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COMPLEMENTS TO HEAVY TRAFFIC LIMIT THEOREMS FOR THE $GI/G/1$ QUEUE

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Abstract

A bound on the rate of convergence and sufficient conditions for the convergence of moments are obtained for the sequence of waiting times in the $GI/G/1$ queue when the traffic intensity is at the critical value $\rho = 1$.

CONGESTION; QUEUES; $GI/G/1$ QUEUE; WAITING TIMES; UNSTABLE QUEUES; HEAVY TRAFFIC; LIMIT THEOREMS; RATE OF CONVERGENCE; CONVERGENCE OF MOMENTS

1. Introduction

This note provides two complements to heavy traffic limit theorems for the sequence of waiting times of successive customers in the standard $GI/G/1$ queue when the traffic intensity is at the critical value $\rho = 1$. In particular, we obtain a bound on the rate of convergence and a condition for convergence of moments. Since the sequence of waiting times $\{W_n, n \geqq 0\}$ can also be viewed as a random walk with an impenetrable barrier at the origin, our results also apply to such random walks.

The two theorems reported in Sections 3 and 4 here come from Sections 4.3 and 4.8 of the author's doctoral dissertation [19] which was written under the direction of Donald L. Iglehart. A rather extensive survey of the heavy traffic literature was provided in Chapter 2 of [19] and will be updated in [22]. The first heavy traffic work was done by Kingman ((1961), (1965)), Prohorov (1963), and Borovkov ((1964), (1965)). Recent heavy traffic work in the context of the weak convergence theory for probability measures on function spaces has been done by Iglehart (1969), Iglehart and Whitt (1970a, b) and Whitt ((1969), (1970), (1971a, b)). Since we treat here only one of the many queueing processes in only one case of heavy traffic, much is yet to be done on the topics discussed in this note.

2. Preliminaries

Let u_n represent the interarrival time between the nth and $(n+1)$th customers; let v_n represent the service time of the nth customer; let $X_n = v_n - u_n$, and let

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$\sigma^2 = \sigma^2(X_n) = \sigma^2(v_n) + \sigma^2(u_n)$, $0 < \sigma^2 < \infty$. Also assume the first (0th) customer arrives at $t = 0$, but the initial conditions are not critical for the heavy traffic limit theorems by virtue of Theorem 4.1 of Billingsley (1968) (cf. [19], Section 4.5).

Theorem 2.1. In the $GI/G/1$ queue with $\rho = 1$ and σ defined above

$$\lim_{n\to\infty} P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} = \begin{cases} (2/\pi)^{\frac{1}{2}} \int_0^x e^{-y^2/2} dy, & x \geqq 0 \\ 0, & x < 0. \end{cases}$$

The limiting distribution in Theorem 2.1 is the positive normal distribution which is known to be the weak convergence limit in R^1 of $\{(\sigma^2 n)^{-\frac{1}{2}} \max_{0 \leqq k \leqq n} S_k$, $n \geqq 1\}$, where $S_0 = 0$ and $S_k = X_0 + \cdots + X_{k-1}$, $k \geqq 1$ (cf. Erdös and Kac (1946)). Since Lindley (1952) and Pollaczek (1952) had observed that

$$P\{W_n \leqq x\} = P\{\max_{0 \leqq k \leqq n} S_k \leqq x\} \tag{2.1}$$

for all $x \geqq 0$ and any $n \geqq 0$, Theorem 2.1 was established early. Theorem 2.1 was also obtained as a special case of more general results by Prohorov (1963).

In [19] and [20] we obtained weak convergence generalizations of Theorem 2.1 based on the relationship

$$W_n = S_n - \min_{0 \leqq k \leqq n} S_k, \qquad n \geqq 0, \tag{2.2}$$

which holds everywhere instead of only in distribution. Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of random functions induced in the function space $D \equiv D[0,1]$ by double sequences $\{S_n^i\}$ and $\{W_n^i\}$ corresponding to a sequence of single-server (not necessarily $GI/G/1$) queueing systems:

$$\xi_n(t) = S_{[nt]}^n/\phi(n), \qquad 0 \leqq t \leqq 1, \tag{2.3}$$

and

$$\eta_n(t) = W_{[nt]}^n/\phi(n), \qquad 0 \leqq t \leqq 1. \tag{2.4}$$

For background on weak convergence, see Billingsley (1968). As a consequence of the continuous mapping theorem (Theorem 5.1 of [1]), we proved (Theorem 4.2 of [19] or Theorem 1 (i) of [20]) for arbitrary first-come-first-served single server queues (the i.i.d. assumptions may be relaxed):

Theorem 2.2. Let ζ be an arbitrary process in $D[0,1]$. If $\xi_n \Rightarrow \zeta$, then $\eta_n \Rightarrow f(\zeta)$, where $f: D \to D$ is the uniformly continuous function, defined for any $x \in D$ by

$$f(x)(t) = x(t) - \inf_{0 \leqq s \leqq t} x(s), \qquad 0 \leqq t \leqq 1.$$

Using Donsker's theorem (Theorem 16.1 of [1]), we then immediately obtained

Theorem 2.3. If $\rho = 1$ in a $GI/G/1$ queue with σ above and $\phi(n) = \sigma n^{\frac{1}{2}}$, then $\eta_n \Rightarrow f(\xi)$ in $D[0,1]$, where f is defined in Theorem 2.2, ξ is the Wiener process in $D[0,1]$, and $f(\xi)$ is equivalent to the one-dimensional Bessel process $|\xi|$.

Theorem 2.1 can then be obtained as a corollary to Theorem 2.3 by applying the continuous mapping theorem with the projection at $t = 1$.

3. Rate of convergence

After having obtained limit theorems, it is natural to look next for asymptotic expansions, correction factors, and bounds on the rate of convergence. Prohorov ((1956), Chapter 4) has done some work in this direction for weak convergence theorems and Prohorov (1963) and Borovkov (1964) have done likewise for queues in heavy traffic, but it appears that much more needs to be done in this area. Here we give a bound on the rate of convergence in Theorem 2.1 which is obtained from Theorem 2.3 and recent results of Rosenkrantz (1968) and Heyde (1969).

Rosenkrantz ((1968), Theorem 5) exploited the representation theorem of Skorohod (1965) to obtain a general theorem on the rates of convergence for functionals of the random functions in $C[0,1]$ associated with Donsker's theorem. Heyde (1969) sharpened the bound. We will not restate their theorem here. It leads to the following result.

Theorem 3.1. In the $GI/G/1$ queue with $\rho = 1$, if $E(|v_n - u_n|^{2+a}) = b$, where $0 < b < \infty$ and $0 < a$, then there exists a constant A such that for all $x \geqq 0$ and $n \geqq 0$

$$\left| P\{W_n/\sigma n^{\frac{1}{2}} \leqq x\} - (2/\pi)^{\frac{1}{2}} \int_0^x e^{-y^2/2}\, dy \right| \leqq A(\log n)^{\lambda} n^{-\mu},$$

where $\lambda = (1 + a/2)/(a+3) < \frac{1}{2}$ and $\mu = \min(a, 1 + a/2)/2(a+3)$.

Proof. Since the function f in Theorem 2.2 is uniformly continuous, it follows that the composite functional obtained by taking the projection at $t = 1$ is also uniformly continuous. The constant $2n(0)$, where n is the standard normal density, supplies the constant L for the Rosenkrantz-Heyde theorem, which is apparent from Theorem 2.1. Finally the queueing functional is the same for the linearly interpolated versions in $C[0,1]$ as it is for (2.3) and (2.4) in $D[0,1]$.

Note that we can also obtain bounds on the rate of convergence for many other functions of the sequence of waiting times by considering the composition $g \circ f\colon C[0,1] \to R$ for uniformly continuous functions $g\colon C[0,1] \to R$ other than the projection at $t = 1$. For example, let

$$g(x) = \sup_{0 \leqq t \leqq 1} x(t). \tag{3.1}$$

Since g in (3.1) is uniformly continuous, Theorem 3.1 also applies to

$$\left| P\{(n\sigma^2)^{-\frac{1}{2}} \max_{1 \leq k \leq n} W_k \leq x\} - P\{\sup_{0 \leq t \leq 1} \left|\xi(t)\right| \leq x\} \right|, \tag{3.2}$$

where

$$P\{\sup_{0 \leq t \leq 1} \left|\xi(t)\right| \leq x\} = 1 - (4/\pi) \sum_{k=1}^{\infty} [(-1)^k/(2k+1)] \exp\{-[\pi^2(2k+1)^2/8x^2]\}. \tag{3.3}$$

This provides a refinement of Theorem 9.1 of [9].

4. Convergence of moments

Since convergence in distribution does not by itself imply convergence of moments (cf. [6], p. 244), we need further conditions to imply that

$$E[(W_n/\sigma n^{\frac{1}{2}})^k] \to E[(f(\xi(1)))^k] \tag{4.1}$$

as $n \to \infty$. The standard tool is uniform integrability (cf. [1], p. 32), which we apply to prove the following theorem.

Theorem 4.1. If $EX_n^{2m} < \infty$ in the $GI/G/1$ queue with $\rho = 1$, then for all $k \leqq 2m$,

$$\lim_{n \to \infty} E[(W_n/\sigma n^{\frac{1}{2}})^k] = \begin{cases} 1(3)\cdots(2j-1), & k = 2j \\ j!\, 2^{j+1}(2\pi)^{-\frac{1}{2}}, & k = 2j+1. \end{cases}$$

Proof. We obtain $(W_n/\sigma n^{\frac{1}{2}})^m \Rightarrow f(\xi(1))^m$ in R from Theorem 2.1 by once again applying the continuous mapping theorem, here with $h: R \to R$ defined by $h(x) = x^m$. By Theorem 5.4 of [1], it thus suffices to show that $(W_n/\sigma n^{\frac{1}{2}})^m$, $n \geqq 1$, are uniformly integrable. From (2.2), $W_n \leqq 2\max_{1 \leqq k \leqq n}\left|S_k\right|$, so that

$$(W_n/\sigma n^{\frac{1}{2}})^m \leqq (2/\sigma)^m \left(\max_{1 \leqq k \leqq n} \left|S_k\right|/n^{\frac{1}{2}}\right)^m. \tag{4.2}$$

To show the uniform integrability of $\{\max_{1 \leqq k \leqq n}\left|S_k\right|^{2m}/n^m\}$, we follow an argument in the proof of Theorem 23.1 of [1].

We begin by truncating the random variables in the sequence $\{X_n, n \geqq 0\}$. Let

$$X_{nj} = \begin{cases} X_n, & \left|X_n\right| \leqq j \\ 0, & \left|X_n\right| > j. \end{cases} \tag{4.3}$$

Now consider for each j the sequence of random variables $\{Y_{nj}, n \geqq 0\}$, where

$$Y_{nj} = X_{nj} - EX_{nj}. \tag{4.4}$$

Observe that if $S_{nj} = Y_{0j} + \cdots + Y_{n-1,j}$, then there exists a constant $K(j)$ (depending on j and m) such that $E(S_{nj}^{4m}) \leqq K(j)n^{2m}$ for $n \geqq 1$. This inequality is easily demonstrated by induction on m, using the fact that Y_{nj} are uniformly bounded and i.i.d. with mean 0 (cf. [4], p. 225). For any random variable Y,

$$E_\alpha(Y) \leqq \alpha^{-1} E(Y^2), \tag{4.5}$$

where

$$E_\alpha(Y) = \int_{|x| \geqq \alpha} |x| \, dF_Y(x),$$

so that

$$\begin{aligned} E_\alpha(n^{-m} \max_{k \leqq n} |S_{kj}|^{2m}) &\leqq \alpha^{-1} E(n^{-2m} \max_{k \leqq n} |S_{kj}|^{4m}) \\ &\leqq \alpha^{-1} \left(\frac{4m}{4m-1}\right)^{4m} n^{-2m} E(S_{nj}^{4m}) \\ &\leqq \alpha^{-1} \left(\frac{4m}{4m-1}\right)^{4m} K(j), \end{aligned} \tag{4.6}$$

the second to last step being a consequence of a submartingale inequality ([4], p. 317). Hence, for any fixed j,

$$\lim_{\alpha \to \infty} \sup_n \; E_\alpha(n^{-m} \max_{k \leqq n} |S_{kj}|^{2m}) = 0. \tag{4.7}$$

Now consider the sequence $\{Z_{nj}\}$ where $Z_{nj} = X_n - X_{nj} + EX_{nj}$. Since $X_n = Y_{nj} + Z_{nj}$, there exists a constant M depending only on m ($M \leqq 2^m$) such that

$$n^{-m} \max_{k \leqq n} |S_k|^{2m} \leqq M \, (n^{-m} \max_{k \leqq n} |S_{kj}|^{2m} + n^{-m} \max_{k \leqq n} |D_{kj}|^{2m}), \tag{4.8}$$

where $D_{kj} = Z_{0j} + \cdots + Z_{k-1,j}$.

Now note that

$$\begin{aligned} E_\alpha(n^{-m} \max_{k \leqq n} |D_{kj}|^{2m}) &\leqq E(n^{-m} \max_{k \leqq n} |D_{kj}|^{2m}) \\ &\leqq n^{-m} \left(\frac{2m}{2m-1}\right)^{2m} E(|D_{nj}|^{2m}) \\ &\leqq n^{-m} \left(\frac{2m}{2m-1}\right)^{2m} K'(j) n^m \\ &\leqq K''(j), \end{aligned} \tag{4.9}$$

where $K''(j)$ is a is a constant only depending on j and m such that $K''(j) \to 0$ as $j \to \infty$ (because $E|Z_{nj}|^p \to 0$ as $j \to \infty$, $1 \leqq p \leqq 2m$). We have used the submartingale inequality again and the fact that Z_{nj} are i.i.d. with mean 0 and $E|Z_{nj}|^{2m} < \infty$.

Finally, from (4.6), (4.8), and (4.9), we have for all $n \geqq 1$

$$E_\alpha(n^{-m} \max_{k \leqq n} |S_k|^{2m}) \leqq M \left(\frac{K(j)(4m/(4m-1))^{4m}}{\alpha} + K''(j) \right). \tag{4.10}$$

Hence, for any $\varepsilon > 0$, we can choose j so that $K''(j)$ is sufficiently small and then choose α sufficiently large, so that

$$\sup_{n \leqq 1} E_\alpha(n^{-m} \max_{k \leqq n} |S_k|^{2m}) < \varepsilon. \tag{4.11}$$

Therefore, we have our desired result:

$$\lim_{\alpha \to \infty} \sup_{n \geqq 1} E_\alpha(n^{-m} \max_{k \leqq n} |S_k|^{2m}) = 0. \tag{4.12}$$

The moments are easy to evaluate. The even moments coincide with those of the normal distribution and the odd moments are related to the integral of the gamma distribution after the change of variables $y = x^2$.

5. Note added in proof

Forthcoming work of Kennedy [10a] significantly extends Section 3.

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