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CONTINUITY OF GENERALIZED SEMI-MARKOV PROCESSES*†

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It is shown that sequences of generalized semi-Markov processes converge in the sense of weak convergence of random functions if associated sequences of defining elements (initial distributions, transition functions and clock time distributions) converge. This continuity or stability is used to obtain information about invariant probability measures. It is shown that there exists an invariant probability measure for any finite-state generalized semi-Markov process in which each clock time distribution has a continuous c.d.f. and a finite mean. For generalized semi-Markov processes with unique invariant probability measures, sequences of invariant probability measures converge when associated sequences of defining elements converge. Hence, properties of invariant measures can be deduced from convenient approximations. For example, insensitivity properties established for special classes of generalized semi-Markov processes by Schassberger (1977), (1978), König and Jansen (1976) and Burman (1981) extend to a larger class of generalized semi-Markov processes.

1. Introduction and summary. Among the most promising stochastic processes for modeling complex phenomena in operations research are the generalized semi-Markov processes introduced by Matthes [19] and investigated further by König, Matthes and Nawrotzki [15], [16], König and Jansen [17], Schassberger [23]–[25], Burman [6] and Fossett [8]. A GSMP moves from state to state with the destination and duration of each transition depending on which of several possible events associated with the occupied state occurs first. Several different events compete for causing the next jump and imposing their own particular jump distribution for determining the next state. An ordinary SMP (semi-Markov process) is the special case in which there is only one event associated with each state. At each transition of a GSMP, new events may be scheduled. For each of these new events, a clock indicating the time until the event is scheduled to occur is set by an independent chance mechanism. An event which is scheduled but does not initiate a transition is either abandoned or it is associated with the next state and its clock just continues running.

We think of a GSMP as a model of discrete-event simulation. A good example of a GSMP is provided by the general multiple-heterogeneous-channel queue studied in Iglehart and Whitt [11]. A state could be the number of customers in the system and an indication of which servers are busy. Possible events associated with such a state would be an arrival in one of the arrival channels or a service completion by one of the occupied servers. With the usual independence assumptions and without any Markov assumptions, as in [11], this representation yields a GSMP which is not a SMP. Furthermore, this GSMP is not regenerative; there does not exist an embedded renewal process. (This statement may be confusing, however, because after appending appropriate supplementary variables to the GSMP we obtain an associated Markov process, and recent results of Athreya, McDonald and Ney [1], [2], and Nummelin [21] show that there will often exist a regenerative structure for this Markov process. At

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present, it does not seem possible to apply the regenerative approach in the full generality of this paper or [11], but the regenerative approach can apparently be used to obtain new proofs of the results in Konig, Matthes and Nawrotzki [15], [16]; some of this has been done by Fossett [8].)

Other examples of GSMPs are given in the references. It will be apparent that a GSMP provides a convenient representation for the familiar technique of supplementary variables; see Cox and Miller [7, §6.3]. However, modifications of the standard definition can easily be introduced to extend the model beyond this setting. Even applications to the setting of supplementary variables are significant. It is well known that supplementary variables can be used to convert non-Markov processes into Markov processes, but the resulting state space becomes so large that the general theory of Markov processes does not contribute much. (However, the recent work in [1], [2], and [21] indicates that the general theory has more to offer.) The idea behind GSMPs is to exploit more of the structure than just the Markov property.

The purpose of this paper is to establish continuity or stability properties for GSMPs. There is now a substantial literature on the continuity of stochastic models; see [4], [9], [12]–[14], [26], [28], and references in these sources. Here we show that sequences of GSMPs converge, in the sense of weak convergence of random functions, if associated sequences of defining elements (initial distributions, transition functions and clock time distributions) converge (Theorem 1). We apply the continuity to obtain information about invariant probability measures. We show that there exists an invariant probability measure for every finite-state generalized semi-Markov process in which all clock time distributions have continuous c.d.f.'s and finite means (Theorem 2); this is an extension of a result proved by König, Matthes, and Nawrotzki [15, p. 15] and [16, p. 381] by a different method. They assumed that the clock time distributions have positive densities and that the GSMP is irreducible, but they also proved uniqueness, which we do not.

We also show for GSMPs with unique invariant probability measures that sequences of invariant probability measures associated with a sequence of these GSMPs converge if associated sequences of defining elements converge (Theorem 3). This implies that insensitivity properties deduced by Schassberger [23]–[25], König and Jansen [17] and Burman [6] for special classes of GSMPs extend to the setting of this paper. The insensitivity property extends to all invariant measures if the limiting GSMP has more than one invariant measure. Our work was motivated by Schassberger's announcement of this open problem. While this work was being done (1975–1976), similar results were also obtained independently by different methods by Hordijk and Schassberger [10].

2. The definition via a discrete-time Markov process. We begin by defining a GSMP in terms of a DTMP (discrete-time Markov process) which describes the process at successive transition epochs. Let S and I be subsets of the positive integers. We regard the elements s of S as possible states of the GSMP and the elements i of I as indices of possible events that can occur. Let E be a function mapping S into the set of all finite subsets of I. We regard $E(s)$ as the set of all events that can occur in state s.

The system evolves from state s by having some event $i \in E(s)$ trigger a transition to another state s'. Let $p(s';s,i)$ be the probability the new state is s' given that event i triggers a transition in state s. We assume $p(\cdot\,;s,i)$ is a probability mass function on S for each $s \in S$ and $i \in E(s)$. The actual triggering event i will depend on clocks associated with the events in state s and speeds at which these clocks run. Let $R_+ = [0,\infty)$ and $R_+^\infty = R_+ \times R_+ \times \cdots$,

$$C_s' = \{c \in R_+^\infty : c_i > 0 \text{ if and only if } i \in E(s)\} \tag{2.1}$$

and $r_s \in R_+^\infty$ with $r_{si} = 0$ if $i \notin E(s)$ for each $s \in S$. The set C_s' is the set of possible clock readings in state s and r_{si} is the (deterministic) rate clock i runs in state s. We assume $r_{si} > 0$ for some $i \in E(s)$. When $r_{si} = 0$ for $i \in E(s)$, event i is regarded as inactive in state s. To avoid having two events simultaneously trigger a transition, we focus attention on the set

$$C_s = \left\{ c \in C_s' : c_i r_{si}^{-1} \neq c_j r_{sj}^{-1} \text{ for } i \neq j \text{ with } c_i c_j r_{si} r_{sj} > 0 \right\}. \tag{2.2}$$

For $s \in S$ and $c \in C_s$, let

$$\begin{aligned} t^* &\equiv t^*(s,c) = \inf\left\{ t \geqslant 0 : \min_{i \in E(s)} \{ c_i - t r_{si} \} = 0 \right\}, \\ c_i^* &\equiv c_i^*(s,c) = c_i - t^* r_{si}, \qquad i \in E(s), \\ i^* &\equiv i^*(s,c) = \min\{ i \in E(s) : c_i^*(s,c) = 0 \}. \end{aligned} \tag{2.3}$$

Our definition of i^* would yield a unique triggering event even if c were in C_s' instead of C_s, but for the continuity results we restrict attention to C_s. The event $i^*(s,c)$ is the unique triggering event and $t^*(s,c)$ is the interval between transitions beginning in state s with clock vector c. At a transition from state s to state s' triggered by event i, new clock values are independently generated for each $j \in N_{s'} \equiv N(s',s,i) \equiv E(s') - (E(s) - \{i\})$. Let $F(x; s', j, s, i)$ be the c.d.f. of such a new clock time. We assume $F(x; s', j, s, i)$ is continuous in x and $F(0; s', j, s, i) = 0$ for each (s', j, s, i). For $j \in 0_{s'} \equiv 0(s', s, i) \equiv E(s') \cap (E(s) - \{i\})$, the old clock reading is kept after the transition, i.e., $c_j = c_j^*(s,c)$. For $j \in (E(s) - \{i\}) - E(s')$, event j ceases to be scheduled after the transition, i.e., c_j is set equal to 0.

The DTMP has state space $\Sigma \times R_+$ where Σ is the topological sum of $\{s\} \times C_s$ for $s \in S$. Let the real line R have the usual topology and let all subsets have the relative topology; let S and I have the discrete topology; and let all product spaces have the product topology. It is easy to see that $\Sigma \times R_+$ with this topology is metrizable as a complete separable metric space. (Use basic properties of Polish spaces; see Bourbaki [5, p. 195].) We now define a Markov kernel K on $\Sigma \times R_+$ by setting

$$K([s,c,t], A) = p(s'; s, i^*) \prod_{i \in N_{s'}} F(a_i; s', i, s, i^*) \prod_{i \in 0_{s'}} 1_{[0, a_i]}(c_i^*) 1_B(t + t^*) \tag{2.4}$$

where

$$A = \{s'\} \times \{ c' \in C_{s'} : c_i' \leqslant a_i, i \in E(s') \} \times B$$

with B a measurable subset of R_+ and 1_B the indicator function of the set B. Since the c.d.f.'s $F(\cdot \mid s', j, s, i)$ are continuous, a legitimate Markov kernel on the state space $\Sigma \times R_+$ is specified by (2.4); see Neveu [20, pp. 73, 162]. The continuity of the c.d.f.'s is used here only to get $K([s,c,t], \Sigma \times R_+) = 1$ for all $[s,c,t]$, i.e., to guarantee that (2.2) is preserved at each transition.

Let $W(k) = [U(k), V(k), T(k)]$ be the coordinate random elements of the DTMP determined by the Markov kernel in (2.4) and an initial random element $W(0) = [U(0), V(0), 0]$. ($U(k)$ gives the state s, $V(k)$ gives the vector c of clock readings and $T(k)$ gives the elapsed time at the epoch of the kth transition.) We assume that

$$P\Big(\sup_{k \geqslant 1} T(k) = \infty \mid W(0) \Big) = 1 \tag{2.5}$$

for all initial random elements $W(0)$. For example, (2.5) is satisfied if S is finite. Define N by $N(t) = \max\{ k : T(k) \leqslant t \}, t \geqslant 0$. Because of the regularity condition (2.5), N is a random element of the function space $D(R_+, R)$; see Lindvall [18] or [27, §2]. It is also integer-valued and increases by unit jumps. Let $X(t) = U(N(t))$,

$$Y_i(t) = \begin{cases} V_i(N(t)) - r_{X(t)i}(t - T(N(t))), & i \in E(X(t)), \\ 0, & i \notin E(X(t)), \end{cases} \tag{2.6}$$

and $Z(t) = [X(t), Y(t)]$, $t \geqslant 0$. The stochastic process X is called the GSMP and the process Z is the associated CTMP (continuous-time Markov process).

3. Continuity of the processes. In this section we establish the continuity of the basic processes W (the DTMP) and Z (the CTMP) as functions of the defining elements: the initial random element $W(0)$, the transition probabilities $p(s'; s, i)$ and the clock time c.d.f.'s $F(x; s', j, s, i)$. Within the framework of §2, the desired results follow directly from results in the literature.

Let $\Rightarrow$ denote weak convergence of random elements, probability measures and c.d.f.'s; see Billingsley [3]. Consider a sequence of GSMP's indexed by n with common sets I and S.

THEOREM 1. *If*
(i) $W_n(0) \Rightarrow W(0)$,
(ii) $p_n(s'; s, i) \to p(s'; s, i)$ *for all* $(s', s, i) \in S^2 \times I$,
(iii) $F_n(x; s', j, s, i) \to F(x; s', j, s, i)$ *for all* $(s', j, s, i) \in (S \times I)^2$ *and* $x \in R_+$,
then $W_n \Rightarrow W$ *in* $(\Sigma \times R_+)^\infty$ *and* $Z_n \Rightarrow Z$ *in* $D(R_+, \Sigma)$.

PROOF. We have noted that the state space $\Sigma \times R_+$ is metrizable as a complete separable metric space, so we can apply Theorem 4 of Karr [12] to get $W_n \Rightarrow W$. As a consequence of S being discrete and Theorem 2.1 of [3], condition (4b) of [12] is equivalent here to $K_n([s, c^n, t^n], \cdot) \Rightarrow K([s, c, t], \cdot)$ whenever $(c^n, t^n) \to (c, t)$ in $C_s \times R_+$ for each $s \in S$, (see the remark following Theorem 4 in [12]) which is easily demonstrated using (2.4). To treat the continuous-time processes, apply the continuous mapping theorem in §5 of [3] together with Theorem 1 in the manner of Lemma 3.1 of Kennedy [13] and Theorem 10 of Karr [12]. The overall continuous function here can be obtained as the composition of several elementary continuous functions. The continuity of (W, N) as a function of W holds because N is the inverse of T where (2.5) holds and $T_k > T_{k-1}$ for all k. Then the continuity of Z as a function of (W, N) is obtained from the definition in (2.6) by applying composition and addition. See [27] for a more extensive study of these functions. ∎

4. Invariant probability measures. Consider an arbitrary CTMP $\{\xi(t), t \geqslant 0\}$ with sample paths in $D(R_+, \Gamma)$ for every possible initial random element $\xi(0)$, where Γ is an arbitrary complete separable metric space. Call a probability measure P on Γ an *invariant probability measure* for the CTMP ξ if $\xi(t)$ has distribution P for all $t > 0$ when $\xi(0)$ is given the distribution P. Continuity can be a powerful tool for establishing existence and other properties of invariant measures, as the next lemma illustrates. (Compare Theorem 6 of Karr [12]; notice that it does not apply.)

LEMMA 1. *Let P_n be an invariant probability measure for the* CTMP $\{\xi_n(t), t \geqslant 0\}$ *for each $n \geqslant 1$. If $\xi_n \Rightarrow \xi$ in $D(R_+, \Gamma)$ whenever $\xi_n(0) \Rightarrow \xi(0)$ in Γ, then any weak convergent limit point of the sequence $\{P_n\}$ is an invariant probability measure for ξ.*

REMARKS. (i) Lemma 1 can be interpreted as saying that the set-valued mapping that maps a CTMP in $D(R_+, \Gamma)$ into its set of invariant probability measures is upper-semicontinuous.

(ii) The condition involving weak convergence $\xi_n \Rightarrow \xi$ with the Skorohod J_1 topology on $D(R_+, \Gamma)$ can be replaced by weak convergence $\xi_n(t) \Rightarrow \xi(t)$ in Γ for each t in a dense subset if ξ is known to be right-continuous in probability; see the proof.

PROOF. Suppose $P_{n'} \Rightarrow P$ on Γ for a subsequence $\{P_{n'}\}$ of $\{P_n\}$. Let $\xi_{n'}(0)$ be given the distribution $P_{n'}$ and let $\xi(0)$ be given the distribution P. Since $\xi_{n'}(0) \Rightarrow \xi(0)$, $\xi_{n'} \Rightarrow \xi$ in $D(R_+, \Gamma)$, which implies that $\xi_{n'}(t) \Rightarrow \xi(t)$ in Γ for each t in a dense subset of $[0, \infty)$; see [3, p. 124]. Since $P_{n'}$ is invariant for $\xi_{n'}$, $\xi_{n'}(t)$ has distribution $P_{n'}$ for each t. Consequently, $\xi(t)$ has distribution P for each t in the dense set where convergence takes place. Since each sample path of ξ is in D, the process ξ is right continuous in probability. Hence $\xi(t)$ has distribution P for all $t \geq 0$. Consequently, P is invariant for ξ. ∎

We now return to GSMPs. We apply an approximation by continuous-time Markov chains with the results already established to obtain the following existence theorem (extending a result in [15] and [16]).

THEOREM 2. *There exists a proper invariant probability measure for any finite-state GSMP in which each clock time* c.d.f. $F(\cdot\,; s', j, s, i)$ *is continuous and has a finite mean* $\mu(s', j', s, i)$.

PROOF. We will represent the given GSMP as the limit in the sense of Theorem 1 of a sequence of GSMPs known to have invariant probability measures. Then we will show that any sequence of invariant probability measures, taking one for each n, has a convergent subsequence. The limit of any such convergent subsequence is an invariant probability measure for the original GSMP by Lemma 1.

We construct the nth GSMP in the converging sequence of GSMPs by approximating each clock time c.d.f. $F \equiv F(\cdot\,; s', j, s, i)$ by a finite mixture of finite convolutions of exponential c.d.f.'s with common parameter n. Let $G_n^k(\cdot)$ denote the c.d.f. of the k-fold convolution of the exponential distribution with mean n^{-1}. Let

$$\pi_n^k = F((k+1)/n) - F(k/n), \qquad 0 \leq k < n^2, \quad \pi_n^{n^2} = 1 - F(n)$$

and

$$F_n(x) = \sum_{k=0}^{n^2} \pi_n^k G_n^k(x), \qquad x \geq 0. \tag{4.1}$$

It is well known (see for example [22, p. 32]) and not difficult to show that $F_n \Rightarrow F$ and $\mu_n \uparrow \mu$ as $n \to \infty$, where μ_n is the mean of F_n.

These specially constructed approximating GSMPs are convenient because they can be represented in terms of continuous-time finite-state Markov chains. Instead of the clock time, we keep track of the number of exponential phases remaining before the clock expires. In particular, we can use the CTMC $\{[X(t), M(t)], t \geq 0\}$ where $X(t)$ is the state of the GSMP and $M(t)$ is the vector with integer-valued coordinates which records the number of exponential phases remaining before each scheduled event will occur. Since the state space of the GSMP is finite and F_n in (4.1) is a finite mixture, the CTMC has a finite state space. Since every finite-state CTMC has an invariant probability measure, the CTMC here does. This in turn implies that the associated GSMP has an invariant probability measure on the space Σ. The distribution of the GSMP at any time t is obtained from the distribution of the CTMC at time t by

$$P(X(t) = s, Y(t) \in B) = \sum_{\mathbf{k}} P(X(t) = s, M(t) = \mathbf{k}) P(Y(t) \in B \mid M(t) = \mathbf{k}) \tag{4.2}$$

where $\mathbf{k}$ is the vector of integers and the conditional probability is determined by (4.1). Any invariant probability measure for (X, M) substituted into (4.2) immediately gives an invariant probability measure for the GSMP.

To complete the proof by applying Lemma 1, it suffices to show that any sequence of invariant probability measures $\{P_n\}$, where P_n is an invariant probability measure for the GSMP associated with the nth CTMC, has a weakly convergent subsequence. By Prohorov's theorem, [3, §6] it suffices to show that $\{P_n\}$ is (uniformly) tight.

The finite state space assumption is included to guarantee that any such sequence $\{P_n\}$ is in fact tight, but an additional argument is needed: we must consider the clock readings. Let K_m be the (finite) topological sum of $\{s\} \times (C_s \cap [0,m]^\infty)$ over all $s \in S$. Since S is finite and $E(s)$ is finite for each $s \in S$, the number of relevant coordinates in $C_s \in R_+^\infty$ is finite. The set K_m is clearly a compact subset of Σ for each m. Our goal is to show that for each $\epsilon > 0$ there exists an m such that $P_n(K_m) \geqslant 1 - \epsilon$ for all n. To accomplish this goal, first let the initial distribution of the nth GSMP be P_n, which makes each GSMP a stationary process. We will get a handle on P_n by looking at the limiting fraction of time each clock reading is outside $[0,m]$. To see that this is sufficient, let $\alpha(m,n,j)$ be the probability under P_n that the clock associated with event j reads more than m at any time, i.e.,

$$\alpha(m,n,j) = \sum_s P_n\big(\{s\} \times \big[C_s \cap \pi_j^{-1}((m,\infty))\big]\big), \tag{4.3}$$

where the summation is over $\{s \in S : j \in E(s)\}$ and $\pi_j : R_+^\infty \to R_+$ is the projection onto the jth coordinate. Clearly

$$P_n(K_m^c) \leqslant \sum_{s \in S} \sum_{j \in E(s)} \alpha(m,n,j). \tag{4.4}$$

Since the sets S and $E(s)$, $s \in S$, are finite, it suffices to show that $\alpha(m,n,j) \to 0$ as $m \to \infty$ uniformly in n for each j. Now let $\beta(m,n,j,s',i,s)$ be the probability under P_n that the clock associated wlth event j reads more than m and event j was initiated in state s triggered by event i and followed by a transition to state s'. Clearly

$$\alpha(m,n,j) \leqslant \sum_{s' \in S} \sum_{s \in S} \sum_{i \in E(s)} \beta(m,n,j,s',i,s), \tag{4.5}$$

so it suffices to show that $\beta(m,n,j,s',i,s) \to 0$ as $m \to \infty$ uniformly in n for each quadruple (s',j,s,i).

Next let $\gamma(m,n,j,s',i,s)$ be the lim sup as $t \to \infty$ of the fraction of time in $[0,t]$ that the jth clock reading is in the set (m,∞) and it was initiated by the quadruple (s',j,s,i), under P_n. To show $\beta(m,n,j,s',i,s) \to 0$ as $m \to \infty$ uniformly in n, it clearly suffices to show that $\gamma(m,n,j,s',i,s) \to 0$ w.p.1. as $m \to \infty$ uniformly in n. Consider a particular quadruple (s',j,s,i) corresponding to event j in state s' triggered by event i in state s. Over time a sequence of such clock readings are generated from the c.d.f. $F_n(\cdot\,; s',j,s,i)$. The successive clock readings generated are independent and identically distributed with this c.d.f. Also there typically will be periods in which the clock reads 0 or the clock reading is initiated by a different quadruple. However, for each sample path, $\gamma(m,n,j,s',i,s)$ would be larger if there were no idle periods for clock j and all clock times were initiated by the quadruple (s',j,s,i). Hence, we can dominate (in the sense of long-run averages) the given clock reading process by the forward excess time process associated with the delayed renewal process based on P_n and the c.d.f. $F_n(\cdot\,; s',j,s,i)$. Let $R_n(s',j,s,i)$ be the stationary forward excess of residual lifetime variable associated with the renewal process with c.d.f. $F_n(\cdot\,; s',j,s,i)$. We thus obtain the bound

$$\begin{aligned} \gamma(m,n,j,s',i,s) &\leqslant P(R_n(s',j,s,i) > m) \\ &\leqslant \mu_n^{-1} \int_m^\infty 1 - F_n(x; s',j,s,i)\,dx \end{aligned} \tag{4.6}$$

which converges to 0 uniformly in n as $m \to \infty$ because $\{F_n\}$ is uniformly integrable. Uniform integrability follows from $F_n \Rightarrow F$ and $\mu_n \to \mu$; see [3, Theorem 5.4]. ∎

Now consider a sequence of finite-state GSMPs indexed by $n \geqslant 1$ with common state and event spaces S and I.

THEOREM 3. *Suppose*
(i) $p_n(s'; s, i) \to p(s', s, i)$,
(ii) $F_n(x; s', j, s, i) \to F(x; s', j, s, i)$, *and*
(iii) $\mu_n(s', j, s, i) \to \mu(s', j, s, i) < \infty$
for all $(s', j, s, i) \in (S \times I)^2$ *and* $x \in R_+$. *If the* CTMP Z_n *has a unique invariant probability measure* P_n *for each* $n \geq 1$, *then the sequence* $\{P_n\}$ *has a convergent subsequence and every weak limit point of* $\{P_n\}$ *is an invariant probability measure for the limiting* CTMP Z.

REMARKS. If the limiting CTMP Z has a unique invariant probability measure P, then obviously $P_n \Rightarrow P$ in Theorem 3. By [16, p. 381], there is a unique invariant probability measure for every irreducible finite-state GSMP in which all clock time distributions have finite means and positive density functions. Theorem 3 thus extends the insensitivity property deduced by Schassberger [23]–[25] beyond mixtures of convolutions of exponential distributions. Similarly, the insensitivity of König and Jansen [17] and Burman [6] extend to the setting of this paper. The insensitivity property holds for all invariant probability measures associated with the limit process.

PROOF. Construct a sequence of finite-state GSMPs generated by CTMCs indexed by k converging to the nth GSMP for each $n \geq 1$ as in the proof of Theorem 2. Let $\{P_{nk}, k \geq 1\}, n \geq 1$, be associated sequences of invariant probability measures. Let P_n', $n \geq 1$, be limit points of these sequences, which exist and are invariant probability measures for the nth GSMP by the proof of Theorem 2. Since we have assumed uniqueness for each $n \geq 1$, $P_n' = P_n$ and $P_{nk} \Rightarrow P_n$ as $k \to \infty$ for each $n \geq 1$. (This is the only place where we use the uniqueness of P_n.) Our object is to show that the sequence $\{P_n\}$ has a convergent subsequence and that every limit point is an invariant probability measure for Z.

As in the proof of Theorem 2, we can apply Theorem 1 and Lemma 1 here plus Prohorov's theorem to conclude that it suffices to show that the sequence $\{P_n\}$ is tight. However, note that the sequence $\{P_n\}$ is contained in the closure of the double sequence $\{P_{nk}, n \geq 1, k \geq 1\}$. By Theorem 2.1 (iii) of [3], it is easy to show that the closure of a tight set of probability measures is again tight. Trivially, a subset of a tight family is tight. Hence, in order to show that the sequence $\{P_n\}$ is tight, it suffices to show that the double sequence $\{P_{nk}\}$ is tight. Now we are in the setting of CTMCs and we can apply the argument used in the proof of Theorem 2. The analog of (4.6) goes to 0 uniformly in n and k by the same reasoning. We can obtain the double limits $F_{nk} \Rightarrow F$ and $\mu_{nk} \to \mu$ as $n \to \infty$ and $k \to \infty$ as needed for uniform integrability by choosing the sequence depending on k appropriately for each n. In particular, we can have $F_{nk} \to F_n$ and $\mu_{nk} \to \mu_n$ as $k \to \infty$ uniformly in n.

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