the same.

Just as with Theorems 6 and 7, a somewhat stronger result follows by the same reasoning in the case of exponentially distributed service times.

Theorem 10. If $c_1+k_1 \leqq c_2+k_2$, $A_1 \leqq_2 A_2$, $\mu_1 \geqq \mu_2$ and $c_1\mu_1 \geqq c_2\mu_2$ in two $A/M/c/k$ systems, then $Q_1 \leqq_4 Q_2$.

With the aid of Theorem 3, it is easy to extend Theorem 9 to systems with multiple heterogeneous arrival channels and service channels, as in Iglehart and Whitt (1970). For this model, the service times are associated with the server and all the channels are independent. Let $M(\lambda)$ be a Poisson process with intensity λ.

Theorem 11. Consider two multiple channel systems in which each has m arrival channels, but the first has c_1 servers and k_1 extra waiting spaces while the second has c_2 and k_2 with $c_1+k_1 \leqq k_2$. Suppose $A_{1j} \leqq_2 A_{2j}$, $1 \leqq j < m$; $S_{1j} \geqq_1 M(\lambda_{1j})$, $1 \leqq j \leqq c_1$; and $S_{2j} \leqq_1 M(\lambda_{2j})$, $1 \leqq j \leqq c_2$. Let the servers be labeled so that $\lambda_{11} \geqq \cdots \geqq \lambda_{1c_1}$, and $\lambda_{21} \geqq \cdots \geqq \lambda_{2c_2}$. If $\lambda_{1c_1} + \cdots + \lambda_{1(c_1-j)} \geqq \lambda_{21} + \cdots + \lambda_{2(j+1)}$ for $0 \leqq j < \min\{c_1, c_2\}$, then $Q_1 \leqq_4 Q_2$.

As a corollary to Theorem 11, we obtain $Q_1 \leqq_4 Q_2$ if the first system is an $A/M/1/\infty$ system where the single server works at rate μ and the second system is an $A/M^{(c)}/c/\infty$ system with $L(A_1)=L(A_2)$ and c heterogeneous servers working at rates $\mu_1, \cdots, \mu_c$ with $\mu \geqq \mu_1 + \cdots + \mu_c$. In other words, Theorem 11 contains Theorems 1 and 2 of Stidham (1970) and the first half of (7) in Theorem 4 of Yu (1974). Since Yu (1974) works with Erlang distributions, it is necessary to first work with exponential phases.

4. Tandem queues

We now compare systems with several stations in series. We assume that the sequences of service times associated with the different stations are independent. One result follows immediately from a comparison of departure processes in a single station; see Section 4 of Stoyan and Stoyan (1976). Let $D_i \equiv \{D_i(t), t \geqq 0\}$ be the counting process recording departures from a single station in the ith system.

Theorem 12. (a) If $A_1 \leqq_3 A_2$ and $L(S_1)=L(S_2)$ in two $A/A/1/\infty$ systems, then $D_1 \leqq_3 D_2$.

(b) If $A_1 \leqq_4 A_2$ and $S_1 \leqq_3 S_2$ in two $A/A/c/\infty$ systems, then $D_1 \leqq_4 D_2$.

(c) If $k_1 \leqq k_2$, $A_1 \leqq_1 A_2$ and $S_1 \leqq_3 S_2$ in two $A/G/c/k$ systems, then $D_1 \leqq_4 D_2$.

Proof. (a) Let $d_i(n)$ be the interval between the nth and $(n+1)$th departures in the ith system. Since

$$\begin{aligned} d_i(n) &= u_i(n) + W_i(n+1) - W_i(n) + v_i(n+1) - v_i(n) \\ &= v_i(n+1) + \max\{0, W_i(n) + v_i(n) - u_i(n)\} - (W_i(n) + v_i(n) - u_i(n)) \\ &= v_i(n+1) - \min\{0, W_i(n) + v_i(n) - u_i(n)\}, \end{aligned}$$

$d_1(n) \geqq d_2(n)$ if $W_1(n) \leqq W_2(n)$ and $u_1(n) \geqq u_2(n)$. Hence, Theorem 4 together with the conditions gives the desired result.

(b) Proceed by induction, using the vector-valued departure sequence in (2.1) of Sonderman (1979a).

(c) Apply Theorems 1 and 2 of Sonderman (1979a) after extending Theorem 2 to the non-Markovian arrival processes using the ordering $\leqq_1$.

Remarks. It is easy to construct examples showing that Theorem 12(a) does not hold for multiserver systems. For further comparisons of departure processes, see Sonderman (1979a, b).

Now consider m stations in series. Let S_{ij} be the service counting process at the jth station in system i and let W_{ij} be the waiting-time sequence at the jth station in the ith system, i.e., $W_{ij}(n)$ is the time the nth arriving customer must wait at the jth station before beginning service there in the ith system. Theorems 4 and 12(a) immediately imply the following corollary due to Niu ((1977), Section 3.6).

Corollary. If $A_1 \leqq_3 A_2$, $L(S_{ij}) = L(S_{2j})$ for $j = 1, \cdots, m-1$, and $S_{1m} \geqq_3 S_{2m}$ in two $A/A/1/\infty \to /A/1/\infty \to \cdots \to /A/c/\infty$ systems, then $W_{ij} \leqq_4 W_{2j}$ for each j, $1 \leqq j \leqq m$.

Remarks. Notice that in the corollary all stations but the last must have only one server, but the last can be general. Theorems 6 and 7 can be extended in the same way since the arrivals at each station are ordered by $\leqq_3$.

Notice that in order to obtain faster departure processes in Theorem 12 we had to assume that the services as well as the arrivals occur more quickly. As a consequence, the corollary comparing waiting times in tandem queues holds only when the service processes at all but the last station of the two systems are identical. We can obtain more interesting comparisons for tandem queues by considering the total waiting time before beginning service at the jth station instead of the waiting time at each station separately. Let $Y_{ij}(n) = W_{i1}(n) + v_{i1}(n) + W_{i2}(n) + \cdots + W_{ij}(n)$ where $v_{ij}(n)$ is the service time of the nth customer at the jth station in the ith system.

Theorem 13. If $A_1 \leqq_3 A_2$ and $S_{1j} \geqq_3 S_{2j}$ for $j = 1, \cdots, m$ in two $A/A/1/\infty \to /A/1/\infty \to \cdots \to /A/c/\infty$ systems, then $Y_{1j} \leqq_4 Y_{2j}$ for $j = 1, \cdots, m$.

Proof. We display the argument only for two single-server stations. For the case we treat, proceed by induction. Since the first station is covered by Theorem 4, we focus on Y_{i2}. Note that

$$\begin{aligned} Y_{i2}(n+1) &= W_{i1}(n+1) + v_{i1}(n+1) + W_{i2}(n+1) \\ &= W_{i1}(n+1) + v_{i1}(n+1) + \max\{0, W_{i2}(n) + v_{i2}(n) - u_i(n) - W_{i1}(n+1) \\ &\qquad - v_{i1}(n+1) + W_{i1}(n) + v_{i1}(n)\}, \\ &= W_{i1}(n+1) + v_{i1}(n+1) + \max\{0, Y_{i2}(n) + v_{i2}(n) - u_i(n) - W_{i1}(n+1) \\ &\qquad - v_{i1}(n+1)\}, \end{aligned}$$

so that $Y_{i2}(n+1)$ is a non-decreasing function of $W_{i1}(n+1)$ and $v_{i1}(n+1)$ as well as $Y_{i2}(n)$, $v_{i2}(n)$ and $-u_i(n)$. Assuming that $u_1(n) \geqq u_2(n)$, $v_{11}(n) \leqq v_{21}(n)$, $v_{12}(n) \leqq v_{22}(n)$ and $W_{11}(n) \leqq W_{21}(n)$ for all n and $Y_{12}(j) \leqq Y_{22}(j)$ for $j \leqq n$, we have $Y_{12}(n+1) \leqq Y_{22}(n+1)$ as desired.

Remark. It is easy to construct examples showing that $W_{1j} \leqq_5 W_{2j}$ for each j

need not hold under the conditions of Theorem 13. It is also easy to construct examples showing that Theorem 13 does not hold for multiserver stations because customers need not depart in the same order as they arrive.

For other processes and stronger conditions, it is possible to obtain comparison results for a series of multiserver stations. For example, we now state the generalization of Theorem 6. For this purpose, let $\boldsymbol{Q}_i^A \equiv (Q_{i1}^A, \cdots, Q_{im}^A)$ be the vector-valued process representing the number of customers at each station at successive arrival epochs. For vectors $\boldsymbol{x} \equiv (x_1, \cdots, x_m)$ and $\boldsymbol{y} = (y_1, \cdots, y_m)$, introduce the partial ordering $\boldsymbol{x} \leqq_s \boldsymbol{y}$ which means that $x_1 + \cdots + x_j \leqq y_1 + \cdots + y_j$ for $j = 1, \cdots, m$. Then the ordering $\boldsymbol{Q}_1^A \leqq_4 \boldsymbol{Q}_2^A$ is the extension of the ordering $\leqq_4$ to vector-valued processes using the ordering $\leqq_s$ on R^m.

Theorem 14. If $c_{1j} \geqq c_{2j}$, $c_{1j} + k_{1j} \leqq c_{2j} + k_{2j}$ and $S_{1j} \geqq_1 S_{2j}$ for $j = 1, \cdots, m$, and $A_1 \leqq_3 A_2$ in two $A/A/c_1/k_1 \rightarrow /A/c_2/k_2 \rightarrow \cdots \rightarrow /A/c_m/k_m$ systems. Then $\boldsymbol{Q}_1^A \leqq_4 \boldsymbol{Q}_2^A$.

Proof. The orderings $S_{1j} \geqq_1 S_{2j}$ guarantee that at most one departure occurs in each system at any time. The thinning argument used in the proof of Theorem 6 applies again here.

Remarks. Corresponding generalizations of Theorems 7–11 are also easy to state and prove. In the same way, acyclic networks of queueing stations can be compared.

5. Generalized semi-Markov processes

The queueing models studied here can all be represented in the framework of (denumerable-state) generalized semi-Markov processes (GSMPS); see Schassberger (1976) plus references there. Moreover, the comparison results here can also be expressed in that framework, but we shall not do so to avoid the complicated notation. Comparisons of GSMPS can be viewed as generalizations of comparisons of semi-Markov processes as treated by Sonderman (1980). When the state space of the GSMP is one-dimensional, the comparison results are similar to those in Section 3 here, especially Theorem 11. When the state space is multidimensional, we introduce a partial ordering to obtain positive results, as in Theorem 14 in Section 4.

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