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LARGE DEVIATIONS OF INVERSE PROCESSES WITH NONLINEAR SCALINGS

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We show, under regularity conditions, that a nonnegative nondecreasing real-valued stochastic process satisfies a large deviation principle (LDP) with nonlinear scaling if and only if its inverse process does. We also determine how the associated scaling and rate functions must be related. A key condition for the LDP equivalence is for the composition of two of the scaling functions to be regularly varying with nonnegative index. We apply the LDP equivalence to develop equivalent characterizations of the asymptotic decay rate in nonexponential asymptotics for queue-length tail probabilities. These alternative characterizations can be useful to estimate the asymptotic decay constant from systems measurements.

1. Introduction. Let $(Z_t\colon t \geq 0)$ be a real-valued stochastic process and v a real-valued function on $\mathbf{R}_+$ which increases to infinity. Then the pair $(Z_t, v(t))$ is said to satisfy a *large deviation principle* (LDP) with rate function I if, for all Borel subsets B of $\mathbf{R}_+$,

$$\begin{aligned}-\inf_{x\in B^0} I(x) &\leq \liminf_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}[Z_t \in B] \\ &\leq \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}[Z_t \in B] \leq -\inf_{x\in \bar{B}} I(x),\end{aligned} \tag{1}$$

where B^o is the interior and $\bar{B}$ the closure of B.

Now suppose in addition that $(Z_t\colon t \geq 0)$ is a nonnegative nondecreasing stochastic process and define its inverse by

$$T_z = \inf\{t \geq 0\colon Z_t \geq z\}, \qquad z \geq 0. \tag{2}$$

Glynn and Whitt [11] studied the relation between the LDPs for these inverse processes with linear scalings. They showed that $(t^{-1}Z_t, t)$ satisfies an LDP with convex rate function I if and only if $(z^{-1}T_z, z)$ satisfies an LDP with convex rate function $J(x) = xI(1/x)$. Russell [22] subsequently produced a similar result requiring only a weakened form of convexity of I, but also requiring a weak mixing condition to be satisfied by Z or T.

The purpose of this paper is to extend these results beyond linear scalings. Suppose for some scaling functions u, v and w increasing on $\mathbf{R}_+$ to $+\infty$ that the pair $(u(Z_t)/w(t), v(t))$ satisfies an LDP with some rate function I. Then it is natural to ask under what circumstances there is a corresponding LDP

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involving T_z and how its rate and and scaling functions are related to those in the LDP for Z_t.

It turns out that for a nontrivial relation to exist, the composition $v \circ w^{-1}$ must be regularly varying with nonnegative index (see Section 2 for definitions). The main result of Section 2 is Theorem 1, in which we show that $(u(Z_t)/w(t), v(t))$ satisfies an LDP with a suitable rate function I if and only if $(w(T_z)/u(z), v \circ w^{-1} \circ u(z))$ satisfies an LDP with a rate function $J(x) = f(x)I(1/x)$, where $f(x) = x^{i(f)}$ with $i(f)$ being the index of the regularly varying function $v \circ w^{-1}$.

We have two applications in mind for LDPs with nonlinear scalings. The first, presented in Section 3, is motivated by the use of scaling functions other than the identity in formulating the large deviation properties of queueing processes with input exhibiting long-range dependence (LRD). Motivation for studying LRD processes comes from empirical traffic studies (see, e.g., [15]). In this context, Z_t is the work arriving at a buffer during the interval $[-t, 0)$. If the queue is served at rate s, then pathwise the queue length (the content in an infinite-capacity buffer) Q can be written as

$$Q = \sup_{t \geq 0}(Z_t - st). \tag{3}$$

It follows from [9] that under a set of conditions which imply that $(t^{-1}Z_t, v(t))$ satisfies an LDP with some rate function I with v regularly varying (in this case u and w are the identity), the asymptotics of the tail probabilities for the queue length are

$$\lim_{b\to\infty} \frac{1}{v(b)} \log \mathbf{P}[Q > b] = -\delta, \tag{4}$$

where $\delta = \inf_{x>0} I(x+s)/f(x)$ with $f(x) = x^{i(f)}$ and $i(f)$ the index of the regularly varying function v. Applying the results of Section 2, we are able to express the decay constant δ directly in terms of expectations over sample paths of Z or the inverse process T. In Theorem 5 we show that

$$\begin{aligned} &\delta = \sup\{\theta\colon \nu(\theta) \leq 0\} \quad \text{where} \\ &\nu(\theta) = \lim_{t\to\infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t \geq st]\}, \end{aligned} \tag{5}$$

when this limit exists. In Theorem 7 we show that

$$\begin{aligned} &\delta = \sup\{\theta\colon \omega(\theta) \leq 0\} \quad \text{where} \\ &\omega(\theta) = \lim_{z\to\infty} \frac{1}{v(z)} \log \mathbf{E}[\exp(\theta v(z - sT_z));\ sT_z \leq z], \end{aligned} \tag{6}$$

a limit which is guaranteed to exist through mild conditions on the rate function I. The restriction to the events $\{Z_t \geq st\}$ and $\{sT_z \leq z\}$ in (5) and (6) appear naturally because, for any $b > 0$, the event $\{Q \geq b\}$ has the same probability as $\bigcup_{t>0}\{Z_t - st \geq b\} = \bigcup_{z>0}\{z - sT_z \geq b\}$ when Z has stationary increments. In the asymptotic regime $b \to \infty$, one of the elements in the union dominates the others in the sense that its probability is overwhelmingly greater. Thus, for large deviations, the only events of interest are contained within the events $\{Z_t \geq st\}$ or, equivalently, $\{sT_z \leq z\}$.

The second queueing application is to time-dependent arrival processes. The nonlinear scalings make it possible to obtain LDPs for queueing processes with time-dependent arrival processes, as we illustrate in our examples in Section 5.

In Section 4 we also briefly discuss the functional (or sample path) large deviations principles (FLDPs) with nonlinear scaling. We are able to obtain some results directly from a recent paper [20], which establishes FLDPs with linear scalings. However, under all the conditions needed to obtain ordinary LDPs for the one-dimensional projections in $\mathbf{R}$ from the FLDPs, we only obtain LDPs in a special case of what we obtain directly for LDPs in $\mathbf{R}$. The approach via FLDPs requires that the composition $v \circ w^{-1}$ must be regularly varying with index 1.

The proofs of all the main theorems are deferred until Section 6.

2. Main results. Recall that a *rate function* on $\mathbf{R}_+$ is a lower semicontinuous function taking values in $[0, \infty]$, and a *good* rate function is one with compact level sets (see, e.g., [5]). For any good rate function I on $\mathbf{R}_+$ there is at least one $x \geq 0$ for which $I(x) = 0$. Following [11], we say that $(W_t, a(t))$ satisfies a *partial* LDP if (1) holds for a proper subclass of Borel subsets.

We say that a rate function I on $(0, \infty)$ has *no peaks* if the following conditions are fulfilled:

1. there exists an $x_I \in [0, \infty)$ such that $I(x_I) = 0$, or $\lim_{x\to\infty} I(x) = 0$, in which case we say $x_I = \infty$;
2. I is nonincreasing on $(0, x_I)$ and nondecreasing on (x_I, ∞) (when the sets are nonempty). We call x_I a *base* of I. Note that x_I need not be unique: we admit the possibility that $I(x)$ is zero for x in some interval. (We might have used the term "unimodal" to describe the no-peaks property, but sometimes "unimodal" is used when there is a unique base.)

We say that a rate function I on $(0, \infty)$ has *no flat spots* if it has no peaks, and in addition I is strictly decreasing on $(0, x_I)$ and strictly increasing on (x_I, ∞) (when the sets are nonempty). It follows that *the base* of I, x_I, is unique if there are no flat spots.

Let $(Z_t\colon t \geq 0)$ be a stochastic process on $\mathbf{R}_+$ with sample paths that are right continuous, nonnegative, nondecreasing and for which $\lim_{t\to\infty} Z_t = \infty$. We define the *inverse process* T of Z to be its pathwise left inverse. That is,

$$T_z = \inf\{t \geq 0\colon Z_t \geq z\}, \qquad z \geq 0. \tag{7}$$

Thus $(T_z\colon z \geq 0)$ is a stochastic process whose sample paths are left continuous, nonnegative and nondecreasing with $T_0 = 0$ and $\lim_{z\to\infty} T_z = \infty$. Conversely, starting instead with a process T with these properties, we could have arrived at Z through the pathwise relation

$$Z_t = \sup\{z \geq 0\colon T_z \leq t\}, \qquad t \geq 0. \tag{8}$$

Such a Z will be called the *inverse process* of T, and we shall say that the pair of processes (Z, T) are *inverse processes*. (We adopt the convention that the first element of the pair is the right-continuous process.) We will exploit the following relations:

$$\{Z_t \geq z\} = \{T_z \leq t\} \quad \text{and hence } \{Z_t < z\} = \{T_z > t\}. \tag{9}$$

To understand inverse processes it is helpful to consider the completed graphs of the sample paths. Let the *completed graph* of Z be

$$\Gamma(Z) = \{(u, t) \in \mathbf{R} \times \mathbf{R}_+ : u \in [Z_{t^-} \wedge Z_t, Z_{t^-} \vee Z_t]\}, \tag{10}$$

where $[x, x] = \{x\}$, Z_{t^-} denotes the left limit, $Z_{0^-} = 0$, $\wedge$ denotes the minimum and $\vee$ denotes the maximum. For inverse processes (Z, T), the completed graph $\Gamma(T)$ is the completed graph $\Gamma(Z)$ with the axes switched, that is,

$$\Gamma(T) = \Gamma^{-1}(Z) \equiv \{(t, u) : (u, t) \in \Gamma(Z)\}. \tag{11}$$

Note that (Z, T) are inverse processes if and only if $\Gamma(T) = \Gamma^{-1}(Z)$, Z is right continuous and T is left continuous.

We define a *scaling function* to be any increasing homeomorphism of $\mathbf{R}_+$. Recall from, for example, Section 1.4 of [1], that a measurable monotone function u is said to be (Baire) *regularly varying* (at infinity) if for all y in a Baire subset of $(0, \infty)$, $\lim_{x\to\infty} u(xy)/u(x)$ exists, in which case

$$\lim_{x\to\infty} u(xy)/u(x) = y^r \tag{12}$$

for all $y > 0$, for some r called the *index* of u. It is worth making clear that the subgroup properties of $(0, \infty)$ under addition mean that any Baire set will suffice: it need not, for example, be dense in the whole interval $(0, \infty)$. Let $\mathscr{S}$ denote the set of scaling functions, and $\mathscr{S}_r$ the set of regularly varying scaling functions with indices in $[0, \infty)$. We will use $i(u)$ to denote the index of a given function u in $\mathscr{S}_r$ and set $\hat{u}(x) = x^{i(u)}$. We will find it useful to denote the composition of real functions u and v by uv. Thus $uv(x) = u(v(x))$. However, $u(x)v(x)$ and $u(x)/v(x)$ will just be the usual products and quotients. Typical scaling functions are positive powers $u(x) = \hat{u}(x) = x^{i(u)}$ and $\log x$, which has index 0. In the latter case we must alter the function on some initial interval $[0, \varepsilon]$ with $\varepsilon > 1$ in order that it is a homeomorphism of $\mathbf{R}_+$. Note that $\widehat{\log}(x) = 1$ is not a scaling function. We collect here some useful properties of $\mathscr{S}$ and $\mathscr{S}_r$.

LEMMA 1. (i) *Here $u, v \in \mathscr{S}$ implies $uv \in \mathscr{S}$.*
(ii) *$u, v \in \mathscr{S}_r$ implies $uv \in \mathscr{S}_r$ with $i(uv) = i(u)i(v)$.*
(iii) *Let $u \in \mathscr{S}_r$ with $i(u) \in (0, \infty)$. Then $u^{-1} \in \mathscr{S}_r$ and $i(u^{-1}) = 1/i(u)$.*

PROOF. Part (i) is trivial. For (ii), note that for $x, y > 0$,

$$\frac{uv(yx)}{uv(x)} = \frac{u(r(y, x)v(x))}{u(v(x))} \quad \text{where } r(y, x) = v(yx)/v(x). \tag{13}$$

Now $\lim_{x\to\infty} r(y,x) = y^{i(v)}$, so, by the uniformity of the convergence in (12) (see Theorem 1.5.2 of [1]),

$$\lim_{x\to\infty} uv(yx)/uv(x) = y^{i(u)i(v)}. \tag{14}$$

For (iii), use Theorem 1.5.12 in [1] together with the fact that u is increasing. □

We can now state the main theorem on LDPs for inverse processes. The proofs of this theorem and the others will be given in Section 6.

THEOREM 1. *Let (Z, T) be an inverse process, and let $u, v, w \in \mathscr{S}$.*

(i) *Let $vw^{-1} \in \mathscr{S}_r$. Then $(u(Z_t)/w(t), v(t))$ satisfies an LDP with a rate function I which has no peaks if and only if $(w(T_z)/u(z), vw^{-1}u(z))$ satisfies an LDP with rate function J which has no peaks. In this case,*

$$J(x) = \widehat{vw^{-1}}(x)I(1/x) \quad \textit{and} \quad I(x) = \widehat{vw^{-1}}(x)\,J(1/x), \tag{15}$$

and I and J at 0 are equal to their right limits there. Moreover, x_I is a base for I if and only if $x_J = 1/x_I$ is a base for J (with the convention for bases that $1/\infty = 0$ and $1/0 = \infty$), and x_I is unique if and only if x_J is unique.

(ii) *Suppose that $(u(Z_t)/w(t), v(t))$ satisfies an LDP with a rate function I which has no peaks, that $(w(T_z)/u(z), vw^{-1}u(z))$ satisfies an LDP with rate function J which has no peaks and that there exist bases for I, J satisfying $x_I = 1/x_J$. Suppose furthermore that there exist $b_\pm$ with $b_- < b_+$ such that $I(x) \in (0,\infty)$ for $x \in (b_-, b_+)$ and $J(x) \in (0,\infty)$ for $x \in (b_+^{-1}, b_-^{-1})$. Then $vw^{-1} \in \mathscr{S}_r$.*

REMARK 2.1. The second part of Theorem 1 shows that the condition $vw^{-1} \in \mathscr{S}_r$ is not just a technical convenience imposed in order to obtain the inverse LDP. Under the conditions of (ii) above, which amount to saying that the LDPs are not trivial, the condition $vw^{-1} \in \mathscr{S}_r$ is necessary.

REMARK 2.2. Sufficient conditions for one of the inverse processes to satisfy an LDP are provided by the Gärtner–Ellis theorem [5]. Assume that for all $\theta \in \mathbf{R}$, the limit

$$\lambda(\theta) = \lim_{t\to\infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v(t)u(Z_t)/w(t)] \tag{16}$$

exists, is lower semicontinuous and essentially smooth. We call λ the *cumulant generating function* (CGF). [In the usual statement $v(t)$ is replaced by t, but that does not alter the result.] Then $(u(Z_t)/w(t), v(t))$ satisfies an LDP with rate function $I = \lambda^*$ where

$$\lambda^*(x) \equiv \sup_\theta(\theta x - \lambda(\theta)) \tag{17}$$

is the *Legendre transform* of λ. Then I is automatically convex and lower semicontinuous, and hence automatically satisfies the conditions in Theorem 1.

That J has no peaks follows a fortiori from Theorem 1 without additional assumptions concerning those of I, for example, that I be convex or Y-shaped. Actually, one can establish this independently for a class of models in which I is convex (as it would be if arising from an application of the Gärtner–Ellis theorem) and $f \equiv \widehat{vw^{-1}}$ is concave; that is, $i(f) \leq 1$, as we now show.

LEMMA 2. *Let I be a nonnegative convex function on $\mathbf{R}_+$ and $f(x) = x^{i(f)}$ for some $i(f) \in (0, 1]$. Then $J(x) = f(x)I(1/x)$ has no peaks.*

PROOF. Note that $f^{-1}J(x) = xf^{-1}(I(1/x))$. Since $i(f) \in (0, 1)$, f^{-1} is convex and increasing. Hence $f^{-1}I$ is convex. Since convexity of a function K implies convexity of $x \mapsto xK(1/x)$, $f^{-1}J(x)$ is convex and so clearly J has no peaks. $\square$

The requirement that vw^{-1} be regularly varying has a consequence that the existence of an LDP for (say) Z with a given set of scaling functions is equivalent to that for a (scaling) function of Z using a second set of scalings which are compatible in the following sense.

THEOREM 2. *Let $p, q, r, s, u, v, w, x \in \mathscr{S}$ and assume that*

$$sr^{-1}p = xw^{-1}u, \tag{18}$$

$$qs^{-1} = vx^{-1}. \tag{19}$$

(i) *If*

$$up^{-1} \in \mathscr{S}_r \quad \text{with } i(up^{-1}) > 0, \tag{20}$$

then $(u(Z_{x(t)})/w(t), v(t))$ satisfies an LDP with rate function I_1 without peaks if and only if $(p(Z_{s(t)})/r(t), q(t))$ satisfies an LDP with rate function I_2 without peaks, in which case,

$$I_2 = I_1\widehat{up^{-1}}. \tag{21}$$

(ii) *Let $(u(Z_{x(t)})/w(t), v(t))$ and $(p(Z_{s(t)})/r(t), q(t))$ satisfy LDPs with rate functions I_1 and I_2, which are without peaks. For $i = 1, 2$, let x_i^- denote the greatest lower bound and x_i^+ the least upper bound of bases for I_i. Furthermore, assume for some $B_i = (b_i^-, b_i^+)$ with $b_i^- < x_i^- \leq x_i^+ < b_i^+$, that I_i is continuous on B_i, strictly decreasing on (b_i^-, x_i^-) or strictly increasing on (x_i^+, b_i^+). Then $up^{-1} \in \mathscr{S}_r$ and $i(up^{-1}) > 0$.*

REMARK 2.3. In (ii) above we admit the possibility that $x_i^- = 0$ or $x_i^+ = \infty$; so b_i^- or b_i^+ will not exist, and we simply omit from consideration the corresponding open intervals which would have them as boundaries.

Theorem 2 helps us contrast Theorem 1 with its analog for linear scalings in [11]. Suppose that the pair $(u(Z_t)/w(t), v(t))$ satisfies an LDP with rate function I. Provided $vw^{-1} \in \mathscr{S}_r$ and $i(wv^{-1}) > 0$, we can bring the LDP into a linear scaling by invoking Theorem 2 with $p = vw^{-1}u$, $s = v^{-1}$ and

r, q equal to the identity. Setting $\tilde{Z}_t = vw^{-1}u(Z_{v^{-1}(t)})$, Theorem 2 then tells us that $(t^{-1}\tilde{Z}_t, t)$ satisfies an LDP with rate function $\tilde{I} = I\widehat{wv^{-1}}$. Returning to the original LDP, we can instead use Theorem 1 to obtain the LDP for $(w(T_z)/u(z), vw^{-1}u(z))$ with rate function $J(x) = \widehat{vw^{-1}}(x)I(1/x)$, and then use Theorem 2 to bring this to a linear scaling of an LDP for the pair $(z^{-1}\tilde{T}_z, z)$, with rate function $\tilde{J} = J\widehat{wv^{-1}}$, where $\tilde{T}_z = v(T_{u^{-1}wv^{-1}(z)})$.

Since, as one can verify, $(\tilde{Z}, \tilde{T})$ are inverse processes, we also could have obtained this last LDP by applying Theorem 1 to the LDP for $\tilde{Z}$. We can summarize this relation between the LDPs by the commutative diagram shown in Figure 1. In it, the notation $(u(Z_t)/w(t), v(t), I)$ is used to denote the statement that $(u(Z_t)/w(t), v(t))$ satisfies an LDP with rate function I with no peaks. On the arrows, we have indicated the theorem used to prove the equivalence, the conditions for this to hold and the relationship between the rate functions for equivalent LDPs.

The relationship between rate functions in the last line follows from the fact that, with $\tilde{I}$, $\tilde{J}$ as defined,

$$I(x) = \widehat{vw^{-1}}(x)J(1/x) \Leftrightarrow \tilde{I}(x) = x\tilde{J}(1/x). \tag{22}$$

This is just the relationship for linear scalings established in [11]. It is worth remarking that, even if $\tilde{I}$ and $\tilde{J}$ are convex I and J need not be convex or even Y-shaped (see [22]) in general. From the commutative diagram, we see that much of Theorem 1 could be deduced from Theorem 2 and the linear scaling result in [11]. However, it would be incorrect to conclude from this that the theory of large deviations for inverse processes with nonlinear scalings reduces to that known for the linear case. There are three reasons for this. The first is the necessity of the condition $vw^{-1} \in \mathscr{S}_r$. This is required either to apply Theorem 1 to obtain the relation between the LDPs for the inverse processes (Z, T), or to use Theorem 2 to convert either of these to an LDP with a linear scaling. The second reason is that, even with linear scalings, Theorem 1 applies without requiring the rate functions to be convex (as does

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\left(\frac{u(Z_t)}{w(t)}, v(t), I\right) \arrow[r, leftrightarrow, "\text{Thm. 1: } vw^{-1} \in \mathcal{S}_r", "I(x) = \widehat{vw^{-1}}(x)J(1/x)"'] \arrow[d, leftrightarrow, "\begin{array}{c}\tilde{Z}_t = vw^{-1}u(Z_{v^{-1}(t)}) \\ \tilde{I} = I\widehat{wv^{-1}}\end{array}", "\begin{array}{c}\text{Thm. 2}: vw^{-1} \in \mathcal{S}_r \\ i(vw^{-1}) > 0\end{array}"'] & \left(\frac{w(T_z)}{u(z)}, vw^{-1}u(z), J\right) \arrow[d, leftrightarrow, "\begin{array}{c}\text{Thm. 2}: vw^{-1} \in \mathcal{S}_r \\ i(vw^{-1}) > 0\end{array}", "\begin{array}{c}\tilde{T}_z = v(T_{u^{-1}wv^{-1}(z)}) \\ \tilde{J} = J\widehat{wv^{-1}}\end{array}"'] \\
\left(\frac{\tilde{Z}_t}{t}, t, \tilde{I}\right) \arrow[r, leftrightarrow, "\tilde{I}(x) = x\tilde{J}(1/x)", "\text{Thm. 1}"'] & \left(\frac{\tilde{T}_z}{z}, z, \tilde{J}\right)
\end{tikzcd}$$

FIG. 1. *Commutative diagram relating LDPs, based on Theorems* 1 *and* 2.

[11]) or Y-shaped (as does [22]). The third and most important reason is that a "known" LDP, which gives the starting point in the diagram, will not generally be one with linear scalings. This becomes evident when one considers that LDPs are usually established through use of the Gärtner–Ellis theorem. The CGF in (16) may be readily calculated for some nonlinear scaling functions (u, v, w), whereas the corresponding putative CGF for linear scalings $\lim_{t\to\infty} t^{-1}\log \mathbf{E}[e^{\theta \tilde{Z}_t}]$ may not be easy to determine. The price of linearizing the scalings has been to introduce nonlinearity into the processes through the relation $\tilde{Z}_t = vw^{-1}u(Z_{v^{-1}(t)})$. We give an example of this in Section 5.

We now investigate the existence of CGFs in more detail. Given an LDP for $(u(Z_t)/w(t), v(t))$ with rate function I satisfying the conditions of Theorem 1, it is not a priori clear that the CGF in (16) exists, or that the analogous CGF exists for any scaling compatible with u, v, w through Theorem 2. However, we are able to demonstrate existence of the CGF for the compatible linear scalings, at least on an interval. In the following theorem we will use the scaling $(vw^{-1}u(Z_t)/v(t), v(t))$, which differs only by a time rescaling from the linear scaling used above.

THEOREM 3. *Let (Z, T) be inverse processes satisfying the hypotheses of Theorem 1 with rate functions I and J, respectively, and let $i(vw^{-1}) \in (0, \infty)$. Then we have the following*:

(i) *The limit*

$$(23)\qquad \nu(\theta) = \lim_{t\to\infty} \nu_t(\theta) \quad \textit{where } \nu_t(\theta) = \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta vw^{-1}u(Z_t))],$$

exists for all $\theta < J(0)$ and is equal to $(\widehat{Iwv^{-1}})^*(\theta)$. If, furthermore, ν is essentially smooth, then $\widehat{Iwv^{-1}}$ is the Legendre transform of ν and is hence convex.

(ii) *The limit*

$$(24)\qquad \omega(\theta) = \lim_{t\to\infty} \omega_t(\theta) \quad \textit{where } \omega_t(\theta) = \frac{1}{vw^{-1}u(z)} \log \mathbf{E}[\exp(\theta v(T_z))],$$

exists for all $\theta < I(0)$ and is equal to $(\widehat{Jwv^{-1}})^*(\theta)$. If, furthermore, ω is essentially smooth, then $\widehat{Jwv^{-1}}$ is the Legendre transform of ω and is hence convex.

Beyond existence, essential smoothness for ν or ω is the remaining condition of the Gärtner–Ellis theorem [5]. Essential smoothness for ν, say, requires that ν is differentiable and either $J(0) = \infty$ or $J(0) < \infty$ with $\lim_{\theta \nearrow J(0)} \nu'(\theta) = \infty$.

To conclude this section on the general theory, we note that the underlying domains for the sample paths of Z and T need not be the whole of $\mathbf{R}_+$. More generally, we could have unbounded subsets χ_1, χ_2 of $\mathbf{R}_+$ with processes $(Z_t\colon t \in \chi_1)$ taking values in χ_2 and conversely $(T_z\colon z \in \chi_2)$ taking values in χ_1. A simple example is when χ_1 or χ_2 is equal to $\mathbf{Z}_+$. Russell [22] has given a general framework within which such cases can be handled. In this paper we will admit a less general extension by allowing also χ_1 and χ_2 to be discrete subsets of $\mathbf{R}_+$ containing 0 and without points of accumulation. A simple example is when χ_1 and/or χ_2 is equal to $\mathbf{Z}_+$. We shall say that that $((Z_i)_{i\in\chi_1}, (T_j)_{j\in\chi_2})$ are discrete inverse processes if they satisfy all the above properties of inverse process apart from right and left continuity. Define for all $t, z, \in \mathbf{R}_+$,

$$\lfloor t\rfloor = \sup\{i \in \chi_1 \colon i \le t\} \quad\text{and}\quad \lceil z\rceil = \inf\{j \in \chi_2 \colon j \ge z\}, \tag{25}$$

and extend Z and T to $\mathbf{R}_+$ by setting

$$Z'_t = Z_{\lfloor t\rfloor} \quad\text{and}\quad T'_z = T_{\lceil z\rceil}. \tag{26}$$

The following theorem shows that in order to treat LDPs for discrete inverse processes (Z, T) it is sufficient to prove the corresponding result for (Z', T'). We prove the result for linear scalings only; we can reduce any other case to this by use of Theorem 2.

THEOREM 4. *Suppose that* $((Z_i)_{i\in\chi_1}, (T_j)_{j\in\chi_2})$ *are discrete inverse processes.*

(i) $((Z'_t)_{t\in\mathbf{R}_+}, (T'_z)_{z\in\mathbf{R}_+})$ *are inverse processes.*

(ii) *Suppose* $\lim_{t\to\infty}\lfloor t\rfloor/t = 1$. *Then* $(Z_i/i, i)_{i\in\chi_1}$ *satisfies an LDP with rate function I with no peaks if and only if* $(Z'_t/t, t)_{t\in\mathbf{R}_+}$ *satisfies an LDP with rate function I with no peaks.*

(iii) *Suppose* $\lim_{z\to\infty}\lceil z\rceil/z = 1$. *Then* $(T_j/j, j)_{j\in\chi_2}$ *satisfies an LDP with rate function I with no peaks if and only if* $(T'_z/z, z)_{z\in\mathbf{R}_+}$ *satisfies an LDP with rate function I with no peaks.*

3. Application to queueing models. We wanted to investigate the relationship between the LDPs for inverse processes because of the applications to single-server queues. For example, let Z_t represent the cumulative work arriving at a queue in the interval $(-t, 0]$. We suppose that this work is served at constant rate s, with the unprocessed excess waiting in a queue with unlimited capacity. Let the excess workload be $W_t = Z_t - st$, $t > 0$, with $W_0 = 0$. Then the queue length Q of unprocessed work at time 0 can be written as the pathwise supremum

$$Q = \sup_{t>0} W_t; \tag{27}$$

see Chapter 1 of [2].

Under very general assumptions, the tail asymptotics of the distribution of Q can be related to the large deviation properties of the arrival process Z. However, in a given system (model or actual) the large deviation properties may be more easily or more accurately determined for the inverse process than for the arrival process itself. For this reason, we want to have the option to determine the queue tail asymptotics directly from the inverse process itself. This relation is already understood for a large class of short-range dependent arrival processes; the same cannot be said for long-range dependent arrival processes. In this section we will use Theorem 1, along with some additional analysis, to supply such a relation. We start with a brief review of the large deviation asymptotics for the tail of the queue length distribution.

Suppose that, for some scaling function v, the CGF of the arrival process

$$\lambda(\theta) = \lim_{t\to\infty} \lambda_t(\theta) = \lim_{t\to\infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v(t) Z_t/t)] \tag{28}$$

exists as an extended real-valued function of θ. The corresponding CGF for the excess workload process when the service rate is s is

$$\lambda_{(s)}(\theta) = \lim_{t\to\infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v(t) W_t/t)] = \lambda(\theta) - s\theta. \tag{29}$$

Suppose also that for an additional scaling function h, the limit

$$f(x) = \lim_{t\to\infty} h(t)/v(t/x) \tag{30}$$

exists. In [9] it is shown (subject to certain technical conditions) that

$$\lim_{b\to\infty} \frac{1}{h(b)} \log \mathbf{P}[Q > b] = -\delta \quad \text{with} \tag{31}$$

$$\delta = \inf_{x>0} \lambda^*_{(s)}(x)/f(x). \tag{32}$$

The stability requirement for this queue is that the service rate is sufficiently large to have

$$\lambda_{(s)}(\theta) < 0 \quad \text{for some } \theta > 0. \tag{33}$$

EXAMPLE 3.1. When v is regularly varying, we can choose $h = v$, and so $f = \widehat{v}$ in (30).

EXAMPLE 3.2. The canonical example of a long-range dependent process to which this theory applies is when Z is fractional Brownian motion (see [16], although this example falls outside the scope of processes considered in this paper since Z_t is not nondecreasing). Then Z is a mean-zero Gaussian process with $\mathrm{Var}(Z_t) = t^{2H}$. Then the appropriate scaling functions are $w(t) = t$, $v(t) = t^{2-2H}$ and so from (32) the distribution of Q is asymptotically Weibullian. The large deviation lower bound was obtained first in [18].

We can contrast Example 3.2 with short-range dependent arrival process, for example, Markov processes. For these, the appropriate scaling for the existence of the CGF in (29) is $v(t) = t$:

$$\lambda_{(s)}(\theta) = \lim_{t\to\infty} t^{-1} \log \mathbf{E}[\exp(\theta(Z_t - st))] \tag{34}$$

The large deviation theory in this scaling was previously worked out by a number of authors; see [12], the LD heuristic in [14] and [3]. Using those results, or applying (32), we can see that the tail of the log queue-length distribution is linear.

Let us review briefly two related results for short-range dependent arrival processes. We wish to determine to what extent these can be generalized for long-range dependent processes. First ([3], [6], [12], [14]), the exponential decay rate δ can be expressed directly in terms of the CGF λ: $\delta = \inf_{x>0} x^{-1}\lambda^*_{(s)}(x)$ can be shown to satisfy

$$\delta = \sup\{\theta\colon \lambda^*_{(s)}(\theta) \le 0\}. \tag{35}$$

Second, if we define the CGF for the inverse process

$$\mu_{(s)}(\theta) = \lim_{t\to\infty} z^{-1} \log \mathbf{E}[\exp(\theta(z - sT_z))], \tag{36}$$

then ([11], [22]) under appropriate technical conditions

$$\lambda_{(s)}(\theta) \le 0 \quad \text{if and only if } \mu_{(s)}(\theta) \le 0. \tag{37}$$

This means that δ can be found by two different routes:

$$\delta = \sup\{\theta\colon \lambda_{(s)}(\theta) \le 0\} = \sup\{\theta\colon \mu_{(s)}(\theta) \le 0\}. \tag{38}$$

We mention some work where these observations have been used to some advantage: [10] proposes predicting cell loss ratios for short-range dependent traffic on the basis of (32), by sampling an arrival process to measure $\lambda_{(s)}$ and hence estimating δ by means of (35). But (38) shows that an alternative strategy is to estimate δ through measuring the CGF $\mu_{(s)}$ due to the inverse process. This was proposed in [4] and in some cases found to be more effective than estimation through measurement of $\lambda_{(s)}$.

It is natural to ask, then, to what extent (35) and (38) can be extended to the case when the arrival process and its inverse satisfy an LDP with more general scalings. The first matter is settled by the following lemma, which appeared in a slightly less general context in [7].

Let K and f be real functions, with f not necessarily invertible. Then we define $(Kf^{-1})^*$ to be

$$(Kf^{-1})^*(\theta) = \sup_{x\in\mathrm{dom}(f)} (f(x)\theta - K(x)). \tag{39}$$

When f is invertible, one sees by changing variable from x to $f^{-1}(x)$ that this coincides with the usual definition of $(Kf^{-1})^*$. Our reason for allowing f to be noninvertible is to allow for the case that v is regularly varying within index 0, for then $\hat{v}(x) = 1$. This happens, for example, when $v(x) \sim \log x$, a scaling which has been used for the large deviation analysis of queues arising from fluid processes driven by $M/G/\infty$ processes: this was proposed in [19]; see also [8] for a further application.

LEMMA 3. *Let K be a function on $\mathbf{R}_+$ and f a scaling function on $\mathbf{R}_+$. Then*

$$\inf_{x>0} K(x)/f(x) = \sup\{\theta \ge 0\colon (Kf^{-1})^*(\theta) \le 0\}, \tag{40}$$

for $(Kf)^*$ as defined in (39).

PROOF. Let $\delta = \inf_{x>0} K(x)/f(x)$. Then

$$\begin{aligned}\theta \le \delta \Leftrightarrow \theta \le K(x)/f(x)\ \forall\ x > 0 \Leftrightarrow \theta f(x) - K(x) \\ \le 0\ \forall\ x > 0 \Leftrightarrow (Kf^{-1})^*(\theta) \le 0.\end{aligned} \tag{41}$$

□

Lemma 3 enables us to give an alternate expression for the queue-tail decay constant δ in (31) when the scaling function v in (28) is regularly varying. Furthermore, it can be expressed directly in terms of a CGF for a rescaled LDP of the arrival process, when the CGF exists.

THEOREM 5. *Suppose that the CGF* λ *in* (28) *exists with* $v \in \mathscr{S}_r$. *Let* δ *be defined by* (31) *with* $h = v$ *and* $f = \hat{v}$. *Then we have the following*:

(i) $\delta = \sup\{\theta \ge 0\colon (\lambda^*_{(s)}\hat{v}^{-1})^*(\theta) \le 0\}$ *for* $\lambda_{(s)}$ *in* (29).

(ii) *Assume also that* λ *is essentially smooth*, $i(v) > 0$ *and that* s *lies in the interior of the effective domain of* λ^*. *Define*

$$\nu_{(s),\,t}(\theta) = \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t \ge st]. \tag{42}$$

Then the limit $\nu_{(s)} = \lim_{t\to\infty} \nu_{(s),\,t}(\theta)$ *exists for all* $\theta < J(0)$, *where* $J(x) = \hat{v}(x)\lambda^*(1/x)$, *and is equal to* $(\lambda^*_{(s)}\hat{v}^{-1})^*(\theta)$.

(iii) *Under the assumptions of* (ii), *a sufficient condition for* $J(0)$ *to be infinite is that* $i(v) < 1$. *A sufficient condition that* s *lies in the interior of the effective domain of* λ^* *is that there exists* $0 < \theta_1 < \theta_2 < \infty$ *for which* $-\infty < \lambda_{(s)}(\theta_1) < 0 < \lambda_{(s)}(\theta_2) < \infty$. *In this case,*

$$\delta = \sup\{\theta \ge 0\colon \nu_{(s)}(\theta) \le 0\}. \tag{43}$$

REMARK 3.1. When $v(t) = t$, (i) reduces to (35).

REMARK 3.2. Fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$ is an example of a process satisfying an LDP with $i(x) < 1$ since $f(x) = v(x) = x^{2-2H}$.

We are also able to obtain a relation analogous to (38) for the inverse process LDP, and an analog of (43) expressing δ directly in terms of a CGF for the inverse process.

THEOREM 6. *Let* I, J *and* f *be functions on* $(0, \infty)$ *with* $f(x) = x^p$ *for some* $p > 0$ *and* $I(x) = f(x)J(1/x)$. *For* $s > 0$ *set*

$$\begin{aligned} I_s(x) &= \begin{cases} I(s + x), & x \ge 0, \\ +\infty, & \textit{otherwise}, \end{cases} \\ J_s(x) &= \begin{cases} J((1 - x)/s), & x \in [0, 1), \\ +\infty, & \textit{otherwise}. \end{cases} \end{aligned} \tag{44}$$

Then

$$\sup\{\theta\colon (I_s f^{-1})^*(\theta) \le 0\} = \sup\{\theta\colon (J_s f^{-1})^*(\theta) \le 0\}. \tag{45}$$

THEOREM 7. *Suppose that the following conditions hold*:

(i) (Z, T) *are inverse processes.*
(ii) $v \in \mathscr{S}_r$ *and* $(Z_t/t, v(t))$ *satisfies an LDP with rate function* I, *which has no peaks and contains* s *within the interior of its effective domain.*

Set $\delta = \inf_{x>0} I_{(s)}(x)/\hat{v}(x)$ *where* $I_{(s)}(x) = I(x+s)$. *Then, for all* θ, *the limit*

$$\begin{aligned}\omega_{(s)}(\theta) &= \lim_{z\to\infty} \omega_{(s),\,z}(\theta)\\ &= \lim_{z\to\infty} \frac{1}{v(z)} \log \mathbf{E}[\exp(\theta v(z - sT_z));\ 0 \le sT_z \le z]\end{aligned} \tag{46}$$

exists and is equal to $(J_s \hat{v}^{-1})^*(\theta)$. *Hence* δ *can be reexpressed as*

$$\delta = \sup\{\theta\colon \omega_{(s)}(\theta) \le 0\}. \tag{47}$$

An interesting feature of the CGFs in (42) and (46) is the extra event in the expectation, $\{Z_t \ge st\}$ in (42) and $\{T_z \le z/s\}$ in (46). We now show that these extra events can be removed by appropriately extending the scaling function v to the entire real line $(-\infty, \infty)$. We assume that v is extended in an arbitrary manner from the increasing homeomorphism on $[0, \infty)$ to an increasing homeomorphism of $(-\infty, \infty)$ with $v(0) = 0$, but we will require that $v(-t) \le -v(t)$ for $t \ge 0$. The CGFs in (5) and (6) correspond to the extension in which $v(t) = -\infty$ for all $t < 0$. We will only state the result for Z.

THEOREM 8. *In addition to the conditions of Theorem* 5(ii), *assume that* $I(a) > 0$, *for some* a *with* $a < s$, *that* $\delta < J(0)$ *and that the scaling function* v *is extended to* $(-\infty, \infty)$ *with* $v(0) = 0$ *and* $v(-t) \le -v(t)$ *for* $t > 0$. *Then there exists an* $\varepsilon > 0$ *for which*

$$\phi_{(s)}(\theta) = \nu_{(s)}(\theta) \quad \text{for all } \theta > \delta - \varepsilon, \tag{48}$$

so that

$$\delta = \sup\{\theta \ge 0\colon \phi_{(s)}(\theta) \le 0\}, \tag{49}$$

where

$$\phi_{(s)}(\theta) = \lim_{t\to\infty} \frac{1}{v(t)} \log E[\exp(\theta v(Z_t - st))]. \tag{50}$$

REMARK 3.3. We cannot expect that (48) will hold for all θ. In particular, for $\theta = 0$, $\phi_{(0)}(0) = 0$, but typically

$$\nu_{(s)}(0) = \lim_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}[Z_t > st] = -I(s) < 0. \tag{51}$$

REMARK 3.4. In applications we anticipate that we will have $t^{-1}Z_t \to m$ for $m < s$ with $I(m) = 0$ and $I(a) > 0$ for all a with $a > m$, which implies the condition on I in Theorem 8.

REMARK 3.5. Theorem 8 only applies simply in the linear case, because then it suffices to have $v(t) = t$ for $t \in (-\infty, \infty)$, which makes the expectations relatively easy to compute. In the linear case, the link between (49) and (43) is already established by (35) and (43).

4. Functional large deviations principles. The purpose of this section is to consider relations between *functional large deviations principles* (FLDPs) with nonlinear scaling for inverse processes. As mentioned in the introduction, we are interested to see to what extent our LDP results can be derived via FLDPs. We are able to establish results via FLDPs, but only in a special case. We apply relations between FLDPs with linear scalings for inverse processes from [20]. We refer to [20] for background.

The FLDPs hold in the function space $D \equiv D[0, \infty)$ of right-continuous real-valued functions on $[0, \infty)$ with limits from the left everywhere (except at 0). It is customary to consider right-continuous versions of the functions, but the topology on D we introduce will be the same for left-continuous and right-continuous functions. As in [20], we consider the M_1' topology, which can be characterized in terms of parameterizations of the completed graphs $\Gamma(x)$ for $x \in D$, defined in (10). (The completed graphs do not depend on the right continuity.) We call a pair of continuous functions $(u, t) \equiv (u(s), t(s))\colon s \geq 0)$ such that $t(s)$ is nondecreasing with $t(0) = 0$ a *parameterization* of $\Gamma(x)$ if

$$\Gamma(x) = \cup\{(u(s), t(s))\colon s \geq 0\}. \tag{52}$$

A sequence $\{x_n\colon n \geq 1\}$ in D converges to x in (D, M_1') if there exist parameterizations (u_n, t_n) of x_n and (u, t) of x such that

$$\sup_{s \leq T}\{|u_n(s) - u(s)| + |t_n(s) - t(s)|\} \to 0 \quad \text{as } n \to \infty \tag{53}$$

for all $T > 0$. The M_1' topology on D is metrizable as a separable metric space. When x is continuous, $x_n \to x$ in D if and only if $x_n(t) \to x(t)$ uniformly on bounded intervals. The Borel σ-field on D generated by the M_1' topology coincides with the usual Kolmogorov σ-field generated by the coordinate projections.

We now apply Theorem 3.3 of [20] to obtain a relation between FLDPs with nonlinear scaling functions. Let $E^{\uparrow}$ be the subset of nonnegative nondecreasing functions unbounded above in D.

THEOREM 9. *Let (Z, T) be inverse processes, and let $u, v, w \in \mathscr{S}$. The random functions*

$$X_n(t) \equiv \frac{u(Z_{w^{-1}(wv^{-1}(n)t)})}{wv^{-1}(n)}, \qquad t \geq 0 \tag{54}$$

obey an FLDP in $(E^{\uparrow}, M_1')$ as $n \to \infty$ with rate function $\hat{I}$ if and only if the random functions

$$X_n^{-1}(t) \equiv \frac{w(T_{u^{-1}(wv^{-1}(n)t)})}{wv^{-1}(n)}, \qquad t \geq 0 \tag{55}$$

obey an FLDP in $(E^{\uparrow}, M_1')$ *as* $n \to \infty$ *with rate function* $\hat{J}$. *If these FLDPs hold, then*

$$\hat{J}(x) = \inf_{\substack{y \in E^{\uparrow} \\ x = y^{-1}}} \hat{I}(y) = \hat{I}(x^{-1}), \qquad x \in E^{\uparrow} \tag{56}$$

and

$$\hat{I}(x) = \inf_{\substack{y \in E^{\uparrow} \\ x = y^{-1}}} \hat{J}(y) = \hat{J}(x^{-1}), \qquad x \in E^{\uparrow} \tag{57}$$

where x^{-1} *is the right-continuous inverse of* x, *defined by*

$$x^{-1}(t) = \inf\{s > 0 \colon x(s) > t\}, \qquad t > 0. \tag{58}$$

In some special cases we can characterize the rate functions. The following is a consequence of Theorem 3.4 of [20].

THEOREM 10. (a) *The process* $\{X_n\}$ *in* (54) *obeys an FLDP in* $(E^{\uparrow}, M_1')$ *with rate function*

$$\hat{I}(x) = \int_0^{\infty} I(\dot{x}(t))\, dt \tag{59}$$

for all absolutely continuous x *with* $x(0) = 0$, *and* $I(x) = \infty$ *otherwise, where* $I(0) = \infty$, *if and only if* $\{X_n^{-1}\}$ *in* (55) *obeys an FLDP in* $(E^{\uparrow}, M_1')$ *with rate function*

$$\hat{J}(x) = \int_0^{\infty} J(\dot{x}(t))\, dt \tag{60}$$

for all absolutely continuous x *with* $x(0) = 0$, *and* $\hat{J}(x) = \infty$ *otherwise, where* $J(0) = \infty$. *If* (59) *and* (60) *hold, then*

$$J(z) = zI(1/z). \tag{61}$$

(b) *If the function* I *in* (59) [*respectively*, J *in* (60)] *is convex, then the sequence of random variables* $\{X_n(1)\colon n \ge 1\}$ [*respectively*, $\{X_n^{-1}(1)\colon n \ge 1\}$] *obeys an LDP in* $\mathbf{R}$ *with rate function* I (*respectively*, J).

In Theorem 9 and 10 we can also let $n \to \infty$ in a continuous manner. The following result relates the FLDPs to the LDPs in Theorem 1.

THEOREM 11. *Let* (Z, T) *be inverse processes and let* $u, v, w \in \mathscr{S}$. *If* $\{X_n\}$ *in* (54) *obeys an FLDP in* $(E^{\uparrow}, M_1')$ *with rate function* $\hat{I}$ *in* (59) *where the local rate functions in* I *in* (59) *and* J *in* (61) *are convex, then* $(u(Z_t)/w(t), v(t))$ *and* $(w(T_z)/u(z), vw^{-1}u(z))$ *obey LDPs in* $\mathbf{R}$ *with rate functions* I *and* J, *where* (61) *holds.*

The significant difference between Theorem 11 and Theorem 1, however, is that here the rate functions J and I are related by (61), which only holds in Theorem 1 for the special case of linear scaling, that is, when $vw^{-1} \in \mathscr{S}_1$.

Evidently the extra conditions here restrict the possible LDPs that can hold in $\mathbf{R}$ for the projections. Here we require (1) that the FLDP hold, (2) that the rate functions $\hat{I}$ and $\hat{J}$ on D have the special integral form in (59) and (60) and (3) that the local rate functions I and J appearing in (59) and (60) be convex. From Theorem 1 we know that we must have $vw^{-1} \in \mathscr{S}_r$ in order to have the LDPs in $\mathbf{R}$. Now we know that we must have $vw^{-1} \in \mathscr{S}_1$ in order to have both the FLDPs and the associated LDPs for the projections. (We have shown that this is a necessary condition.) It still remains to determine what forces the relation (61) here.

5. Examples. In this section we give several examples of LDPs with nonlinear scaling.

The supremum of fractional Gaussian noise. Let B_t be fractional Brownian noise [16], that is, the restriction of fractional Brownian motion to integer times. In particular $B_0 = 0$ and $(B_t)_{t \in \mathbf{Z}_+}$ is a Gaussian process with $\operatorname{Var} B_t = t^{2H}$ for some $H \in (0, 1)$, known as the Hurst parameter. Let Z_t be the nondecreasing process $Z_t = \sup_{0 \le s \le t} B_t$. We first establish the large deviation properties of Z_t, then use Theorem 1 to establish the LDP associated with the hitting time

$$T_z = \inf\{t \ge 0\colon Z_t \ge z\} = \inf\{t \ge 0\colon B_t \ge z\}. \tag{62}$$

PROPOSITION 1. (i) $(Z_t/t, t^{2-2H})$ *satisfies an LDP with rate function* $I(x) = x^2/2$.

(ii) $(T_z/z, z^{2-2H})$ *satisfies an LDP with rate function* $J(x) = x^{-2H}/2$.

PROOF. (i) The LDP for $(Z_t/t, t^{2-2H})$ follows from the Gärtner–Ellis theorem if we can show that the CGF

$$\lambda(\theta) = \lim_{t\to\infty} t^{2H-2} \log \mathbf{E}[\exp(\theta Z_t)] = \theta^2/2 \tag{63}$$

exists for all $\theta \in \mathbf{R}$. For $\theta = 0$ this is trivial. For $\theta > 0$,

$$\exp(\theta B_t) \le \exp(\theta Z_t) \le \sum_{0 \le s \le t} \exp(\theta B_s) \tag{64}$$

and hence

$$\begin{aligned} \exp(\theta^2 t^{2-2H}/2) &= \mathbf{E}[\exp(\theta B_t)] \le \mathbf{E}[\exp(\theta Z_t)] \le \sum_{0 \le s \le t} \mathbf{E}[\exp(\theta B_s)] \\ &= \sum_{0 \le s \le t} \exp(\theta^2 s^{2-2H}/2) \le (s+1)\exp(\theta^2 t^{2-2H}/2), \end{aligned} \tag{65}$$

from which (63) is established for $\theta > 0$. For $\theta < 0$,

$$\frac{\sum_{0 \le s \le t} \exp(\theta B_s)}{s+1} \le \exp(\theta Z_t) \le \exp(\theta B_t) \tag{66}$$

and hence

$$\begin{aligned}\frac{\exp(\theta^2 t^{2-2H}/2)}{s+1} &\le \frac{\sum_{0\le s\le t}\exp(\theta^2 s^{2-2H}/2)}{s+1}\\ &= \frac{\sum_{0\le s\le t}\mathbf{E}[\exp(\theta B_s)]}{s+1} \le \mathbf{E}[\exp(\theta Z_t)]\\ &\le E[\exp(\theta B_t)] = \exp(\theta^2 t^{2-2H}/2),\end{aligned} \tag{67}$$

from which (63) is established for $\theta < 0$.

(ii) Combine (i) with Theorem 1. □

Time-transformed stationary processes. As in Section 7 of [17], we can construct nonstationary processes to serve as models for nonstationary phenomena by performing a deterministic time transformation to a stationary stochastic process. Suppose that we have a stationary stochastic process $(S_t: t \ge 0)$ for which $(t^{-1}S_t, v(t))$ obeys an LDP with rate function I. We can then construct a nonstationary stochastic process $(Z_t: t \ge 0)$ from S_t and a scaling function w by letting

$$Z_t = S_{w(t)}, \qquad t \ge 0. \tag{68}$$

Given the construction in (68), we see that trivially $(Z_t/w(t), vw(t))$ obeys an LDP as $T \to \infty$ with rate function I. We now can apply Theorem 1 to deduce that the inverse process of Z_t obeys an LDP as well. In particular, if $v \in \mathscr{S}_r$, then Theorem 1 implies that $(w(T_z)/z, v(z))$ obeys an LDP as $z \to \infty$ with rate function $J(x) = \hat{v}(x)I(1/x)$, where $(T_z: z \ge 0)$ is the inverse process of $(Z_t: t \ge 0)$.

The analysis applies directly if Z_t is a nonhomogeneous Poisson process with deterministic intensity $\alpha(t)$. If we let

$$w(t) = \int_0^t \alpha(s)\,ds, \tag{69}$$

then (68) holds, where S_t is a homogeneous Poisson process with rate 1. Moreover $(t^{-1}S_t, t)$ obeys an LDP with rate function

$$I(x) = x\log x - x + 1. \tag{70}$$

Let $(T_z: z \ge 0)$ be the inverse process of the nonhomogeneous Poisson process, that is, representing the successive arrival times if Z_t is an arrival counting process. Then $(w(T_z)/z, z)$ obeys an LDP as $z \to \infty$ with rate function $J(x) = xI(1/x)$ for I in (70).

Compound processes. In a standard queueing model, the input Z_t is often a random sum of service times, that is,

$$Z_t = \sum_{i=1}^{A_t} S_i, \qquad t \ge 0, \tag{71}$$

where S_i is the service time of the ith customer and A_t counts the number of arrivals in $[0, t]$. We now show how the CGF of $(Z_t: t \geq 0)$ can be obtained from CGFs of $\{S_i\}$ and $(A_t: t \geq 0)$. The following extends Proposition 7 of [12] to nonlinear scalings.

PROPOSITION 2. *Suppose that* $(Z_t: t \geq 0)$ *has the special form* (71) *with* $\{S_i: i \geq 1\}$ *and* $(A_t: t \geq 0)$ *independent. Let* u, v *and* w *be scaling functions and suppose that*

$$\frac{1}{w(n)} \log \mathbf{E}\left[\exp\left(\theta u\left(\sum_{i=1}^{n} S_i\right)\right)\right] \to \psi_S(\theta) \quad \text{as } n \to \infty \tag{72}$$

for all θ *in a neighborhood of* θ_0 *and*

$$\frac{1}{v(t)} \log E[\exp(\theta w(A(t)))] \to \psi_A(\theta) \quad \text{as } t \to \infty \tag{73}$$

for all θ *in a neighborhood of* $\psi_S(\theta_0)$, *where* $\psi_S(\theta)$ *and* $\psi_A(\theta)$ *are finite. Then*

$$\frac{1}{v(t)} \log E[\exp(\theta_0 u(Z_t))] \to \psi_A(\psi_S(\theta_0)). \tag{74}$$

PROOF. The assumptions imply that $\psi_S(\theta)$ is continuous at θ_0 and $\psi_A(\theta)$ is continuous at $\psi_S(\theta_0)$: the functions are convex by Hölder's inequality; then apply Theorem 10.1 of [21]. For any $\varepsilon > 0$, there is an M such that

$$\begin{aligned}
\mathbf{E}[\exp(\theta_0 u(Z_t))] &= \sum_{n=1}^{\infty} \mathbf{E}\left[\exp\left(\theta_0 u\left(\sum_{i=1}^{n} S_i\right)\right)\right] \mathbf{P}[A(t) = n] \\
&\leq \sum_{n=1}^{\infty} \exp(w(n)[\psi_S(\theta_0) + \varepsilon])\, \mathbf{P}[A(t) = n] + M \\
&\leq \mathbf{E}[\exp(w(A(t))[\psi_S(\theta_0) + \varepsilon])] + M \\
&\leq \exp(v(t)[\psi_A(\psi_S(\theta_0) + \varepsilon) + \varepsilon]) + M
\end{aligned}$$

for t suitably large. Hence

$$\limsup_{t \to \infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta u(Z_t))] \leq \psi_A(\psi_S(\theta_0) + \varepsilon) + \varepsilon. \tag{75}$$

Since ε was arbitrary and ψ_A is continuous at $\psi_S(\theta_0)$,

$$\limsup_{t \to \infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta u(Z_t))] \leq \psi_A(\psi_S(\theta_0)). \tag{76}$$

The reasoning in the other direction is similar. □

EXAMPLE. Now we consider the case in which $(Z_t: t \geq 0)$ is a compound Poisson process with the Poisson process being nonhomogeneous with intensity $\alpha(t)$. As indicated above, $(A_t/v(t), v(t))$ obeys an LDP with rate function I in (70). Moreover,

$$\frac{1}{v(t)} \log \mathbf{E}[\exp(\theta A_t)] \to \psi_A(\theta) = (I^*)(\theta). \tag{77}$$

Now suppose that $Z_t = \sum_{i=1}^{A_t} S_i$, where $\{S_i: i \geq 1\}$ is a sequence of nonnegative service times satisfying

$$\frac{1}{n} \log \mathbf{E}\left[\exp\left(\theta u\left(\sum_{i=1}^{n} S_i\right)\right)\right] \to \psi_S(\theta). \tag{78}$$

Then, by Proposition 2,

$$\frac{1}{v(t)} \log \mathbf{E}[\exp(\theta u(Z_t))] \to \psi_A(\psi_S(\theta)). \tag{79}$$

In the standard case with $\{S_i\}$ i.i.d., $\psi_S(\theta) = \mathbf{E}[\exp(\theta S_1)]$ and u is the identity map.

Renewal processes. Let $(Z_t: t \geq 0)$ be a (nondelayed) renewal process with interrenewal distribution F. In the inverse process $(T_n: n \in \mathbf{Z}_+)$, $T_0 = 0$ and T_n is the time of the nth renewal. (Here we could apply Theorem 4.) The independence of the interrenewal times means that an LDP for T is easy to obtain. Define $\mu(\theta) = \log \int dF(x) e^{\theta x}$. The LDPs for Z and T involve only linear scalings, and are proved in [11]. The LDP for T follows directly from Cramer's theorem (2.2.3 in [5]) applied to the sums of the i.i.d. random variables $T_i - T_{i-1}$. We include this result here in order to emphasize that the LDP applies with a linear scaling, even if the distribution F is long tailed. The CGF $\mu(\theta)$ will then be finite for all $\theta \leq 0$ but not for $\theta > 0$. Furthermore, we are able classify the qualitative features of the rate functions for Z and T in terms of μ.

PROPOSITION 3. *Let Z be a nondelayed renewal process with μ the interarrival time CGF. Then the following statements hold*:

(i) $(T_n/n, n)$ *satisfies an LDP with rate function* $J = \mu^*$;
(ii) $(Z_t/t, t)$ *satisfies an LDP with rate function* $I(x) = x\mu^*(1/x)$;
(iii) $J(x) = \infty$ *for* $x < \underline{x} \equiv \inf\{x: F(x) > 0\}$;
(iv) $J(\underline{x}) < \infty$ *if and only if* $F(\underline{x}) > 0$;
(v) $\mathbf{E}[T_1]$ *is a base for* J. *If* $\mu(\theta) = \infty$ *for all* $\theta > 0$, *then* $I(1/x) = J(x) = 0$ *for* $x \geq \mathbf{E}[T_1]$.

PROOF. First, (i) follows from Cramér's theorem. Then (ii) follows from Theorem 1 here or from Theorem 4 of [11].

(iii) $\mu(\theta) \le \underline{x}\theta$ for $\theta < 0$, and so for $x < \underline{x}$, $\mu^*(x) \ge \sup_{\theta<0}(\theta x - \mu(\theta)) \ge \sup_{\theta<0} \theta(x - \underline{x}) = \infty$.

(iv) Since $\underline{x} \le \mathbf{E}[T_1] = \mu'(0^-)$, then $\mu^*(\underline{x}) = \sup_{\theta<0}(\theta\underline{x} - \mu(\theta))$. If $F(\underline{x}) > 0$ then $\mu(\theta) \ge \theta\underline{x} + \log F(\underline{x})$ for all $\theta < 0$, and hence

$$\mu^*(\underline{x}) \le -\log F(\underline{x}). \tag{80}$$

Now for any $\varepsilon > 0$ and $\theta < 0$,

$$\begin{aligned}\mu(\theta) &\le \log(F(\underline{x}+\varepsilon)\exp(\theta\underline{x}) + F^c(\underline{x}+\varepsilon)\exp(\theta(\underline{x}+\varepsilon))) \\ &= \theta\underline{x} + \log(1 + F^c(\underline{x}+\varepsilon)(\exp(\theta\varepsilon)-1)).\end{aligned} \tag{81}$$

Hence

$$\mu^*(\underline{x}) = \sup_{\theta<0}(\underline{x}\theta - \mu(\theta)) \ge \limsup_{\theta\to-\infty}(\underline{x}\theta - \mu(\theta)) \ge -\log F(\underline{x}+\varepsilon) \tag{82}$$

by (81). Taking $\varepsilon \to 0$, then when $F(\underline{x}) > 0$, we combine with (80) to conclude that $\mu^*(\underline{x}) = -\log F(\underline{x})$. When $F(\underline{x}) = 0$, we conclude from the right continuity of F that $\mu^*(\underline{x}) = \infty$.

(v) That $\mathbf{E}[T_1]$ is a base for J (even when infinite) follows from the fact that it is the derivative from the left of μ at 0. If $x > \mathbf{E}[T_1]$, then by convexity of μ, $\theta x - \mu(\theta) \le 0$ for all $\theta \le 0$, with equality at 0. Since $\mu(\theta) = \infty$ for $\theta > 0$, then $\mu^*(x) = 0$ for such x. □

We can also directly bound the CGF of the renewal counting process Z.

PROPOSITION 4. *For any renewal process* $(Z_t\colon t \ge 0)$ *and any* $\varepsilon > 0$,

$$\mathbf{E}[\exp(\theta Z_t)] \le \left(\frac{\mathbf{P}[T_1 \ge \varepsilon]\exp(\theta\varepsilon)}{1 - \mathbf{P}[T_1 < \varepsilon]\exp(\theta\varepsilon)}\right)^{\lfloor t/\varepsilon \rfloor}. \tag{83}$$

PROOF. We bound the renewal process above by the renewal process with smaller interarrival times $\hat{T}_1$, where

$$\mathbf{P}[\hat{T}_1 = 0] = \mathbf{P}[T_1 < \varepsilon] = 1 - \mathbf{P}[\hat{T}_1 = \varepsilon] = 1 - \mathbf{P}[T_1 \ge \varepsilon]. \tag{84}$$

It is easy to see that the displayed formula is $\mathbf{E}[\exp(\theta\hat{Z}_t)]$ for the renewal process $(\hat{Z}_t\colon t \ge 0)$ associated with $\hat{T}_1$. □

EXAMPLE (Pareto interrenewal times). Consider a renewal process with F Pareto, that is, with $1 - F(x) = (1+x)^{-\alpha}$ some $\alpha > 0$. We can distinguish two cases: $\alpha \in (0,1]$ and $\alpha > 1$.

When $\alpha > 1$, the mean interrenewal time $\mathbf{E}[T_1]$ is finite and so $I(1/x) = J(x) = \mu^*(x) = 0$ for $x \ge \mathbf{E}[T_1]$, $F(0) = 0$, and clearly $J(x)$ increases to ∞ as x decreases from $\mathbf{E}[T_1]$ to 0. But if $\alpha \in (0,1]$, then $\mathbf{E}[T_1] = \mu'(0^-)$ is infinite and so ∞ is the only base for J: since $\mathbf{E}[T_1] = \mu'(0^-) = \infty$, then $J(x) = \mu^*(x) > 0$ for any $x < \infty$.

We conclude this section by pointing out that the usual LDP with linear scaling can hold for the input to a standard queue when the interarrival times have a long-tailed distribution, but the service times do not. We can apply Proposition 2 with scaling functions u, v and w all equal to the identity. Assuming that the service times are i.i.d. with a finite moment generating function $\check{\psi}(\theta) = \log \mathbf{E}[\exp(\theta S_1)]$ for θ less that some positive $\check{\theta}$, then (72) holds with $\psi_S = \check{\psi}$. We know from [11] that the MGFs ψ_A and ψ_T of the inverse processes (A, T) are related under very general conditions by $\psi_A(\theta) = -\psi_T^{-1}(-\theta)$. Thus the finiteness of $\psi_T(\theta)$ for all $\theta \le 0$ implies the finiteness of $\psi_A(\theta)$ for all $\theta \ge 0$. Hence, (74) can hold for the compound net input process Z defined by (71). Thus, under regularity conditions, the Gärtner–Ellis theorem will hold for Z and the queue asymptotics in (4) to (6) will hold with v being the identity function. In summary, the long-tailed interarrival time distribution does not in itself prevent exponential tails in the queue-length distribution.

6. Proofs of theorems. The proof of Theorem 1 proceeds via a number of subsidiary results. In the following, $h(x^-)$ and $h(x^+)$ denote the limits from the left and right of a function h at x, when these exist.

THEOREM 12. *Let (Z, T) be inverse processes, and let $u, v, w, \in \mathscr{S}$ with $vw^{-1} \in \mathscr{S}_r$.*

(i) *We have*

$$\begin{aligned} -f_+(x^+) &\le \liminf_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} \ge x\right] \\ &\le \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} \ge x\right] \le -f_+(x) \end{aligned} \tag{85}$$

for all $x \in (0, \infty)$ and some nondecreasing lower semicontinuous function f_+ on $(0, \infty)$, if and only if

$$\begin{aligned} -g_-(x^-) &\le \liminf_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) \le x]}{vw^{-1}u(z)} \\ &\le \limsup_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) \le x]}{vw^{-1}u(z)} \le -g_-(x) \end{aligned} \tag{86}$$

for all $x \in (0, \infty)$ and some nonincreasing lower semicontinuous function g_- on $(0, \infty)$, in which case,

$$g_-(x) = \widehat{vw^{-1}}(x) f_+(1/x). \tag{87}$$

(ii) *We have*

$$\begin{aligned} -f_-(x^-) &\le \liminf_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} < x\right] \\ &\le \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} < x\right] \le -f_-(x) \end{aligned} \tag{88}$$

for all $x \in (0,\infty)$ *and some nonincreasing lower semicontinuous function* f_- *on* $(0,\infty)$, *if and only if*,

$$\begin{aligned}(89)\qquad -g_+(x^+) &\le \liminf_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) > x]}{vw^{-1}u(z)} \\ &\le \limsup_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) > x]}{vw^{-1}u(z)} \le -g_+(x)\end{aligned}$$

for all $x \in (0,\infty)$ *and some nondecreasing lower semicontinuous function* g_+ *on* $(0,\infty)$, *in which case*

$$(90)\qquad g_+(x) = \widehat{vw^{-1}}(x) f_-(1/x).$$

(iii) *If, with* f_- *as in* (ii), (88) *holds for all* $x \in (0,\infty)$, *then for all* $x \in (0,\infty)$,

$$\begin{aligned}(91)\qquad -g_+(x^+) &\le \liminf_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) \ge x]}{vw^{-1}u(z)} \\ &\le \limsup_{z\to\infty} \frac{\log \mathbf{P}[w(T_z)/u(z) \ge x]}{vw^{-1}u(z)} \le -g_+(x).\end{aligned}$$

(iv) *If, with* g_+ *as in* (ii), (89) *holds for all* $x \in (0,\infty)$, *then for all* $x \in (0,\infty)$,

$$\begin{aligned}(92)\qquad -f_-(x^-) &\le \liminf_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} \le x\right] \\ &\le \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} \le x\right] \le -f_-(x).\end{aligned}$$

REMARK 6.1. The role of parts (iii) and (iv) here is to extend the result of part (ii) for open semiinfinite intervals to the corresponding closed semiinfinite intervals. This is required for Theorem 13 which follows.

PROOF. We prove (i) in one direction; the reverse is similar. Assume (85). Note that $f_+(x^+)$ exists by assumption that f_+ is nondecreasing. The key observation is that, for $x \in (0,\infty)$,

$$(93)\qquad \frac{1}{vw^{-1}u(z)} \log \mathbf{P}\left[\frac{w(T_z)}{u(z)} \le x\right] = \frac{1}{v(t)} \frac{vw^{-1}w(t)}{vw^{-1}(w(t)/x)} \log \mathbf{P}\left[\frac{u(Z_t)}{w(t)} \ge 1/x\right],$$

where $t = w^{-1}(u(z)x)$. Then (86) follows upon taking $z \to \infty$ (and hence $t \to \infty$) and using (12). Since vw^{-1} is continuous and f_+ lower semicontinuous, g_- is lower semicontinuous; but it remains to be shown that g is nonincreasing. A lower semicontinuous function is discontinuous at most on a meager set (i.e., a countable union of nowhere dense sets: see Section 231 of [13]), and so is continuous on a dense set Δ in $\mathbf{R}_+$. For $x \in \Delta$ the lim inf and lim sup in (86) are both equal to $-g_-(x)$. So, by construction, the restriction of g_- to Δ, as a limit of the nonincreasing functions $-\log \mathbf{P}[w(T_z)/u(z) \le x]/vw^{-1}u(z)$, is also nonincreasing.

We show that g_- is nonincreasing on the whole of $\mathbf{R}_+$. Since Δ is dense, then for all $x, x' \in \Delta^c$ with $x < x'$ there exists $y \in \Delta$ with $x < y < x'$. Thus it suffices to show for all such x, x', y that $g_-(x) \geq g_-(y)$ and $g_-(y) \geq g_-(x')$. For the first inequality, observe that $g_-(x) \geq \liminf_{\Delta \ni a \nearrow y} g_-(a) \geq g_-(y)$ by lower semicontinuity of g_-. For the second inequality, since f_+ is nondecreasing [hence $\exists \lim_{a \nearrow b} f_+(a) \leq f_+(b)$] and lower semicontinuous [hence $\lim_{a \nearrow b} f_+(a) \geq f_+(b)$], it is continuous from the left, so that g_- is continuous from the right. Hence $g_-(x') \geq \lim_{\Delta \ni a \searrow y} g_-(a) = g_-(y)$.

The proof of (ii) is similar. For (iii), if (88) holds, then from (ii) we have the trivial lower bound

$$\liminf_{t\to\infty} \frac{1}{vw^{-1}u(z)} \log \mathbf{P}\left[\frac{w(T_z)}{u(z)} \geq x\right] \geq -g_+(x^+). \tag{94}$$

To obtain the complementary upper bound, note that for any $\varepsilon > 0$,

$$\mathbf{P}\left[\frac{w(T_z)}{u(z)} \geq x\right] \leq \mathbf{P}\left[\frac{w(T_z)}{u(z)} > x - \varepsilon\right] \tag{95}$$

so that

$$\limsup_{t\to\infty} \frac{1}{vw^{-1}u(z)} \log \mathbf{P}\left[\frac{w(T_z)}{u(z)} \geq x\right] \leq -g_+(x - \varepsilon). \tag{96}$$

The result follows on taking $\varepsilon \to 0$ by the lower semicontinuity of g_+. The proof of (iv) is similar. □

THEOREM 13. *Let I be a rate function on $\mathbf{R}_+$ without peaks. Then $(W_t, a(t))$ satisfies an LDP with rate function I if and only if it satisfies a partial LDP with rate function I with respect to all semiinfinite intervals $[0, y_1]$ and $[y_2, \infty)$ with $0 < y_1 < x_I < y_2 < \infty$, for some base x_I of I.*

Note that if $x_I = 0$ or ∞, only one of the sets of intervals enters.

PROOF. This follows from Theorem 3 in [11], modifying the scaling functions as appropriate throughout. The hypotheses of closedness, convexity stated there are not used in the proof, and the condition used there that I has no flat spots can be weakened to I having no peaks. □

LEMMA 4. (i) *Let h_+ be a nondecreasing nonnegative function on $\mathbf{R}_+$ and h_- a nonincreasing nonnegative function on $\mathbf{R}_+$ and suppose that for some $\bar{x} \in [0, \infty]$,*

$$h_-(x) = 0 \quad \text{if } x \geq \bar{x}; \qquad h_+(x) = 0 \quad \text{if } x \leq \bar{x}. \tag{97}$$

Then $h = h_- + h_+$ has no peaks and $\bar{x}$ is a base for h; h is lower semicontinuous if $h_\pm$ are lower semicontinuous.

(ii) *Let h be a function on $\mathbf{R}_+$ which has no peaks and which is lower semicontinuous. Then $h = h_+ + h_-$ where*

$$h_-(x) = \inf_{y \le x} h(y) \quad \textit{and} \quad h_+(x) = \inf_{y \ge x} h(y) \tag{98}$$

are lower semicontinuous functions.

PROOF. The proof of (i) follows by inspection. The proof of (ii) follows from (i) since, because h has no peaks,

$$\begin{aligned} h_-(x) &= \begin{cases} 0, & \text{if } x \ge x_h, \\ h(x), & \text{if } x < x_h, \end{cases} \\ h_+(x) &= \begin{cases} 0, & \text{if } x \le x_h, \\ h(x), & \text{if } x > x_h, \end{cases} \end{aligned} \tag{99}$$

where x_h is a base for h. □

PROOF OF THEOREM 1. (i) This follows by combining Theorems 12, 13 and Lemma 4. We show this in the forward direction; the reverse is similar. Assuming the LDP with rate function I and base x_I for $(u(Z_t)/(w(t), v(t))$, then in Theorem 12 we take

$$f_-(x) = \inf_{y \le x} I(y) \quad \text{and} \quad f_+(x) = \inf_{y \ge x} I(y). \tag{100}$$

Then with $x_J = 1/x_I$, we get

$$\begin{aligned} g_-(x) &= \widehat{vw^{-1}}(x) \inf_{y \ge 1/x} I(y) \\ &= \begin{cases} 0, & \text{if } x \ge x_J, \\ \widehat{vw^{-1}}(x) I(1/x) = J(x), & \text{if } 0 < x \le x_J, \end{cases} \end{aligned} \tag{101}$$

while

$$\begin{aligned} g_+(x) &= \widehat{vw^{-1}}(x) \inf_{y \le 1/x} I(y) \\ &= \begin{cases} 0, & \text{if } 0 < x \le x_J, \\ \widehat{vw^{-1}}(x) I(1/x) = J(x), & \text{if } x \ge x_J. \end{cases} \end{aligned} \tag{102}$$

Thus x_J is a base for J, which by Lemma 4 and Theorem 12 is the rate function for the restricted LDP for $(w(T_z)/u(z), vw^{-1}u(z))$ and hence also for the full LDP which then holds by Theorem 13. Since g_- is nonincreasing, we are free to extend J to zero by taking the limit from above. The symmetry of the relation (15) is due to $\widehat{vw^{-1}}$ being a power. Finally, for the uniqueness property, note that $(I(x) = 0 \Leftrightarrow x = x_I) \Leftrightarrow (J(x) = 0 \Leftrightarrow x_J = 1/x_I)$.

(ii) Without loss of generality we can suppose that either $x_I > b_+$ or $x_I < b_-$. We give a proof for the former case: the proof for the latter is similar. Denote (b_-, b_+) by B and set

$$\begin{aligned} \Delta_I &= \{x \in B: I \text{ continuous at } x\}, \\ \Delta_J &= \{x \in B: J \text{ continuous at } 1/x\}. \end{aligned} \tag{103}$$

Consider the relation (93) with $x \in \Delta_I \cap \Delta_J$. Then, taking the limit $t \to \infty$ (and hence $z \to \infty$), using the LDPs for Z and T, and that $x < x_I$ and $1/x > x_J$, we find that

$$-I(x) = -J(1/x) \lim_{t\to\infty} \frac{vw^{-1}w(t)}{vw^{-1}(w(t)/x)}, \tag{104}$$

where this limit must now exist since $I(x)$ and $J(1/x)$ lie in $(0, \infty)$. But $B \setminus \Delta_I$ and $B \setminus \Delta_J$ are meager, and hence so is $(B \setminus \Delta_I) \cup (B \setminus \Delta_J) = B \setminus (C_I \cap C_J)$. Thus $C_I \cap C_J$, being the complement of a meager set in a open set, is Baire. To summarize, $\lim_{t\to\infty}(vw^{-1}w(t)/vw^{-1}(w(t)/x))$ exists for all x in a Baire set, and so we are done. □

PROOF OF THEOREM 2. (i) We establish the proof in one direction only; the proof in the reverse direction is similar. Assume that $(u(Y_{x(t)})/w(t), v(t))$ satisfies an LDP with rate function I_1. By Theorem 13 it is enough to establish the partial LDP for $(p(Y_{s(t)})/r(t), q(t))$ with rate function I_2 on sets of the form $[0, y_1]$ and $[y_2, \infty)$ with $0 < y_1 < x_{I_2} < y_2 < \infty$, where x_{I_2} is a base for I_2. We perform the proof for sets of the form $[0, y]$ only; the proof for sets $[y, \infty)$ is similar. That I_2 is a rate function follows simply the continuity of $\widehat{up^{-1}}$ and I_1 being a rate function. If x_{I_1} is a base for I_1 then $x_{I_2} = \widehat{pu^{-1}}(x_{I_1})$ is a base for I_2.

One shows that

$$\begin{aligned} &\frac{1}{q(t)} \log \mathbf{P}\left[\frac{p(Y_{s(t)})}{r(t)} \le y\right] \\ &\qquad = \frac{1}{qs^{-1}x(t')} \log \mathbf{P}\left[\frac{u(Y_{x(t')})}{w(t')} \le up^{-1}\frac{(rs^{-1}x(t')y)}{w(t')}\right], \end{aligned} \tag{105}$$

where $s(t) = x(t')$. By (20) and (18), $up^{-1}(rs^{-1}x(t')y)/w(t')$ converges to $\widehat{up^{-1}}(y)$ as $t' \to \infty$. Then by the assumed LDP, for $0 < \varepsilon < y < x_{I_2}$,

$$-\inf_{b<y-\varepsilon} I_2(b) \le \liminf_{t\to\infty} \frac{1}{q(t)} \log \mathbf{P}\left[\frac{p(Y_{s(t)})}{r(t)} \le y\right] \tag{106}$$

$$\le \limsup_{t\to\infty} \frac{1}{q(t)} \log \mathbf{P}\left[\frac{p(Y_{s(t)})}{r(t)} \le y\right] \le -\inf_{b\le y+\varepsilon} I_2(b). \tag{107}$$

To complete the proof we must obtain these same bounds with ε replaced by zero. For the lower bound we can just take $\varepsilon \to 0$. For the upper bound we do the same, but additionally invoke the lower semicontinuity of I_2.

(ii) Suppose $b \in (b_1^-, x_1^-)$; the proof for $b \in (x_1^+, b_1^+)$ is similar. From (18) and (19) it follows that

$$\frac{1}{v(t')}\log \mathbf{P}\left[\frac{u(Z_{x(t')})}{w(t')} \le b\right] = \frac{1}{q(t)}\log \mathbf{P}\left[\frac{p(Z_{s(t)})}{r(t)} \le h_t(b)\right], \tag{108}$$

where

$$x(t') = s(t) \quad \text{and} \quad h_t(b) = \frac{pu^{-1}(wx^{-1}s(t)b)}{pu^{-1}wx^{-1}s(t)}. \tag{109}$$

Since I_1 is continuous on B_1, we have the existence of the limit

$$\begin{aligned} -I_1(b) &= \lim_{t'\to\infty} \frac{1}{v(t')}\log \mathbf{P}\left[\frac{u(Z_{x(t')})}{w(t')} \le b\right] \\ &= \lim_{t\to\infty} \frac{1}{q(t)}\log \mathbf{P}\left[\frac{p(Z_{s(t)})}{r(t)} \le h_t(b)\right]. \end{aligned} \tag{110}$$

Suppose we can show that, as $t \to \infty$, the possible limit points of $h_t(b)$ lie in (b_2^-, x_2^-). Let c be any such limit point. Then by the assumed continuity of I_1 and I_2, we have $I_1(b) = I_2(c)$. Since I_2 is strictly monotonic on (b_2^-, x_2^-), then c is uniquely determined, and is hence the only possible limit point. Thus $\lim_{t\to\infty} h_t(b)$ exists in $(0, \infty)$ on a interval, and hence $pu^{-1} \in \mathscr{S}_r$ with $i(pu^{-1}) \ge 0$. Reversing the roles of I_1 and I_2, we get $up^{-1} \in S_r$ with $i(up^{-1}) \ge 0$, and so, using Lemma 1, we find that $i(up^{-1}) = 1/i(pu^{-1}) \in (0, \infty)$.

It remains to establish the required boundedness property of the limit points of $h_t(b)$. First, $h_t(b)$ cannot have a limit point $c > x_2^-$, for then we would have, for small enough $\varepsilon < c - x_2^-$, that

$$\begin{aligned} -I_1(b) &= \liminf_{t\to\infty} \frac{1}{q(t)}\log \mathbf{P}\left[\frac{p(Z_{s(t)})}{r(t)} \le h_t(b)\right] \\ &\ge \liminf_{t\to\infty} \frac{1}{q(t)}\log \mathbf{P}\left[\frac{p(Z_{s(t)})}{r(t)} \le c - \varepsilon\right] = 0, \end{aligned} \tag{111}$$

where the limit is taken on a subsequence of t along which $h_t(b)$ converges to c. But then $I(b) = 0$, in contradiction with $b \ne x_1^-$.

Finally, suppose $h_t(b)$ has a limit point $c < b_2^- + \delta$ for some arbitrarily small $\delta > 0$. Then, by an argument similar to that used for (111), we find that $I_1(b) \ge I_2(b_2^-) > 0$. But since I_1 is continuous, then by replacing B_1 with some open subinterval also containing $[x_1^-, x_1^+]$, we can get a contradiction for b close enough to x_1^-. □

PROOF OF THEOREM 3. We prove (i) only; the proof of (ii) is the same.

We first prove the existence of a finite limit $\nu(\theta)$ for $\theta < J(0)$. The existence of $\nu(\theta)$ for $\theta \le 0$ follows by Theorem 4.3.1 in [5] since $\limsup_{t\to\infty} \nu_t(\theta) < \infty$ for all $\theta \le 0$. If $J(0) = 0$, we are done. So assume now $J(0) > 0$ and hence that the smallest base x_J for J is strictly positive. We need only consider $\theta \in (0, J(0))$. We first show that

$$\limsup_{t\to\infty} \nu_t(\theta) < \infty \tag{112}$$

for $\theta < J(0)$. Using Lemma 1 in [11], then for any $\varepsilon > 0$,

$$\mathbf{E}[\exp(\theta v w^{-1} u(Z_t))] = 1 + \theta \int_0^\infty dx \exp(\theta x) \mathbf{P}[vw^{-1}u(Z_t) \geq x] \tag{113}$$

$$= 1 + \theta \int_0^\infty dx \exp(\theta x) \mathbf{P}[T_{u^{-1}wv^{-1}(x)} \leq t] \tag{114}$$

$$\begin{aligned} &\leq 1 + \theta \int_0^{vw^{-1}(w(t)/r)} dx \exp(\theta x) \\ &\quad + \theta \int_{vw^{-1}(w(t)/r)}^\infty dx \exp(\theta x) \exp(-x(J(r) - \varepsilon)), \end{aligned} \tag{115}$$

for $r < x_J$ and t sufficiently large, by the LDP for T. Then (112) follows for any $\theta < J(0)$ on taking $t \to \infty$ and choosing r small enough.

From the above bound, it follows, using Theorem 4.3.1 of [5], that, for $\theta < J(0)$,

$$\lim_{M\to\infty} \limsup_{t\to\infty} \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v w^{-1} u(Z_t));\ vw^{-1}u(Z_t) \geq Mv(t)] = -\infty \tag{116}$$

and hence that

$$\limsup_{t\to\infty} \nu_t(\theta) \leq \lim_{M\to\infty} \limsup_{t\to\infty} \nu_t^M(\theta) \tag{117}$$

where

$$\nu_t^M(\theta) = \frac{1}{v(t)} \log \mathbf{E}[\exp(\theta v w^{-1} u(Z_t));\ vw^{-1}u(Z_t) \leq Mv(t)] \tag{118}$$

$$= \frac{1}{v(t)} \log \int_0^{g_t^{-1}(M)} \mathbf{P}_{u(Z_t)/w(t)}(dx) \exp(\theta v(t) g_t(x)), \tag{119}$$

where $P_{u(Z_t)/w(t)}$ is the distribution of

$$u(Z_t)/w(t) \quad \text{and} \quad g_t(x) = vw^{-1}(w(t)x)/v(t).$$

Since $i(vw^{-1}) > 0$, $i(wv^{-1}) < \infty$ and so according to Theorem 1.5.2 in [1], $g_t^{-1}(M)$ converges to $\check{M} = \widehat{wv^{-1}}(M)$. In particular, $g_t^{-1}(M)$ is bounded as $t \to \infty$ and so, by the same theorem, $g_t(x)$ converges uniformly to the continuous function $g(x) = \widehat{vw^{-1}}(x)$ on $\bigcup_{t>t_0}[0, g_t^{-1}(M)]$ for any t_0. Combining the uniform convergence with the extension of Varadhan's theorem in Exercise 4.3.11 of [5], we then find that

$$\begin{aligned} \limsup_{t\to\infty} \nu_t(\theta) &\leq \lim_{M\to\infty} \sup_{x\in[0,\check{M}]} (\theta \widehat{vw^{-1}}(x) - I(x)) \\ &= \sup_{x\geq 0} (\theta \widehat{vw^{-1}}(x) - I(x)) = (I\widehat{wv^{-1}})^*(\theta). \end{aligned} \tag{120}$$

The complementary lower bound follows, since by the same argument,

$$\liminf_{t\to\infty} \nu_t(\theta) \geq \lim_{M\to\infty} \liminf_{t\to\infty} \nu_t^M(\theta) \geq \lim_{M\to\infty} \sup_{x\in(0,\check{M})} (\theta \widehat{vw^{-1}}(x) - I(x)) \tag{121}$$

$$= \sup_{x>0}(\theta \widehat{vw^{-1}}(x) - I(x)) = (I\widehat{wv^{-1}})^*(\theta), \tag{122}$$

the last equality by continuity of I at zero. This concludes the proof of the existence and finiteness of $\nu(\theta)$ for $\theta < J(0)$.

Since $(u(Z_t)/w(t), v(t))$ satisfies an LDP with rate function I, then applying Theorem 2 with $p = vw^{-1}u$, $r = q = v$ and x, s the identity map, tells us that $(vw^{-1}u(Z_t)/v(t), v(t))$ satisfies an LDP with rate function $I\widehat{wv^{-1}}$. When ν is essentially smooth, we conclude from the Gärtner–Ellis theorem that ν^* is the rate function for this LDP and is hence equal to $I\widehat{wv^{-1}}$. As a Legendre transform, $I\widehat{wv^{-1}}$ is convex. □

PROOF OF THEOREM 4. (i) By using the definitions of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, it is easy to see that $\Gamma(T') = \Gamma^{-1}(Z')$ and that Z' and T' have appropriate right and left continuity. The proof of (ii) and (iii) is essentially by the same arguments as used in the proof of Theorem 2 if one observes, for instance, that (i) implies that for all $k > 1$ and t sufficiently large, $kZ_i/i \leq Z'_t/t \leq Z_i/i$ where $i = \lfloor t\rfloor$. □

PROOF OF THEOREM 5. Part (i) is a direct corollary of Lemma 3.

(ii) Since λ satisfies the conditions of the Gärtner–Ellis theorem, $(Z_t/t, v(t))$ satisfies an LDP with good rate function $I = \lambda^*$. Since I is good, it achieves its infimum, 0, at some finite x_I. Since I is convex, it has no peaks and x_I is a base for I. The rest of the proof now follows Theorem 3(i) applied to the case that $u(x) = w(x) = x$. Thus the existence of the $\nu_{(s)}(\theta)$ for $\theta < J(0)$ follows since $i(v) > 0$. This is because $\nu_{(s),t}(\theta) < \nu_t(\theta)$, and so conclude that $\limsup_{t\to\infty} \nu_{(s),t}(\theta) \leq \limsup_{t\to\infty} \nu_t(\theta)$ for $\theta < J(0)$. The form of the limit also follows by use of Varadhan's theorem as in the proof of Theorem 3(i), replacing the l.h.s. of (119) by $(1/v(t))\log\int_0^{g_t^{-1}(M)} \mathbf{P}_{Z_t/t}(dx)\exp(\theta v(t) g_t(x-s))$. Repeating the steps from the proof of Theorem 3 we find

$$\begin{aligned} \sup_{x>s}(\theta\hat{v}(x-s) - \lambda^*(x)) &\leq \liminf_{t\to\infty} \nu_{(s),t}(\theta) \leq \limsup_{t\to\infty} \nu_{(s),t}(\theta) \\ &\leq \sup_{x\geq s}(\theta\hat{v}(x-s) - \lambda^*(x)). \end{aligned} \tag{123}$$

The equality of the bounds follows from Theorem 10.1 of [21] by observing that s, being in the interior of the effective domain of λ^*, is a continuity point of λ^*. Thus

$$\begin{aligned} \nu_{(s)}(\theta) &= \sup_{x>s}(\theta\hat{v}(x-s) - \lambda^*(x)) = \sup_{x>s}(\theta\hat{v}(x) - \lambda^*_{(s)}(x)) \\ &= (\lambda^*_{(s)}\hat{v}^{-1})^*(\theta). \end{aligned} \tag{124}$$

(iii) We have seen that $I = \lambda^*$ has finite base x_I. Since I is also convex, for some $k > 0$, $I(x) > kx$ for sufficiently large x. Thus $J(0) = \lim_{x\to 0}\hat{v}(x)I(1/x) =$

∞ when $i(v) < 1$. Since $\lambda_{(s)}$ is convex, bounded below (by $-s\theta$) on $[0, \theta_2]$ and $\lambda_{(s)}(0) = 0$, we have for any $x \in [x_1, x_2]$ [where $x_i = \lambda_{(s)}(\theta_i)/\theta_i$, $i = 1, 2$] that

$$\lambda^*_{(s)}(x) = \sup_{0 \le \theta \le \theta_2} (\theta x - \lambda_{(s)}(\theta)) < \infty.$$

Hence 0 lies in the interior of the effective domain of $\lambda^*_{(s)}$. The form (43) then follows from (i) and (ii). □

PROOF OF THEOREM 6. Note that

$$(I_s f^{-1})^*(\theta) \le 0 \tag{125}$$

$$\Leftrightarrow \quad f(x)\theta - I(x+s) \le 0 \quad \text{for all } x > 0 \tag{126}$$

$$\Leftrightarrow \quad f(x)\theta - f(x+s)J(1/(x+s)) \le 0 \quad \text{for all } x > 0 \tag{127}$$

$$\Leftrightarrow \quad f(x+s)(\theta f(x/(x+s)) - J_s(x/(x+s))) \le 0 \quad \text{for all } x > 0 \tag{128}$$

$$\Leftrightarrow \quad \theta f(y) - J_s(y) \le 0 \quad \text{for all } y \in (0,1) \tag{129}$$

$$\Leftrightarrow \quad (J_s f^{-1})^*(\theta) \le 0. \tag{130}$$

□

PROOF OF THEOREM 7. By (ii) we can adapt Theorem 3 in the same manner as we did during the proof of Theorem 5, replacing $w(T_z)/u(z) = T_z/z$ by $1 - sT_z/z$. But in this case there is a simplification, since $(1 - sT_z/z)$ is automatically bounded on $\{T_z\colon 0 \le sT_z \le z\}$ as $z \to \infty$. This enables us to conclude that $\limsup_{z\to\infty} \omega_{(s),t}(\theta) < \infty$ for all $\theta \in \mathbf{R}$ and so

$$\omega_{(s),t}(\theta) = \frac{1}{v(z)} \log \int_0^{1/s} \mathbf{P}_{T_z/z}(dx) \exp(\theta v(z) g_z(1 - sx)), \tag{131}$$

where $\mathbf{P}_{T_z/z}$ is the distribution of T_z/z and $g_z(1 - xs) = v(z(1 - xs))/v(z)$, which converges uniformly on $(0, 1/s)$ to $\hat{v}(1 - xs)$ even if $i(v) = 0$. Now $J(x)$ is by definition continuous as $x \searrow 0$, while since s lies in the interior of the effective domain of I, I is continuous at s and since f, being a power, is continuous, J is continuous at $1/s$. Thus using the uniform convergence of g_z, Varadhan's Theorem and the continuity of the rate function as before, we conclude $\mu_{(s)}(\theta)$ exists for all $\theta \in \mathbf{R}$ and

$$\begin{aligned} \omega_{(s)}(\theta) &= \sup_{x \in (0, 1/s]} (\theta \hat{v}(1 - xs) - J(x)) = \sup_{x \in [0,1)} (\theta \hat{v}(x) - J_s(x)) \\ &= (J_s \hat{v}^{-1})^*(\theta). \end{aligned} \tag{132}$$

The expression for δ then follows from Lemma 3 and Theorem 6. □

PROOF OF THEOREM 8. Note that

$$\mathbf{E}[\exp(\theta v(Z_t - st))] = \mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t \ge st] \tag{133}$$

$$\begin{aligned} &+ \mathbf{E}[\exp(\theta v(Z_t - st));\ at \le Z_t \le st] \\ &+ \mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t < at], \end{aligned} \tag{134}$$

where, for any $\varepsilon' > 0$, $i((\ )v) > \varepsilon'' > 0$ and $\kappa > 1$. Then for sufficiently large t,

$$\mathbf{E}[\exp(\theta v(Z_t - st));\ at \le Z_t \le st] \le \mathbf{P}[Z_t \ge a_t] \le \exp(-\theta v(t)(I(a) - \varepsilon')) \tag{135}$$

and

$$\mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t < at] \le \exp(\theta v((a - s)t)) \tag{136}$$
$$\le \exp(-\theta v((s - a)t))$$
$$\le \exp(-\theta v(t)[v((s - a)t)/v(t)]) \tag{137}$$
$$\le \exp(-\theta v(t)\kappa(s - a)^{i(()v) - \varepsilon''})$$

because $v \in S_r$. (The second estimate uses Potter bounds: Theorem 1.5.6 in [1].) On the other hand, by Theorem 5, for any $\varepsilon''' > 0$, then for t sufficiently large,

$$\mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t \ge st] \ge \exp(v(t)[\nu_{(s)}(\theta) - \varepsilon''']). \tag{138}$$

Since $\delta < J(0)$, $\nu_{(s)}$ is finite, convex and hence continuous in a neighborhood of δ (see Theorem 10.1 in [21]). So by choosing θ close enough to δ, we have

$$\mathbf{E}[\exp(\theta v(Z_t - st));\ Z_t \ge st] \ge \exp(-v(t)\varepsilon) \tag{139}$$

for arbitrary ε. Hence this last term exponentially dominates the other two terms, so that the limit (50) exists and (48) holds. □

PROOF OF THEOREM 9. We apply the inverse map x^{-1} defined in (58), which is continuous on $(E^{\uparrow}, M_1')$ and the contraction principle. Note that we can write

$$X_n(t) \equiv (\phi_{2n} \circ Y_n \circ \phi_{1n}^{-1})(t) \tag{140}$$

and

$$X_n^{-1}(t) \equiv (\phi_{1n} \circ Y_n^{-1} \circ \phi_{2n}^{-1})(t) \tag{141}$$
$$= (\phi_{2n} \circ Y_n \circ \phi_{1n}^{-1})^{-1}(t) = (X_n)^{-1}(t), \tag{142}$$

where

$$Y_n(t) = \frac{Z_{b_n t}}{a_n}, \qquad Y_n^{-1}(t) = \frac{Z^{-1}_{a_n t}}{b_n}, \tag{143}$$

$$\phi_{1n}(t) = \frac{w(b_n t)}{wv^{-1}(n)}, \qquad \phi_{1n}^{-1}(t) = \frac{w^{-1}(wv^{-1}(n)t)}{b_n}, \tag{144}$$

$$\phi_{2n}(t) = \frac{u(a_n t)}{wv^{-1}(n)}, \qquad \phi_{2n}^{-1}(t) = \frac{u^{-1}(wv^{-1}(n)t)}{a_n}, \tag{145}$$

with a_n and b_n arbitrary. □

PROOF OF THEOREM 11. The conditions here imply the conditions of both Theorems 9 and 10. Hence both

$$(u(Z_{v^{-1}(t)})/wv^{-1}(t), t)$$

and

$$(w(T_{u^{-1}wv^{-1}(t)})/wv^{-1}(t), t)$$

obey LDPs in **R**. By transforming time, these are equivalent to $(u(Z_t)/w(t), v(t))$ and $(w(T_z)/u(z), vw^{-1}u(z))$ obeying LDPs as $t \to \infty$ and $z \to \infty$, respectively, just as in Theorem 1. □

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