Heavy-Traffic Limits for Infinite-Server Queues in Series with Time-Varying Arrival Rates and Splitting, Allowing Dependent Service Times at Each Queue

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Abstract

We study a stochastic network with two service stations in series, each equipped with infinitely many servers, together with a probabilistic and time-dependent splitting mechanism after service completions at the first station. External arrivals enter the system at the first station according to a general arrival process with time-varying arrival rate, assumed to satisfy a functional central limit theorem (FCLT). The service-time distributions are allowed to be non-exponential. At each station, the service times are identically distributed but allowed to be weakly dependent. We establish heavy-traffic limits (first a FWLLN and then a FCLT refinement) for the two-parameter stochastic processes $\{(Q^e_1(t,y), Q^e_2(t,y)) : t \geq 0, 0 \leq y \leq t\}$, where $Q^e_l(t,y)$, $l = 1,2$, represents the number of customers in the l^{th} service station at time t with elapsed service times less than or equal to y. The FCLT limit is a continuous two-parameter Gaussian processes (random field). We give explicit formulas for the time-dependent means and variances of the resulting Gaussian approximation when the arrival limit process is a Brownian motion.

Key words: stochastic network, infinite-server queues, two-parameter processes, time-varying arrivals, time-dependent splitting of counting processes, martingales, weakly dependent service times, ϕ-mixing, S-mixing, functional central limit theorems, Gaussian (random field) approximation, generalized Kiefer process

1 Introduction

This paper is a sequel to Pang and Whitt (2010), in which we established heavy-traffic limits for the stochastic processes describing performance of the $G_t/GI/\infty$ infinite-server (IS) model, allowing a non-Poisson arrival process with time-varying arrival rate and a non-exponential service-time distribution. Extending Krichagina and Puhalskii (1997), we established heavy-traffic limits for two-parameter stochastic processes, such as $\{(Q^e(t,y) : t \geq 0, 0 \leq y \leq t\}$, where $Q^e(t,y)$ represents the number of customers in the service station at time t with elapsed service times less than or equal to y. A key assumption was that the arrival process satisfy a functional central limit theorem (FCLT), which includes many cases with dependence among the interarrival times.

In the present paper we establish new heavy-traffic limits that extend our previous results in three ways: First, we consider two IS systems in series. Second, in addition to time-varying arrivals, we consider time-varying stochastic splitting after service completion at the first station, allowing only some of the customers to continue on to the second station. Both of the first two features are depicted in Figure 1. Third, in addition to allowing the non-exponential service-time distributions we considered before, we allow the service times at each station to be weakly dependent. However, here we require that the service times at the two stations be mutually independent. Thus, we call our model a $G_t/G^D/\infty \xrightarrow{p_t} G^D/\infty$ queueing network.

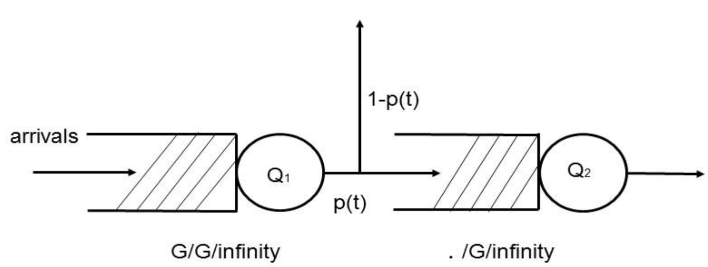

Figure 1: Two Infinite-Server Queues with Time-Dependent Splitting

It is significant that the features we consider make it possible to directly treat more general models. First, the analysis extends to recursively treat more than two stations in series and more general feed-forward networks. The analysis also covers feedback to the same station, because we can represent the content of one station with a single direct feedback, with time-varying feedback probability, as the sum of the contents of two stations in series in the model we consider.

We are motivated by potential applications in service systems. First, we are motivated to consider network structure because many service systems, such as hospitals, directly have such network structure. In healthcare, patients often need to visit several units consecutively or revisit a medical unit several times in order to get the proper treatment; e.g., see the abalysis of an urgent care center considered by Jiang and Giachetti (2006). In manufacturing, defective products after the first-time processing can sometimes be reprocessed to reach the production standard. Our model captures the possibility of second-time service requests in these examples and also of the time-varying feature for these requests.

There is also strong motivation for network structure from traditional customers contact centers, which are designed to have only a single service experience, because customers may not get their needs met during their first service experience. Customers often need to call back one or more times before their needs are met. Thus, there is concern about having first-call resolution, see de Véricourt and Zhou (2005), and there is a need to understand performance when it is not achieved. Call centers often have an Interactive Voice Response (IVR) system so that some customers will complete their service at IVR but others will go through the series of IVR and agents, see Khudyakov et al. (2010) for a Markovian model of $M/M/N/N \xrightarrow{p} M/M/S$ queues with iid splitting.

We are motivated to study dependence among service times by several applications. First, in hospitals, several patients can have similar medical conditions, requiring similar treatment. That occurs with seasonal or epidemic diseases and with multi-person transportation accidents, as with cars or trains. Second, in customer contact centers, new products may have defects that lead to many customers calling with similar needs. These patients or customers will have service requests that are highly dependent upon each other. Third, service times can be affected by common events in the service mechanism. For instance, service interruptions are inevitable in many large-scale service systems, e.g., Pang and Whitt (2009), and interruptions can cause all service times to become longer or stimulate the servers to interact with each other in order to reduce the effect. Moreover, there is empirical evidence in call centers that service times can be dependent, see Brown et al. (2005).

We analyze this stochastic network model in the heavy-traffic regime by scaling up the arrival rates while fixing the service-time distributions. We consider the two-dimensional two-parameter stochastic processes $\{(Q^e_1(t,y), Q^e_2(t,y)) : t \geq 0, 0 \leq y \leq t\}$, where $Q^e_l(t,y)$, $l = 1, 2$, represents the number of customers in the l^{th} service station at time t with elapsed service times less than or equal to y. We prove a functional weak law of large numbers (FWLLN, Theorem 3.1) and a functional central limit theorem (FCLT, Theorem 3.2) for these processes jointly with the departure processes from both service stations and the split arrival process entering the second service station. The FWLLN limits are simple deterministic two-parameter functions and the FCLT limits are continuous two-parameter Gaussian processes (random fields). The weak dependence among service times has no impact on the fluid limits, but the time-dependent splitting after service completion at the first service station plays a prominent role in the fluid limits. Propositions 3.2 and 3.3 provide explicit variance formulas for the Gaussian limit processes when the arrival limit process is a Brownian motion (BM). Dependence among the service times has no impact upon the fluid limit (the mean), but has a clear impact upon the variances; we study this impact further in Pang and Whitt (2011). Our analysis here shows that the methodology in Pang and Whitt (2010) can be extended to analyze networks of queues with the extra features of splitting and dependence among the service times.

In order to allow dependence among the service times, we apply previous FCLT's for the sequential empirical process of weakly dependent random variables, exploiting results by Berkes and Philipp (1977) and Berkes, Hörmann and Schauer (2009). As in Pang and Whitt (2010), one key step in proving our limits is to show that the sequential empirical processes with the underlying weakly dependent service times converge in distribution to a continuous generalized Kiefer process, in the space of D_D endowed with the Skorohod J_1 topology, see Theorem 2.1. The previous results were established in the space of $D([0,1]\times[0,1],\mathbb{R})$ endowed with the generalized Skorohod topology by Bickel and Wichura (1971) and Straf (1971). Here we need to extend the convergence to the larger space D_D because the two-parameter queueing processes $(Q^e_1(t,y), Q^e_2(t,y))$ are not in the space $D([0,T]\times[0,T],\mathbb{R}^2)$.

Time-dependent splitting of general counting processes is an important feature of our model, applied here to the departure process from the first station. We prove a FWLLN (Theorem 4.1) and a FCLT (Theorem 4.2) in §4.2 for the general setting. We assume that the splitting events are conditionally independent, given the arrival process, but our framework allows for weakly dependent splitting; the proofs apply results for martingale difference sequences, e.g., Theorem 6 in Rootzén

(1980). Time-varying demand patterns have been well studied, see Green, Kolesar and Whitt (2007). Our results show the joint effects of time-varying arrivals and time-varying splitting.

There has been considerable work on IS models and associated networks of IS models. Since we already reviewed earlier work on IS queues in Pang and Whitt (2010), here we only discuss networks. Much has been done for stationary models, but some also has been done for time-varying arrival rates. Much has also been done for the exact queueing model, but some also has been done on heavy-traffic approximations. We first discuss explicit results for stationary models. In that context, Boxma (1984) studied a tandom of $M/G/\infty$ queues and obtained the joint time-dependent distribution of queue lengths and residual service times at each queue. Mechata and Deivamoney Selvam (1984) studied the covariance structure of a tandom of $M_t/G/\infty$ queues. Schmidt (1987) considered a tandom of $GI/G/\infty$ queues with renewal arrivals and obtained the generating function of the joint customer-stationary distribution of the successive number of customers a randomly chosen customer finds at his arrival epochs at two queues of the system. For a single IS model, Liu and Templeton (1993) give explicit formulas capturing complex structure including dependence.

For explicit results about networks of IS queues with time-varying arrival rates, we refer to Massey and Whitt (1993) and Nelson and Taaffe (2004). The second paper is notable for providing an algorithmic approach to treat non-Poisson arrival processes and non-exponential (i.i.d.) service times. By focusing on the relatively simple fluid and Gaussian approximations, our approach is quite different. Previous heavy-traffic limits for Markovian service systems with time-varying arrival rates were obtained by Mandelbaum et al. (1998). They treat finite-capacity systems as well as infinite-capacity systems, but the model is Markovian. For a different asymptotic perspective, Zajic (1998) obtained the large deviations principle and moderate deviations principle for the joint distribution of the queue-length processes and departure processes for tandem $M_t/G/\infty$ queues via Poissonized empirical processes.

Here is how the rest of this paper is organized. In §2, we give the detailed model description and assumptions. We also establish some preliminary results including the representation of the queue-length process in terms of the sequential empirical processes, Lemma 2.1. In §3, we state our main results, the FWLLN in §3.1, the FCLT in §3.2, and the characterization of Gaussian properties in §3.3. We collect the proofs for the main results in §4. In §4.1, we prove Theorem 2.1 for the convergence of sequential empirical processes with underlying weakly dependent sequences in D_D. In §4.2, we state and prove our FWLLN and FCLT results for time-dependent splitting of general counting processes. In §4.3, we prove some results for the characterization of the FCLT limits; the rest are given in the appendix. In §§4.4 and 4.5, we prove the FWLLN and FCLT.

2 Model Description and Preliminaries

2.1 The Model Assumptions

We consider a sequence of stochastic networks as depicted in Fig. 1 indexed by n and then let $n \to \infty$. We assume the system starts empty at time 0. As in Pang and Whitt (2010), we would analyze other initial content separately, which can be done because capacity is unlimited. For the $n^{\rm th}$ system, the $i^{\rm th}$ customer arrives at the time $\tau^n_{i,1}$ with the service time $\eta_{i,1}$ and receives service upon arrival. Upon completing service at the time $\delta^n_{i,1} \equiv \tau^n_{i,1} + \eta_{i,1}$, the customer will leave the system or continue to receive a service of length $\eta_{i,2}$ at the second station. The departure time from the system for the $i^{\rm th}$ customer δ^n_i is either equal to $\delta^n_{i,1}$ or $\delta^n_{i,2} \equiv \tau^n_{i,1} + \eta_{i,1} + \eta_{i,2}$. If the $i^{\rm th}$ customer decides to leave the system after service completion at service station 1, we set $\delta^n_{i,2} \equiv \infty$.

In order to count the number of customers that receive service at the second station, we define the sequence $\{\tilde{\tau}^n_{k,2} : k \geq 1\}$ and the associated sequence $\{\tilde{\delta}^n_{k,2} : k \geq 1\}$, where

$$\tilde{\tau}^n_{1,2} \equiv \delta^n_{j_1,1}, \quad j_1 \equiv \min\{i \geq 1 : \delta^n_{i,2} < \infty\},$$

$$\tilde{\tau}^n_{k,2} \equiv \delta^n_{j_k,1}, \quad j_k \equiv \min\{i \geq j_{k-1}+1 : \delta^n_{i,2} < \infty\}, \quad k = 2, 3, ...$$

and

$$\tilde{\delta}^n_{k,2} = \tilde{\delta}^n_{k,2} + \eta_{j_k,2}, \quad k = 1, 2, ...$$

We also define a sequence of random variables $\{\zeta^n_i : i \geq 1\}$ for each n by

$$\zeta^n_i = \begin{cases} 0, & \text{if} \quad \delta^n_i = \delta^n_{i,1}, \quad (i.e., \quad \delta^n_{i,2} = \infty), \\ 1, & \text{if} \quad \delta^n_i = \delta^n_{i,2}, \quad (i.e., \quad \delta^n_{i,2} < \infty), \end{cases} \tag{2.1}$$

and the associated partial sum process $Z_n \equiv \{Z_{n,k} : k \geq 0\}$ defined by

$$Z_{n,k} \equiv \zeta^n_1 + \cdots + \zeta^n_k, \quad \text{for all} \quad k \geq 1, \quad Z_{n,0} \equiv 0. \tag{2.2}$$

We next define several sequences of counting processes: for new arrivals to the first station, $A_{n,1} \equiv \{A_{n,1}(t) : t \geq 0\}$; for service completions at the first station $D_{n,1} \equiv \{D_{n,1}(t) : t \geq 0\}$, for customers continuing in the system to receive service at the second station, $A_{n,2} \equiv \{A_{n,2}(t) : t \geq 0\}$; for service completions at the second station, $D_{n,2} \equiv \{D_{n,2}(t) : t \geq 0\}$; and for the total departure process $D_n \equiv \{D_n(t) : t \geq 0\}$, including departures from both service stations 1 and 2. Formally, these are defined by

$$A_{n,1}(t) \quad \equiv \quad \max\{k \geq 0 : \tau^n_{0,1} + \cdots + \tau^n_{k,1} \leq t\} \tag{2.3}$$

$$
\begin{aligned}
D_{n,1}(t) &\equiv \max\{k \geq 0 : \delta^n_{0,1} + \cdots + \delta^n_{k,1} \leq t\} \\
A_{n,2}(t) &\equiv Z_{n,D_{n,1}(t)} = \sum_{i=1}^{D_{n,1}(t)} \zeta^n_i = \max\{k \geq 0 : \tilde{\tau}^n_{0,2} + \cdots + \tilde{\tau}^n_{k,2} \leq t\} \\
D_{n,2}(t) &\equiv \max\{k \geq 0 : \tilde{\delta}^n_{0,2} + \cdots + \tilde{\delta}^n_{k,2} \leq t\} \\
D_n(t) &= D_{n,1}(t) - A_{n,2}(t) + D_{n,2}(t)
\end{aligned}
$$

where $\tau^n_{0,1} = \delta^n_{0,1} = \tilde{\tau}^n_{0,2} = \tilde{\delta}^n_{0,2} = 0$.

We assume that the sequence of arrival processes into service station 1 satisfies a FCLT.

Assumption 1: FCLT for arrivals. There exist: (i) a continuous nondecreasing deterministic real-valued function $\bar{a}_1$ on $[0,\infty)$ with $\bar{a}_1(0) = 0$ and (ii) a stochastic process $\hat{A}_1$ in D with continuous sample paths, such that

$$
\hat{A}_{n,1}(t) \equiv n^{-1/2}(A_{n,1}(t) - n\bar{a}_1(t)) \Rightarrow \hat{A}_1(t) \quad \text{in} \quad D \quad \text{as} \quad n \to \infty. \tag{2.4}
$$

■

As an immediate consequence of Assumption 1, we have the associated FWLLN

$$
\bar{A}_{n,1} \equiv n^{-1}A_{n,1}(t) \Rightarrow \bar{a}_1(t) \quad \text{in} \quad D \quad \text{as} \quad n \to \infty. \tag{2.5}
$$

We will allow the service times in each service station to be weakly dependent and consider two types of weak dependence for stationary stochastic sequences: ϕ-mixing and S-mixing. The ϕ-mixing is a common condition for weakly dependent stationary sequence, see Billingsley (1999) and Whitt (2002). Here we restate the definition of S-mixing, first introduced by Berkes, Hörmann and Schauer (2009). A stationary stochastic sequence $\{x_i : i \geq 1\}$ is called S-mixing if (i) for any $i, m \geq 1$, there exists a random variable x_{im} such that $P(|x_i - x_{im}| \geq \beta_m) \leq \epsilon_m$ for some constant sequences $\beta_m \to 0$ and $\epsilon_m \to 0$ as $m \to \infty$; (ii) for any disjoint intervals $I_1, ..., I_r$ of positive integers and any positive integers $m_1, ..., m_r$, the vectors $\{x_{im_1} : i \in I_1\}$,...,$\{x_{im_r} : i \in I_r\}$ are independent provided that the separation between $I_{r'}$ and $I_{r''}$, $1 \leq r', r'' \leq r$, is greater than $m_{r'} + m_{r''}$. Berkes, Hörmann and Schauer (2009) show that neither of the two mixing condition includes the other, but the S-mixing condition is relatively easy to verify because it is restricted to random sequences $\{x_i : i \geq 1\}$ with representations that $x_i = \psi(y_i, y_{i+1}, ...)$ for iid sequences $\{y_i : i \geq 1\}$ and Borel measurable functions $\psi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$.

Assumption 2: weakly dependent service times. We assume that the successive service times at the first service station $\{\eta_{i,1} : i \geq 1\}$ are weakly dependent and constitute a one-sided stationary sequence, and the same for the service times at the second service station $\{\eta_{j,2} : j \geq 1\}$

(where this sequence now refers to only those that enter service at the second station). We also assume that the two sequences are independent. For $l = 1, 2$, we also assume that $\eta_{i,l}$'s have the same continuous c.d.f. F_l and p.d.f. f_l with $F_l(0) = 0$, and $E[\eta_{1,l}^2] < \infty$, and

$$\sum_{i=1}^{\infty}(E[(E[\eta_{i+k,l}|\mathcal{F}_{k,l}^s])^2])^{1/2} < \infty, \quad k = 1, 2, ...,$$

where $\mathcal{F}_{k,l}^s \equiv \sigma\{\eta_{i,l} : 1 \le i \le k\}$. We let

$$\mu_l \equiv E[\eta_{1,l}], \quad \sigma_l^2 \equiv Var(\eta_{1,l}) + 2\sum_{i=1}^{\infty} Cov(\eta_{1,l}, \eta_{1+i,l}) < \infty, \quad l = 1, 2.$$

Moreover, we assume that one of the following two types of mixing conditions holds for both $\{\eta_{i,1} : i \ge 1\}$ and $\{\eta_{j,2} : j \ge 1\}$:

(i) (ϕ-mixing) Define

$$\phi_{k,l} \equiv \sup\{|P(B|A) - P(B)| : A \in \mathcal{F}_{m,l}^s, P(A) > 0, B \in \mathcal{G}_{m+k,l}^s, m \ge 1\}, \quad l = 1, 2,$$

where $\mathcal{G}_{k,l}^s = \sigma\{\eta_{i,l} : i \ge k\}$ for $l = 1, 2$. The two sequences satisfy the ϕ-mixing condition:

$$\sum_{k=1}^{\infty} \phi_{k,l} < \infty, \quad l = 1, 2.$$

(ii) (S-mixing) Each of the two sequences is S-mixing.

■

The splitting process after completing service at the first service station can be time-dependent. We are primarily thinking of stochastically independent and time-dependent splitting, but we present a more general framework that allows weak dependence in §4.2. We also consider its special case of iid constant splitting.

Assumption 3: stochastically independent and time-dependent splitting. Let the splitting probability be specified by a deterministic function $p \in D([0, \infty), [0, 1])$, independent of n, such that p is piecewise-smooth, by which we mean that in any interval $[0, T]$, there exist finitely many time points $0 < t_1 < \cdots < t_k < T$ such that on each subinterval (t_{j-1}, t_j), the function $p(t)$ has a continuous derivative $\dot{p}(t)$, both p and its derivative $\dot{p}$ have left and right limits at each endpoint of the subinterval. Let both p and $\dot{p}$ be right continuous. We also require that, almost surely, the discontinuity points t_i do not coincide with any departure time from the first station.

Moreover, the sequence $\{\zeta_i^n : i \geq 1\}$ is a sequence of mutually conditionally independent random variables given $\mathcal{F}_{\infty,1}^n$ for each n, and

$$E[\zeta_{i+1}^n | \mathcal{F}_{i,1}^n] = p(\tau_{i,1}^n), \quad i \geq 1, \tag{2.6}$$

where $\mathbf{F}_1^n \equiv \{\mathcal{F}_{k,1}^n : k \geq 1\}$ be the filtration generated by the service completion times at the first service station, i.e., $\mathcal{F}_{k,1}^n \equiv \sigma\{\delta_{i,1}^n : 1 \leq i \leq k\} \vee \mathcal{N} = \sigma\{\tau_{i,1}^n, \eta_{i,1} : 1 \leq i \leq k\} \vee \mathcal{N}$ and $\mathcal{F}_{\infty,1}^n = \sigma\{\tau_{i,1}^n, \eta_{i,1} : i \geq 1\} \vee sN$ with $\mathcal{N}$ being the null set. ■

Definition 2.1 *For the piecewise smooth function* $p \in D([0,\infty),[0,1])$ *defined in Assumption 2, and for any function* $f \in D([0,T],\mathbb{R})$ *that is continuous at the time points* $0 < t_1 < \cdots < t_k < T$*, we define the integral*

$$\int_0^t f(s)dp(s) \quad \equiv \quad \int_0^t f(s)\dot{p}(s)ds + \sum_{k=1}^m \mathbf{1}(t_k \in [0,t])f(t_k)[p(t_k) - p(t_k-)], \tag{2.7}$$

for each $t \in [0,T]$.

The Standard Case

The standard case concerns a stationary model in which the limit of the arrival process FCLT is Brownian motion.

(i) In the assumed arrival FWLLN, $\bar{a}_1 = \lambda_1 t$, $t \geq 0$, for some positive constant λ_1. The limit in the FCLT is $\hat{A}_1 = \sqrt{\lambda_1 c_{a,1}^2} B_{a,1}$, i.e., a Brownian motion (BM), where $c_{a,1}^2$ is variability parameter, which for a renewal arrival process is the squared coefficient of variation (SCV) of an interarrival times, and $B_{a,1}$ is a standard BM.

(ii) Assume that ζ_i^n's are iid, with distribution $P(\zeta_i^n = 1) = 1 - p > 0$ and $P(\zeta_i^n = 0) = p > 0$ for some constant $p \in [0,1]$.

■

2.2 Preliminaries

Let $Q_{n,1}^e(t,y)$ represent the number of new arrivals in the first service station at time t in the n^{th} model that have elapsed service times less than or equal to y, and $Q_{n,2}^e(t,y)$ be the number of customers in the second service station that have elapsed service times less than or equal to y, $0 \leq y \leq t$. Then we can express $Q_{n,1}^e$ and $Q_{n,2}^e$ as

$$Q_{n,1}^e(t,y) = \sum_{i=A_{n,1}(t-y)}^{A_{n,1}(t)} \mathbf{1}(\tau_{i,1}^n + \eta_{i,1} > t), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{2.8}$$

and

$$Q^e_{n,2}(t,y) = \sum_{j=A_{n,2}(t-y)}^{A_{n,2}(t)} \mathbf{1}(\tilde{\tau}^n_{j,2} + \eta_{j,2} > t), \quad t \geq 0, \quad 0 \leq y \leq t. \tag{2.9}$$

Note that $Q^e_{n,1}(t,t)$ counts the total number of new arrivals in the first service station, and $Q^e_{n,2}(t,t)$ counts the total number of customers in the second service station. Evidently, we have the balance equation

$$A_{n,1}(t) = Q^e_{n,1}(t,t) + Q^e_{n,2}(t,t) + D_n(t), \quad t \geq 0.$$

The processes $Q^e_{n,1}$ and $Q^e_{n,2}$ and their limits (after scaling) to be established lie in the space $D_D \equiv D([0,\infty), D([0,\infty),\mathbb{R}))$, where $D \equiv D([0,\infty), S)$, for a separable metric space S, is the space of all right-continuous S-valued functions with left-limits in $(0,\infty)$; see Billingsley (1999) and Whitt (2002) for background. We will be using the standard Skorohod J_1 topologies on both D spaces in D_D. For a discussion of D_D, see Talreja and Whitt (2008) and Pang and Whitt (2010).

Following Krichagina and Puhalskii (1998) and Pang and Whitt (2010), we can rewrite the random sums in (2.8) and (2.9) as integrals with respect to the random fields by

$$Q^e_{n,1}(t,y) = n \int_{t-y}^{t} \int_0^{\infty} \mathbf{1}(s + x > t) d\bar{K}_{n,1}(\bar{A}_{n,1}(s), x), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{2.10}$$

and

$$Q^e_{n,2}(t,y) = n \int_{t-y}^{t} \int_0^{\infty} \mathbf{1}(s + x > t) d\bar{K}_{n,2}(\bar{A}_{n,2}(s), x), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{2.11}$$

where $(\bar{A}_{n,1}, \bar{A}_{n,2}) \equiv n^{-1}(A_{n,1}, A_{n,2})$, the two-parameter random fields $(\bar{K}_{n,1}, \bar{K}_{n,2})$ in $D^2_D \equiv D_D \times D_D$ are defined by

$$\bar{K}_{n,1}(t,x) \equiv \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \mathbf{1}(\eta_{i,1} \leq x), \quad \bar{K}_{n,2}(t,x) \equiv \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \mathbf{1}(\eta_{j,2} \leq x), \quad t \geq 0, \quad x \geq 0. \tag{2.12}$$

The integrals in (2.10) and (2.11) are well defined as Stieltjes integrals for functions of bounded variation as integrators.

These two-parameter random fields are often called sequential empirical processes. For the case of iid service times for IS queues, the FWLLN and FCLT for such random fields is discussed in Pang and Whitt (2010). Here, for weakly dependent service times, the corresponding FCLT was established by Berkes and Philipp (1977) for ϕ-mixing sequences and by Berkes, Hörmann and Schauer (2009) for S-mixing sequences, where the convergence is in the space of $D([0,1]\times[0,1],\mathbb{R})$ with the generalized Skorohod J_1 topology on two-parameter processes (Bickel and Wichura (1971) and Straf (1971)). Here we first extend their results to the space D_D with the Skorohod J_1 topology on both D spaces (recall that the space $D([0,1]\times[0,1],\mathbb{R})\subset D([0,1],D([0,1],\mathbb{R})))$. The proof is in §4.1.

Theorem 2.1 *(FCLT in D_D for the sequential empirical process with weakly dependent random variables) Let $\{\xi_k : k\geq 1\}$ be a weakly dependent stationary sequence, either (i) ϕ-mixing or (ii) S-mixing. Assume that ξ_k's are uniformly distributed on $[0,1]$, and*

$$\sum_{i=1}^{\infty}\|E[\xi_{i+k}|\mathcal{F}_k]\|_{L^2}=\sum_{i=1}^{\infty}(E[(E[\xi_{i+k}|\mathcal{F}_k])^2])^{1/2}<\infty \tag{2.13}$$

where $\mathcal{F}_k\equiv\sigma\{\xi_i : 1\leq i\leq k\}$ for each $k\geq 1$. Then, the series

$$\Gamma(x,y)=E[\gamma_1(x)\gamma_1(y)]+\sum_{k=2}^{\infty}\Big(E[\gamma_1(x)\gamma_k(y)]+E[\gamma_1(y)\gamma_k(x)]\Big)<\infty,\quad x,y\in[0,1] \tag{2.14}$$

converges absolutely, where $\gamma_k(x)\equiv\mathbf{1}(\xi_k\leq x)-x$, and the diffusion-scaled sequential empirical processes $\hat{U}_n(t,x)$ defined by

$$\hat{U}_n(t,x)\equiv\frac{1}{\sqrt{n}}\sum_{k=1}^{\lfloor nt\rfloor}\gamma_k(x),\quad t\geq 0,\quad x\in[0,1] \tag{2.15}$$

converge

$$\hat{U}_n\Rightarrow\hat{U}\quad in\quad D([0,\infty),D([0,1],\mathbb{R}))\quad as\quad n\to\infty, \tag{2.16}$$

where $\hat{U}$ is a generalized Kiefer process (continous two-parameter Gaussian process) with $E[\hat{U}(t,x)]=0$ and $E[\hat{U}(t,x)\hat{U}(s,y)]=(t\wedge s)\Gamma(x,y)$ with $\Gamma(x,y)$ defined in (2.14) for any $t,s\geq 0$ and $x,y\in[0,1]$. Moreover, the convergence is uniform in the second parameter $x\in[0,1]$.

The convergence in (2.16) implies that the fluid-scaled sequential processes satisfy the FWLLN:

$$\bar{U}_n(t,x)\equiv\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\mathbf{1}(\xi_k\leq x)\Rightarrow\bar{u}(t,x)\equiv tx,\quad\text{in}\quad D([0,\infty),D([0,1],\mathbb{R}))\quad\text{as}\quad n\to\infty \tag{2.17}$$

Moreover, in Theorem 2.1, when the sequence $\{\xi_k\}$ is iid, the limit process $\hat{U}$ becomes a standard Kiefer process, where $\Gamma(x,y)=x\wedge y-xy$ for $x,y\in[0,1]$.

Thus, the two-parameter random fields in (2.12) satisfy the FWLLN:

$$(\bar{K}_{n,1},\bar{K}_{n,2})\Rightarrow(\bar{k}_1,\bar{k}_2)\quad\text{in}\quad D_D^2\quad\text{as}\quad n\to\infty,$$

where $\bar{k}_1(t,x)=tF_1(x)$, $\bar{k}_2(t,x)=tF_2(x)$, and the convergence is uniform over sets of the form $[0,T]\times[0,\infty)$ and there is uniformity in the second argument x over $[0,\infty)$. Define the scaled processes

$$\hat{K}_{n,l}(t,x)\quad\equiv\quad\sqrt{n}(\bar{K}_{n,l}(t,x)-\bar{k}_l(t,x))=\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}(\mathbf{1}(\eta_{i,l}\leq x)-F_l(x))\stackrel{\mathrm{d}}{=}\hat{U}_{n,l}(t,F_l(x)),\quad t,x\geq 0,$$

for $l=1,2$, where

$$\hat{U}_{n,l}(t,x)=\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}(\mathbf{1}(\xi_{i,l}\le x)-x),\quad t\ge 0,\quad x\in[0,1],\quad l=1,2,$$

with $\{\xi_{i,l}: i\ge 1\}$ being weakly dependent and stationary satisfying either ϕ-mixing or S-mixing conditions with uniform distribution on $[0,1]$ for each $l=1,2$, and $\xi_{i,1}$'s and $\xi_{i,2}$'s also being mutually independent. Here we require the joint convergence of $(\hat{U}_{n,1},\hat{U}_{n,2})$ in D_D^2, and this is direct since they are independent and so are their limits. Hence,

$$(\hat{U}_{n,1},\hat{U}_{n,2})\Rightarrow(\hat{U}_1,\hat{U}_2)\quad\text{in}\quad D_D^2\quad\text{as}\quad n\to\infty,$$

where $\hat{U}_1$ and $\hat{U}_2$ are independent generalized Kiefer processes with $E[\hat{U}_1(t,x)]=E[\hat{U}_2(t,x)]=0$, $E[\hat{U}_1(t,x)\hat{U}_1(s,y)]=(t\wedge s)\Gamma_1(x,y)$ and $E[\hat{U}_2(t,x)\hat{U}_2(s,y)]=(t\wedge s)\Gamma_2(x,y)$ for each $t,s\ge 0$ and $x,y\ge 0$, where

$$\Gamma_l(x,y)=E[\gamma_{1,l}(x)\gamma_{1,l}(y)]+\sum_{k=2}^{\infty}\Big(E[\gamma_{1,l}(x)\gamma_{k,l}(y)]+E[\gamma_{1,l}(y)\gamma_{k,l}(x)]\Big),\quad l=1,2,\tag{2.18}$$

and $\gamma_{k,l}(x)\equiv\mathbf{1}(\eta_{k,l}\le x)-F_l(x)$ for $l=1,2$. This implies that the FCLT for $(\hat{K}_{n,1},\hat{K}_{n,2})$ holds

$$(\hat{K}_{n,1},\hat{K}_{n,2})\Rightarrow(\hat{K}_1,\hat{K}_2)\quad\text{in}\quad D_D^2\quad\text{as}\quad n\to\infty,\tag{2.19}$$

where $\hat{K}_1$ and $\hat{K}_2$ are independent time-changed generalized Kiefer processes

$$\hat{K}_1(t,x)=\hat{U}_1(t,F_0(x)),\quad \hat{K}_2(t,x)=\hat{U}_2(t,F_2(x)),\quad t,x\ge 0,$$

independent of $\hat{A}_1$, with mean 0 and covariances

$$E[\hat{K}_l(t,x)\hat{K}_l(s,y)]=(t\wedge s)\Gamma_{K,l}(x,y),\quad l=1,2,\quad t,s,x,y\ge 0,\tag{2.20}$$

$$\Gamma_{K,l}(x,y)=[F_l(x)\wedge F_l(y)-F_l(x)F_l(y)]+\Gamma^c_{K,l}(x,y)<\infty,\tag{2.21}$$

$$\Gamma^c_{K,l}(x,y)=\sum_{k=2}^{\infty}\Big(E[\gamma_{1,l}(x)\gamma_{k,l}(y)]+E[\gamma_{1,l}(y)\gamma_{k,l}(x)]\Big)<\infty,\tag{2.22}$$

for each $x,y\ge 0$.

In the case of iid service times, $\hat{U}_l(t,x)$ is the standard Kiefer process for each $l=1,2$, and $U_l(t,x)=W_l(t,x)-xW_l(t,1)$ for standard Brownian sheet W_l, and W_1 and W_2 are independent, so that $\hat{K}_1$ and $\hat{K}_2$ are standard Kiefer processes with the second parameter having time changes by service-time distributions, $\Gamma_{K,l}(x,y)=F_l(x)\wedge F_l(y)-F_l(x)F_l(y)$.

Then we obtain the following representation of the processes $Q^e_{n,1}$ and $Q^e_{n,2}$. The proof follows from the same argument as Lemma 2.1 in Pang and Whitt (2010) and thus is omitted. Let $\bar{a}_2$ be the fluid limit for $\bar{A}_{n,2}\equiv A_{n,2}/n$ to be established and define $\hat{A}_{n,2}\equiv\sqrt{n}(\bar{A}_{n,2}-\bar{a}_2)$.

Lemma 2.1 (*Queue-length representation by sequential empirical processes*) *The processes* $Q^e_{n,1}$ *and* $Q^e_{n,2}$ *defined in* (2.8)*and* (2.9)*, respectively, can be represented as*

$$Q^e_{n,1}(t,y) = n\int_{t-y}^{t} F_1^c(t-s)d\bar{a}_1(s) + \sqrt{n}(\hat{X}^e_{n,1}(t,y) + \hat{X}^e_{n,2}(t,y)), \tag{2.23}$$

and

$$Q^e_{n,2}(t,y) = n\int_{t-y}^{t} F_2^c(t-s)d\bar{a}_2(s) + \sqrt{n}(\hat{Y}^e_{n,1}(t,y) + \hat{Y}^e_{n,2}(t,y)), \tag{2.24}$$

where

$$\begin{aligned} \hat{X}^e_{n,1}(t,y) &= \int_{t-y}^{t} F_1^c(t-s)d\hat{A}_{n,1}(s) \qquad (2.25)\\ &= \hat{A}_{n,1}(t) - F_1^c(y)\hat{A}_{n,1}(t-y) - \int_{t-y}^{t} \hat{A}_{n,1}(s-)dF_1^c(t-s), \end{aligned}$$

$$\hat{X}^e_{n,2}(t,y) = \int_{t-y}^{t}\int_0^\infty \mathbf{1}(s+x>t)d\hat{R}_{n,1}(s,x) = -\int_{t-y}^{t}\int_0^\infty \mathbf{1}(s+x\le t)d\hat{R}_{n,1}(s,x), \tag{2.26}$$

$$\begin{aligned} \hat{Y}^e_{n,1}(t,y) &= \int_{t-y}^{t} F_2^c(t-s)d\hat{A}_{n,2}(s) \qquad (2.27)\\ &= \hat{A}_{n,2}(t) - F_2^c(y)\hat{A}_{n,2}(t-y) - \int_{t-y}^{t} \hat{A}_{n,2}(s-)dF_2^c(t-s), \end{aligned}$$

$$\hat{Y}^e_{n,2}(t,y) = \int_{t-y}^{t}\int_0^\infty \mathbf{1}(s+x>t)d\hat{R}_{n,2}(s,x) = -\int_{t-y}^{t}\int_0^\infty \mathbf{1}(s+x\le t)d\hat{R}_{n,2}(s,x), \tag{2.28}$$

with the integrals in (2.25)-(2.28) *all defined as Stieltjes integrals for functions of bounded variation as integrators, and*

$$\begin{aligned} \hat{R}_{n,l}(t,x) &= \hat{K}_{n,l}(\bar{A}_{n,l}(t),x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{A_{n,l}(t)} \gamma_{i,l}(x)\\ &= \sqrt{n}\bar{K}_{n,l}(\bar{A}_{n,l}(t),x) - \hat{A}_{n,l}(t)F_l(x) - \sqrt{n}\bar{a}_l(t)F_l(x), \quad l=1,2. \qquad (2.29) \end{aligned}$$

3 Main Results

In this section, we will present the main results, the heavy-traffic FWLLN and FCLT limits for the joint queue-length processes at the two service stations. We will also discuss characterizations of the limit processes.

3.1 FWLLN Limits

We first define the LLN-scaled processes associated with $(D_{n,1}, A_{n,2}, D_{n,2}, D_n, Q^e_{n,1}, Q^e_{n,2})$:

$$(\bar{D}_{n,1}, \bar{A}_{n,2}, \bar{D}_{n,2}, \bar{D}_n, \bar{Q}^e_{n,1}, \bar{Q}^e_{n,2}) \equiv n^{-1}(D_{n,1}, A_{n,2}, D_{n,2}, D_n, Q^e_{n,1}, Q^e_{n,2}).$$

By Lemma 2.1, these LLN-scaled processes can be represented as

$$\bar{Q}^e_{n,1}(t,y) = \int_{t-y}^t F_1^c(t-s)d\bar{a}_1(s) + \frac{1}{\sqrt{n}}(\hat{X}^e_{n,1}(t,y) + \hat{X}^e_{n,2}(t,y)), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{3.1}$$

$$\bar{Q}^e_{n,2}(t,z) = \int_{t-y}^t F_2^c(t-s)d\bar{a}_2(s) + \frac{1}{\sqrt{n}}(\hat{Y}^e_{n,1}(t,z) + \hat{Y}^e_{n,2}(t,z)), \quad t \geq 0, \quad 0 \leq z \leq t, \tag{3.2}$$

$$\bar{D}_{n,l}(t) = \bar{A}_{n,l}(t) - \bar{Q}^e_{n,l}(t,t), \quad l = 1,2, \quad t \geq 0, \tag{3.3}$$

$$\bar{A}_{n,2}(t) = n^{-1} \sum_{i=1}^{n\bar{D}_{n,1}(t)} \zeta_i^n, \quad t \geq 0, \tag{3.4}$$

$$\bar{D}_n(t) = \bar{D}_{n,1}(t) - \bar{A}_{n,2}(t) + \bar{D}_{n,2}(t), \quad t \geq 0. \tag{3.5}$$

The FWLLN limits for these processes are given in the following theorem. The proof for the convergence of the processes $(Q^e_{n,1}, Q^e_{n,2})$ simply follows from tightness of the processes $\hat{X}^e_{n,1}$, $\hat{X}^e_{n,2}$, $\hat{Y}^e_{n,1}$, $\hat{Y}^e_{n,2}$ to be established as a main component in proving the FCLT limits. The convergence of the processes $\hat{A}_{n,2}$ follows from Theorem 4.1 for time-dependent split counting processes. The convergence of other processes follows from applying the continuous mapping theorem (CMT). Thus, the proof for the following theorem is omitted.

Theorem 3.1 (FWLLN with weakly dependent service times and time-dependent splitting) *Under Assumptions 1 - 3,*

$$(\bar{D}_{n,1}, \bar{A}_{n,2}, \bar{D}_{n,2}, \bar{D}_n, \bar{Q}^e_{n,1}, \bar{Q}^e_{n,2}) \Rightarrow (\bar{d}_1, \bar{a}_2, \bar{d}_2, \bar{d}, \bar{q}^e_1, \bar{q}^e_2) \quad in \quad D^4 \times D^2_D \quad as \quad n \to \infty \tag{3.6}$$

where the limits are all deterministic functions,

$$\bar{q}^e_1(t,y) = \int_{t-y}^t F_1^c(t-s)d\bar{a}_1(s), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{3.7}$$

$$\bar{d}_1(t) = \bar{a}_1(t) - \bar{q}^e_1(t,t) = \int_0^t F_1(t-s)d\bar{a}_1(s), \quad t \geq 0, \tag{3.8}$$

$$\bar{a}_2(t) = \int_0^t p(s)d(\bar{d}_1(s)) = \int_0^t p(s)\int_0^s f_1(s-u)d\bar{a}_1(u)ds, \quad t \geq 0, \tag{3.9}$$

$$\bar{q}^e_2(t,y) \quad = \quad \int_{t-y}^t F_2^c(t-s)d\bar{a}_2(s),$$

$$= \int_{t-y}^{t} F_2^c(t-s)p(s)\int_0^s f_1(s-u)d\bar{a}_1(u)ds, \quad t \geq 0, \quad 0 \leq y \leq t, \tag{3.10}$$

$$\begin{aligned} \bar{d}_2(t) &= \bar{a}_2(t) - \bar{q}_2^e(t,t) = \int_0^t F_2(t-s)d\bar{a}_2(s) \\ &= \int_0^t F_2(t-s)p(s)\int_0^s f_1(s-u)d\bar{a}_1(u)ds, \quad t \geq 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \bar{d}(t) &= \bar{d}_1(t) - \bar{a}_2(t) + \bar{d}_2(t) = \int_0^t (1 - F_2^c(t-s)p(s))d(\bar{d}_1(s)) \\ &= \int_0^t (1 - F_2^c(t-s)p(s))\int_0^s f_1(s-u)d\bar{a}_1(u)ds, \quad t \geq 0. \end{aligned} \tag{3.12}$$

We remark that the weak dependence among service times does not affect the fluid limits, which are the same as the case of iid service times, while the impact of the time-dependent splitting mechanism for customers completing service at the first station is captured in the fluid limits. In particular, the arrival rate at the second service station, $\bar{a}_2$ in (3.9), is affected only by the time-dependent splitting, but not by the weak dependence of service times in the first service station.

Corollary 3.1 (FWLLN in the standard case) *In the standard case, the limits in* (3.6) *simplify as follows,*

$$\bar{q}_1^e(t,y) = \lambda_1 \int_{t-y}^t F_1^c(t-s)ds = \lambda_1 \int_0^y F_1^c(s)ds \equiv \bar{q}_1^e(\infty, y), \tag{3.13}$$

$$\bar{d}_1(t) = \lambda_1 \int_0^t F_1(t-s)ds = \lambda_1 \int_0^t F_1(s)ds, \quad t \geq 0, \tag{3.14}$$

$$\bar{a}_2(t) = p\bar{d}_1(t) = \lambda_1 p \int_0^t F_1(s)ds, \quad t \geq 0, \tag{3.15}$$

$$\begin{aligned} \bar{q}_2^e(t,y) &= \lambda_1 p \int_{t-y}^t F_2^c(t-s)F_1(s)ds = \lambda_1 p \int_0^y F_2^c(s)F_1(t-s)ds, \quad t \geq 0, \quad 0 \leq y \leq t, \\ &\to (\lambda_1 p/\mu_2)F_{2,e}(y), \quad y \geq 0, \quad \textit{as} \quad t \to \infty, \end{aligned} \tag{3.16}$$

$$\bar{d}_2(t) = \lambda_1 p \int_0^t F_2(t-s)F_1(s)ds, \quad t \geq 0, \tag{3.17}$$

$$\bar{d}(t) = \lambda_1 \int_0^t (1 - pF_2^c(t-s))F_1(s)ds, \quad t \geq 0, \tag{3.18}$$

where $F_{2,e}$ is the stationary-excess (or residual-lifetime) cdf associated with the service-time cdf F_2, defined by $F_{2,e}(x) = \mu_2 \int_0^x F_2^c(s)ds$ for each $x \geq 0$. Moreover, the long-run average rates for the counting processes $D_{n,1}$, $A_{n,2}$, $D_{n,2}$ and D_n are given by λ_1, $\lambda_1 p$, $\lambda_1 p$, and λ_1, respectively, and the total queue lengths at the two service stations have steady-state values λ_1/μ_1, and $\lambda_1 p/\mu_2$.

3.2 FCLT Limits

We first define the FCLT-scaled processes associated with $(D_{n,1}, A_{n,2}, D_{n,2}, D_n, Q^e_{n,1}, Q^e_{n,2})$:

$$\begin{aligned}
\hat{D}_{n,1} &\equiv \sqrt{n}(\bar{D}_{n,1} - \bar{d}_1), \quad \hat{A}_{n,2} \equiv \sqrt{n}(\bar{A}_{n,2} - \bar{a}_2), \\
\hat{D}_{n,2} &\equiv \sqrt{n}(\bar{D}_{n,2} - \bar{d}_2), \quad \hat{D}_n \equiv \sqrt{n}(\bar{D}_n - \bar{d}), \\
\hat{Q}^e_{n,1} &\equiv \sqrt{n}(\bar{Q}^e_{n,1} - \bar{q}_1), \quad \hat{Q}^e_{n,2} \equiv \sqrt{n}(\bar{Q}^e_{n,2} - \bar{q}_2),
\end{aligned} \tag{3.19}$$

where $\bar{d}_1$, $\bar{a}_2$, $\bar{d}_2$, $\bar{d}$, $\bar{q}_1$ and $\bar{q}_2$ are defined in Theorem 3.1. By Lemma 2.1, the processes $Q^e_{n,1}$ and $Q^e_{n,2}$ can be represented as

$$\hat{Q}^e_{n,1}(t,y) = \hat{X}^e_{n,1}(t,y) + \hat{X}^e_{n,2}(t,y), \quad t \ge 0, \quad 0 \le y \le t, \tag{3.20}$$

$$\hat{Q}^e_{n,2}(t,y) = \hat{Y}^e_{n,1}(t,y) + \hat{Y}^e_{n,2}(t,y), \quad t \ge 0, \quad 0 \le z \le t, \tag{3.21}$$

and it is clear that

$$\hat{D}_n(t) = \hat{D}_{n,1}(t) - \hat{A}_{n,2}(t) + \hat{D}_{n,2}(t), \quad t \ge 0. \tag{3.22}$$

The limits of the processes $\hat{X}^e_{n,2}(t,y)$ and $\hat{Y}^e_{n,2}(t,y)$ are given as mean-square integrals of the time-changed generalized Kiefer processes $\hat{K}_1(t,x)$ and $\hat{K}_2(t,x)$ in (2.19). Here we first give the definitions of their limits.

Definition 3.1 *The two-parameter processes $\hat{X}^e_2$ and $\hat{Y}^e_2$ written as*

$$\hat{X}^e_2(t,y) = \int_{t-y}^t \int_0^\infty \mathbf{1}(s+x > t) d\hat{K}_1(\bar{a}_1(s), x) = -\int_{t-y}^t \int_0^\infty \mathbf{1}(s+x \le t) d\hat{K}_1(\bar{a}_1(s), x), \tag{3.23}$$

and

$$\hat{Y}^e_2(t,y) = \int_{t-y}^t \int_0^\infty \mathbf{1}(s+x > t) d\hat{K}_2(\bar{a}_2(s), x) = -\int_{t-y}^t \int_0^\infty \mathbf{1}(s+x \le t) d\hat{K}_2(\bar{a}_2(s), x), \tag{3.24}$$

are defined by mean-square integrals, that is,

$$\lim_{k\to\infty} E[(\hat{X}^e_2(t,y) - \hat{X}^e_{2,k}(t,y))^2] = 0, \quad \lim_{k\to\infty} E[(\hat{Y}^e_2(t,y) - \hat{Y}^e_{2,k}(t,y))^2] = 0, \quad t \ge 0, \quad 0 \le y \le t, \tag{3.25}$$

with

$$\hat{X}^e_{2,k}(t,y) = \int_{t-y}^t \int_0^\infty \mathbf{1}_{k,t,y}(s,x) d\hat{K}_1(\bar{a}_1(s), x), \quad t \ge 0, \quad 0 \le y \le t, \tag{3.26}$$

$$\mathbf{1}_{k,t,y}(s,x) \equiv \sum_{i=1}^k [\mathbf{1}(s^k_{i-1} < s \le s^k_i) \mathbf{1}(t - s^k_i < x \le t)] \tag{3.27}$$

$t - y = s^k_0 < s^k_1 < \cdots < s^k_k = t$ and $\max_{1 \le i \le k} |s^k_i - s^k_{i-1}| \to 0$ as $k \to \infty$, and similarly for $\hat{Y}^e_{2,k}(t,y)$. Write $\hat{X}^e_2(t,y) = l.i.m._{k\to\infty} \hat{X}^e_{2,k}(t,y)$ and $\hat{Y}^e_2(t,y) = l.i.m._{k\to\infty} \hat{Y}^e_{2,k}(t,y)$.

Theorem 3.2 (FCLT with weakly dependent service times and time-dependent splitting) *Under Assumptions 1-3,*

$$(\hat{D}_{n,1}, \hat{A}_{n,2}, \hat{D}_{n,2}, \hat{D}_n, \hat{Q}^e_{n,1}, \hat{Q}^e_{n,2}) \Rightarrow (\hat{D}_1, \hat{A}_2, \hat{D}_2, \hat{D}, \hat{Q}^e_1, \hat{Q}^e_2) \quad in \quad D^4 \times D^2_D \quad as \quad n \to \infty, \tag{3.28}$$

where

$$\hat{Q}^e_1(t,y) \equiv \hat{X}^e_1(t,y) + \hat{X}^e_2(t,y), \quad \hat{Q}^e_2(t,y) \equiv \hat{Y}^e_1(t,y) + \hat{Y}^e_2(t,y), \quad t \geq 0, \quad 0 \leq y \leq t, \tag{3.29}$$

$$\hat{X}^e_1(t,y) = \int_{t-y}^{t} F^c_1(t-s) d\hat{A}_1(s) = \hat{A}_1(t) - F^c_1(y)\hat{A}_1(t-y) - \int_{t-y}^{t} \hat{A}_1(s) dF^c_1(t-s), \tag{3.30}$$
$$\hat{Y}^e_1(t,y) = \hat{A}_2(t) - F^c_1(y)\hat{A}_2(t-y) - \int_{t-y}^{t} \hat{A}_2(s) dF^c_2(t-s),$$

$\hat{X}^e_2$ *is defined in* (3.23) *and* $\hat{Y}^e_2$ *in* (3.24)*,*

$$\begin{aligned} \hat{D}_1(t) &= \hat{A}_1(t) - \hat{Q}^e_1(t,t) = \int_0^t F_1(t-s) d\hat{A}_1(s) - \hat{X}^e_2(t,t) \\ &= \int_0^t \hat{A}_1(s) dF^c_1(t-s) - \hat{X}^e_2(t,y), \end{aligned} \tag{3.31}$$

$$\begin{aligned} \hat{A}_2(t) &= B_s\left(\int_0^t p(s)(1-p(s)) \int_0^s f_1(s-u) d\bar{a}_1(u) ds\right) \\ &\quad + p(t)\hat{D}_1(t) - \int_0^t \hat{D}_1(s) dp(s), \end{aligned} \tag{3.32}$$

$$\hat{D}_2(t) = \hat{A}_2(t) - \hat{Q}^e_2(t,t) = \int_0^t \hat{A}_2(s) dF^c_2(t-s) - \hat{Y}^e_2(t,y), \tag{3.33}$$

$$\hat{D}(t) = \hat{D}_1(t) - \hat{A}_2(t) + \hat{D}_2(t), \tag{3.34}$$

where $\hat{A}_1$ *is given in Assumption 1,* $\hat{X}^e_1$ *and* $\hat{D}_1$ *take the first expression in* (3.30) *and* (3.31) *respectively if* $\hat{A}_1$ *is a BM and the second if* $\hat{A}_1$ *is a general Gaussian process, and* B_s *is a standard BM, independent of* $\hat{A}_1$*,* $\hat{K}_1$ *and* $\hat{K}_2$*.*

We remark about the impact of weak dependence of service times and time-dependent splitting mechanism upon various processes. Weak dependence of service times in the first service station affects its queue-length, in particular, in the $\hat{X}^e_2$ term with $\hat{K}_1$ capturing the effect, see its covariance formula $\Gamma^c_{K,1}(x,y)$ in (2.22). The arrival process into the second service station is affected by both weak dependence of service times and the time-dependent splitting. The queue-length process at the second service station is affected by three factors, weak dependence of service times at the first station and at the second station, and time-dependent splitting. If we only consider two service stations in series without splitting, by considering the second station alone, the arrival process entering the second station is simply $\hat{D}_1$, the departure process from the first queue. Weak dependence of service times in the first station affects the departure process $\hat{D}_1$ in the term $\hat{X}_2^e$ in (3.31). These effects are all captured in the variance formulas for these processes when the arrival limit process is a Brownian motion, see Propositions 3.2 and 3.3.

Special Case I: EARMA(1,1) Service Times. Jacobs and Lewis (1977) proposed an approach to generate a stationary sequence of dependent random variables from a sequence of iid exponential random variables, the so-called EARMA(1,1) sequence, and Jacobs (1980) applied such stationary sequences to study single server queues with dependent service and interarrival times. Here we apply to the IS setting with dependent service times.

We construct the two sequences of service times at the two service stations, $\{\eta_{i,l} : i \geq 1\}$, $l = 1, 2$, from the following mutually independent sequences of iid random variables, $\{\tilde{\eta}_{i,l} : i \geq 1\}$ as a sequence of iid exponential random variables with mean μ_l^{-1}, $\{\psi_{i,l} : i \geq 1\}$ as a sequence of iid random variables with $P(\psi_{i,l} = 0) = \alpha_l \in (0, 1)$ and $P(\psi_{i,l} = 1) = 1 - \alpha_l$, and $\{\varphi_{i,l} : i \geq 1\}$ as a sequence of iid random variables with $P(\varphi_{i,l} = 0) = \beta_l \in (0, 1)$ and $P(\varphi_{i,l} = 1) = 1 - \beta_l$. Define

$$\eta_{i,l} \equiv \alpha_l \tilde{\eta}_{i,l} + \psi_i \xi_{i-1,l}, \quad \xi_{i,l} = \beta_l \xi_{i-1,l} + \varphi_{i,l} \tilde{\eta}_{i,l} \quad i = 1, 2, ... \tag{3.35}$$

with $\xi_{0,l}$ being an independent random variable with mean μ_l^{-1} for each $l = 1, 2$. Then, by Jacobs and Lewis (1977), for each $l = 1, 2$, $\{\eta_{i,1} : i \geq 1\}$ is a stationary sequence of dependent exponential random variables with mean μ_l^{-1} and correlation

$$Corr(\eta_{1,l}, \eta_{k,l}) = \beta_l^{k-2}(1 - \alpha_l)(\alpha_l(1 - \beta_l) + (1 - \alpha_l)\beta_l), \quad k = 2, 3, ... \tag{3.36}$$

Thus, each pair $(\eta_{1,l}, \eta_{k,l})$, $k = 2, 3, ...$, is a bivariate exponentially distributed random variable with mean (μ_l^{-1}, μ_l^{-1}) and covariance matrix $[\sigma_{\eta,l,i,j} : i, j = 1, 2]$ with

$$\sigma_{\eta,l,1,1} = \sigma_{\eta,l,2,2} = \mu_l^{-1}, \tag{3.37}$$

and

$$\sigma_{\eta,l,1,2} = \sigma_{\eta,l,2,1} = \beta_l^{k-2}(1 - \alpha_l)(\alpha_l(1 - \beta_l) + (1 - \alpha_l)\beta_l)\mu_l^{-1}. \tag{3.38}$$

Let $F_{k,l}(\cdot, \cdot)$, $k = 2, 3, ..., l = 1, 2$, be the joint distribution function of each pair $(\eta_{1,l}, \eta_{k,l})$ with the covariance structure in (3.37) and (3.38). Then the covariance $\Gamma^c_{K,l}(x, y)$ in (2.22) can be written as

$$\Gamma^c_{K,l}(x,y) = \sum_{k=2}^{\infty} \Big(F_{k,l}(x,y) + F_{k,l}(y,x) - 2(1-e^{-\mu_l x})(1-e^{-\mu_l y}) \Big) < \infty. \tag{3.39}$$

■

Special Case II: Batch Arrivals. Suppose that at each arrival time $\tau^n_{i,1}$, $i = 1, 2, ...,$ there are a random number $\mathcal{B}_i$ of service requests entering the system at the same time, where $\{\mathcal{B}_i : i = 1, 2, ...\}$ is a sequence of iid random variables with a common distribution. Let $p_{\mathcal{B},k} = P(\mathcal{B}_i = k)$ and $\sum_{k=1}^{\infty} p_{\mathcal{B},k} = 1$. Suppose that $E[\mathcal{B}_i] = \sum_{k=1}^{\infty} k p_{\mathcal{B},k} < \infty$ and $E[\mathcal{B}_i^2] = \sum_{k=1}^{\infty} k^2 p_{\mathcal{B},k} < \infty$. The stationary excess distribution of $\mathcal{B}_i$ is given by $p^*_{\mathcal{B},k} = (E[\mathcal{B}_i])^{-1} \sum_{j=k}^{\infty} p_{\mathcal{B},j}$ for $k = 1, 2, ...,$ and $E[\mathcal{B}_i^*] = (E[\mathcal{B}_i^2] + E[\mathcal{B}_i])/(2E[\mathcal{B}_i])$.

For the arrivals in the i^{th} batch, the service requirements $\{\eta_{i_1,l}, \eta_{i_2,l}, ..., \eta_{i_{\mathcal{B}_i},l}\}$ are correlated at the l^{th} station, $l = 1, 2$, but mutually independent for the two stations (if the service requests will occur at the second station) and moreover, for any i^{th} and j^{th} batches of arrivals, the service requirements $\{\eta_{i_1,l}, \eta_{i_2,l}, ..., \eta_{i_{\mathcal{B}_i},l}\}$ and $\{\eta_{j_1,l}, \eta_{j_2,l}, ..., \eta_{j_{\mathcal{B}_j},l}\}$ are independent. Then, the covariance function $\Gamma^c_{K,l}(x,y)$ in (2.22) becomes

$$\begin{aligned} \Gamma^c_{K,l}(x,y) &= \sum_{i=1}^{\infty} \Big[p^*_{\mathcal{B},i} \sum_{k=2}^{i} \Big(E[\gamma_{1_1,l}(x)\gamma_{1_k,l}(y)] + E[\gamma_{1_1,l}(y)\gamma_{1_k,l}(x)] \Big) \Big] \\ &= \sum_{i=1}^{\infty} \Big[p^*_{\mathcal{B},i} \sum_{k=2}^{i} \Big(F_{k,l}(x,y) + F_{k,l}(y,x) - 2F_l(x)F_l(y) \Big) \Big], \end{aligned} \tag{3.40}$$

where $\gamma_{i_k,l}(x) = \mathbf{1}(\eta_{i_k,l} \le x) - F_l(x)$ for each service requirement $k = 1, ..., \mathcal{B}_i$ in the i^{th} batch, and $F_{k,l}(x,y)$ is the joint distribution function for each pair $(\eta_{i_1,l}, \eta_{i_k,l})$ of the i^{th} batch. Note that the job 1 in batch i is not necessarily the first job in the batch, but instead an arbitrary job in the batch, and thus we use the stationary-excess batch size distribution. For a comparison of the difference between the first job delay and an arbitrary job delay in a batch for single-server queues, see Whitt (1983).

Suppose, in addition, that the dependence between any two service requests among the arrivals in a batch is the same, that is, $F_l(x,y) = F_{k,l}(x,y)$ for each pair $(\eta_{i_1,l}, \eta_{i_k,l})$ of the i^{th} batch. Then the covariance function $\Gamma^c_{K,l}(x,y)$ in (3.40) can be simplified

$$\Gamma^c_{K,l}(x,y) = \Big(F_l(x,y) + F_l(y,x) - 2F_l(x)F_l(y) \Big)(E[\mathcal{B}_1^*] - 1). \tag{3.41}$$

■

3.3 Characterizing the FCLT Limit Processes

In this section, we give the characterizations of the limit processes in Theorem 3.2. First, we give the Gaussian property of the processes $\hat{X}_2^e$ and $\hat{Y}_2^e$ defined in Definition 3.1. Recall that these processes do not involve the limit process $\hat{A}$.

Proposition 3.1 (Gaussian property of $\hat{X}_2^e$ and $\hat{Y}_2^e$) *The two-parameter processes $\hat{X}_2^e$ and $\hat{Y}_2^e$ in* (3.30) *are well-defined continuous Gaussian processes with mean* 0 *and covariances*

$$\begin{aligned} E[\hat{X}_2^e(t_1,y_1)\hat{X}_2^e(t_2,y_2)] &= \int_{(t_1-y_1)\vee(t_2-y_2)}^{t_1\wedge t_2} \Big(F_1(t_1\wedge t_2 - s) - F_1(t_1-s)F_1(t_2-s) \\ &\qquad +\Gamma_{K,1}^c(t_1-s,t_2-s)\Big)d\bar{a}_1(s), \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} E[\hat{Y}_2^e(t_1,y_1)\hat{Y}_2^e(t_2,y_2)] &= \int_{(t_1-y_1)\vee(t_2-y_2)}^{t_1\wedge t_2} \Big[\Big(F_2(t_1\wedge t_2 - s) - F_2(t_1-s)F_2(t_2-s) \\ &\qquad +\Gamma_{K,2}^c(t_1-s,t_2-s)\Big)p(s)\int_0^s f_1(s-u)d\bar{a}_1(u)\Big]ds. \end{aligned} \tag{3.43}$$

Proposition 3.2 (Gaussian property with time-varying arrivals) *Under Assumptions in Theorem 3.2, if, in addition, $\hat{A}_1(t) = \sqrt{c_{a,1}^2}B_{a,1}(\bar{a}_1(t))$, where $B_{a,1}$ is a standard BM, $\bar{a}_1(t) = \int_0^t \lambda_1(s)ds$ and $c_{a,1}^2$ is the SCV of the interarrival times, then the limit processes are all continuous Gaussian processes*

$$\begin{aligned} &\hat{D}_1(t) \stackrel{\mathrm{d}}{=} N(0,\sigma_{D_1}^2(t)), \quad \hat{A}_2(t) \stackrel{\mathrm{d}}{=} N(0,\sigma_{A_2}^2(t)), \quad \hat{D}_2(t) \stackrel{\mathrm{d}}{=} N(0,\sigma_{D_2}^2(t)), \\ &\hat{Q}_1^e(t,y) \stackrel{\mathrm{d}}{=} N(0,\sigma_{Q_1,e}^2(t,y)), \quad \hat{Q}_2^e(t,y) \stackrel{\mathrm{d}}{=} N(0,\sigma_{Q_2,e}^2(t,y)), \end{aligned} \tag{3.44}$$

where

$$\sigma_{Q_1,e}^2(t,y) = \int_{t-y}^t \lambda_1(s)\Big(F_1^c(t-s) + (c_{a,1}^2-1)(F_1^c(t-s))^2 + \Gamma_{K,1}^c(t-s,t-s)\Big)ds, \tag{3.45}$$

$$\sigma_{D_1}^2(t) = \int_0^t \lambda_1(s)\Big(F_1(t-s) + (c_{a,1}^2-1)(F_1(t-s))^2 + \Gamma_{K,1}^c(t-s,t-s)\Big)ds, \tag{3.46}$$

$$\begin{aligned} \sigma_{Q_2,e}^2(t,y) &= \sigma_{A_2}^2(t) + (F_2^c(y))^2\sigma_{A_2}^2(t-y) + \int_{t-y}^t\int_{t-y}^t C_{A_2}(u,v)dF_2^c(t-u)dF_2^c(t-v) \\ &\quad -2F_2^c(y)C_{A_2}(t,t-y) - 2\int_{t-y}^t C_{A_2}(t,s)dF_2^c(t-s) \\ &\quad +2F_2^c(y)\int_{t-y}^t C_{A_2}(t-y,s)dF_2^c(t-s) \end{aligned} \tag{3.47}$$

$$+\int_{t-y}^{t}\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)p(s)\int_0^s f_1(s-u)\lambda_1(u)duds,$$

$$\begin{aligned}\sigma_{D_2}^2(t) &= \int_0^t\int_0^t C_{A_2}(u,v)dF_2^c(t-u)dF_2^c(t-v) \\ &\quad +\int_0^t\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)p(s)\int_0^s f_1(s-u)\lambda_1(u)duds,\end{aligned} \tag{3.48}$$

$$\begin{aligned}\sigma_{A_2}^2(t) &= p(t)^2\sigma_{D_1}^2(t)+\int_0^t\sigma_{D_1}^2(s)dp(s)-2p(t)\int_0^t C_{D_1}(t,s)dp(s) \\ &\quad +\int_0^t p(s)(1-p(s))\int_0^s f_1(s-u)\lambda_1(u)duds,\end{aligned} \tag{3.49}$$

with

$$\begin{aligned}C_{D_1}(t_1,t_2) &= c_{a,1}^2\int_0^{t_1\wedge t_2}\lambda_1(s)F_1(t_1-s)F_1(t_2-s)ds, \\ &\quad +\int_0^{t_1\wedge t_2}\lambda_1(s)\Big(F_1^c(t_1\wedge t_2-s)-F_1^c(t_1-s)F_1^c(t_2-s)+\Gamma_{K,1}^c(t_1-s,t_2-s)\Big)ds,\end{aligned} \tag{3.50}$$

$$\begin{aligned}C_{A_2}(t_1,t_2) &= p(t_1)p(t_2)C_{D_1}(t_1,t_2)+\int_0^{t_1}\int_0^{t_2}C_{D_1}(u,v)dp(u)dp(v) \\ &\quad -p(t_1)\int_0^{t_2}C_{D_1}(t_1,v)dp(v)-p(t_2)\int_0^{t_1}C_{D_1}(u,t_2)dp(u) \\ &\quad +\int_0^{t_1\wedge t_2}p(s)(1-p(s))\int_0^s f_1(s-u)\lambda_1(u)duds.\end{aligned} \tag{3.51}$$

Proposition 3.3 (Gaussian property in the standard case) *Under Assumptions in Theorem 3.2 and in the standard case,* (3.44) *holds with*

$$\begin{aligned}\sigma_{Q_1,e}^2(t,y) &= \lambda_1\int_{t-y}^{t}\Big(F_1^c(t-s)+(c_{a,1}^2-1)(F_1^c(t-s))^2+\Gamma_{K,1}^c(t-s,t-s)\Big)ds \\ &= \lambda_1\int_0^y\Big(F_1^c(s)+(c_{a,1}^2-1)(F_1^c(s))^2+\Gamma_{K,1}^c(s,s)\Big)ds,\end{aligned} \tag{3.52}$$

$$\sigma_{D_1}^2(t) = \lambda_1\int_0^t\Big(F_1(s)+(c_{a,1}^2-1)(F_1(s))^2+\Gamma_{K,1}^c(s,s)\Big)ds \tag{3.53}$$

$$\lim_{t\to\infty}\frac{\sigma_{D_1}^2(t)}{t}=\lim_{t\to\infty}\lambda_1\Big[F_1(t)+(c_{a,1}^2-1)(F_1(t))^2+\Gamma_{K,1}^c(t,t)\Big)\Big]=\lambda_1c_{a,1}^2, \tag{3.54}$$

$$\sigma_{A_2}^2(t)=p^2\sigma_{D_1}^2(t)+p(1-p)\lambda_1\int_0^t F_1(s)ds \tag{3.55}$$

$$\lim_{t\to\infty}\frac{\sigma_{A_2}^2(t)}{t}=p(1-p)\lambda_1+p^2\lambda_1c_{a,1}^2, \tag{3.56}$$

$$
\begin{aligned}
\sigma^2_{Q_2,e}(t,y) \quad &= \quad p\lambda_1 \int_{t-y}^{t} \Big(F_2^c(t-s) + \Gamma^c_{K,2}(t-s,t-s)\Big) F_1(s)ds \\
&\quad +p^2\lambda_1 \Big\{ \int_{t-y}^{t}\int_{t-y}^{t} \Big(F_2^c(t-u\vee v)F_2^c(t-u\wedge v)\Big(f_1(u\vee v - u\wedge v) \\
&\qquad +(c_a^2-1)\int_0^{u\wedge v} f_1(u-s)f_1(v-s)ds\Big)\Big)dudv \\
&\quad + \int_{t-y}^{t}\int_{t-y}^{t} F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v}\Big(\frac{\partial^2}{\partial v\partial u}\Gamma^c_{K,1}(s-u,s-v)\Big)ds\Big]dudv\Big\} \\
&\xrightarrow{t\to\infty} \quad p\lambda_1 \int_0^{y} \Big(F_2^c(s) + \Gamma^c_{K,2}(s,s)\Big)ds \\
&\quad p^2\lambda_1 \Big\{ \int_0^{y}\int_0^{y} \Big(F_2^c(u\vee v)F_2^c(u\wedge v)\Big(f_1(u\vee v - u\wedge v) \\
&\qquad +(c_a^2-1)\int_{u\wedge v}^{\infty} f_1(s-u)f_1(s-v)ds\Big)\Big)dudv \\
&\quad + \int_0^{y}\int_0^{y} F_2^c(u)F_2^c(v)\Big[\int_{u\wedge v}^{\infty}\Big(\frac{\partial^2}{\partial v\partial u}\Gamma^c_{K,1}(s-u,s-v)\Big)ds\Big]dudv\Big\}
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
\sigma^2_{D_2}(t) \quad &= \quad p^2\lambda_1\Big[\int_0^t (F_1(s) + (c_{a,1}^2-1)F_1(s)^2)ds - \int_0^t (F_2^c(t-v))^2 F_1(v)dv \\
&\quad -2\int_0^t F_2^c(t-u)\Big(F_1(t-u) + (c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s))ds\Big)du \\
&\quad +\int_0^t\int_0^t F_2^c(t-v)F_2^c(t-u)\Big(f_1(u\vee v - u\wedge v) \\
&\qquad +(c_{a,1}^2-1)\int_0^{u\wedge v} f_1(u-s)f_1(v-s))ds\Big)du\Big]dv \\
&\quad +\int_0^t\int_0^t \Big[\int_0^{u\wedge v}\Gamma^c_{K,1}(u-s,v-s)\Big)ds\Big] f_2(t-u)f_2(t-v)dudv\Big] \\
&\quad +p(1-p)\lambda_1\int_0^t F_2(t-s)^2F_1(s)ds \\
&\quad +p\lambda_1\int_0^t \Big(F_2^c(t-s) - (F_2^c(t-s))^2 + \Gamma^c_{K,2}(t-s,t-s)\Big)F_1(s)ds,
\end{aligned}
\tag{3.58}
$$

and

$$
\lim_{t\to\infty}\frac{\sigma^2_{D_2}(t)}{t} = p(1-p)\lambda_1 + p^2\lambda_1 c_{a,1}^2. \tag{3.59}
$$

4 Proofs

4.1 Proof of Theorem 2.1

We first showing the convergence of the finite-dimensional distributions (f.d.d.'s) and then we show tightness of $\{\hat{U}_n : n\geq 1\}$ in the space $D([0,\infty), D([0,1],\mathbb{R}))$.

For the convergence of f.d.d.'s, we can apply Theorem 1 of Berkes and Philipp (1977) under the ϕ-mixing condition and Theorem A of Berkes, Hormann and Schauer (2009) under the S-mixing condition to deduce that, for $0 \le t_1 < t_2 < \cdots < t_k$,

$$(\hat{U}_n(t_1,\cdot),...,\hat{U}_n(t_k,\cdot)) \Rightarrow (\hat{U}(t_1,\cdot),...,\hat{U}(t_k,\cdot)) \quad \text{in} \quad D([0,1],\mathbb{R})^k \quad \text{as} \quad n\to\infty, \tag{4.1}$$

where the k elements in the limit are random elements in the functional space $D([0,1],\mathbb{R})$. Then, by those two theorems above, for each t_i, we have that for each $x_{t_i,1},...,x_{t_i,j_{t_i}}$,

$$\begin{aligned} & (\hat{U}_n(t_1,x_{t_1,1}),...,\hat{U}_n(t_1,x_{t_1,j_{t_i}}),\cdots,\hat{U}_n(t_k,x_{t_k,1}),...,\hat{U}_n(t_k,x_{t_k,j_{t_k}})) \qquad (4.2)\\ \Rightarrow\ & (\hat{U}(t_1,x_{t_1,1}),...,\hat{U}(t_1,x_{t_1,j_{t_i}}),\cdots,\hat{U}(t_k,x_{t_k,1}),...,\hat{U}(t_k,x_{t_k,j_{t_k}})) \quad \text{in} \quad \mathbb{R}^{j_{t_1}+\cdots+j_{t_k}} \quad \text{as} \quad n\to\infty. \end{aligned}$$

We next show the tightness of $\{\hat{U}_n : n\ge 1\}$ in $D([0,\infty),D([0,1],\mathbb{R}))$ by applying the tightness criteria in Theorem 6.2 in Pang and Whitt (2010). First, the stochastic boundedness of $\{\hat{U}_n : n\ge 1\}$ in $D([0,\infty),D([0,1],\mathbb{R}))$ follows easily from the convergence in $D([0,1]^2,\mathbb{R})$ under either the ϕ-mixing condition or the S-mixing condition.

Then, it suffices to show that

$$\lim_{\vartheta\to 0}\limsup_{n\to\infty}\sup_{\kappa_n} P\left(\sup_{t\le\vartheta} d_{J_1}(\hat{U}_n(\kappa_n+t,\cdot),\hat{U}_n(\kappa_n,\cdot)) \ge \varsigma\right) = 0 \tag{4.3}$$

where $\{\kappa_n : n\ge 1\}$ is a sequence of uniformly bounded stopping times with respect to the natural filtration $\mathbf{G}_n \equiv \{\mathcal{G}_n(t) : t\in[0,T]\}$ with $\mathcal{G}_n(t) = \sigma\{\hat{U}_n(s,\cdot) : 0\le s\le t\}\vee\mathcal{N}$ satisfying the usual conditions (complete, increasing and right continuous). Due to the fact that the Shokorod J_1 metric for any two functions in D is less than the uniform metric (§3.3, Whitt (2002)), and moreover, by easily observing that

$$\begin{aligned} & P\left(\sup_{t\le\vartheta}\sup_{x\in[0,1]}\left|\hat{U}_n(\kappa_n+t,x)-\hat{U}_n(\kappa_n,x)\right| \ge \varsigma\right)\\ \le\ & 2P\left(\sup_{t\le\vartheta}\sup_{x\in[0,1/2]}\left|\hat{U}_n(\kappa_n+t,x)-\hat{U}_n(\kappa_n,x)\right| \ge \varsigma\right), \end{aligned}$$

we only need to prove that

$$\lim_{\vartheta\to 0}\limsup_{n\to\infty}\sup_{\kappa_n} P\left(\sup_{t\le\vartheta}\sup_{x\in[0,1/2]}\left|\hat{U}_n(\kappa_n+t,x)-\hat{U}_n(\kappa_n,x)\right| \ge \varsigma\right) = 0. \tag{4.4}$$

The sequence $\{\gamma_k(x) : k\ge 1\}$ for each $x\in[0,1]$ is stationary and ergodic, because $\{\xi_k : k\ge 1\}$ is stationary and ergodic under either the ϕ-mixing condition or the S-mixing condition, and moreover,

$$E[\gamma_k(x)] = 0, \quad E[\gamma_k(x)^2] = x(1-x) \le \frac{1}{4}, \quad \text{for all} \quad x\in[0,1].$$

We now construct a martingale difference sequence from the sequence $\{\gamma_k(\cdot) : k \geq 1\}$. We follow the idea in the proof of Theorem 19.1 in Billingsley (1999). Let $\mathbf{F} \equiv \{\mathcal{F}_k : k \geq 1\}$ be the natural filtration generated by the sequence $\{\xi_k : k \geq 1\}$, defined by $\mathcal{F}_k \equiv \sigma\{\xi_i : i \leq k\}$. Define

$$\hat{\gamma}_k(x) \equiv \sum_{i=1}^{\infty} E[\gamma_{k+i}(x)|\mathcal{F}_k], \quad x \in [0,1], \quad k = 1, 2, ... \tag{4.5}$$

and

$$\tilde{\gamma}_k(x) \equiv \gamma_k(x) + \hat{\gamma}_k(x) - \hat{\gamma}_{k-1}(x), \quad x \in [0,1], \quad k = 1, 2, ... \tag{4.6}$$

Then, the sequence $\{\tilde{\gamma}_k(x) : k \geq 1\}$ for each $x \in [0,1]$ is a martingale difference sequence, because for each $k \geq 1$,

$$\begin{aligned}
E[\tilde{\gamma}_{k+1}(x)|\mathcal{F}_k] &= E[\gamma_{k+1}(x) + \hat{\gamma}_{k+1}(x) - \hat{\gamma}_k(x)|\mathcal{F}_k] \\
&= E[\gamma_{k+1}(x)|\mathcal{F}_k] + E\Big[\sum_{i=1}^{\infty} E[\gamma_{k+1+i}(x)|\mathcal{F}_{k+1}]\Big|\mathcal{F}_k\Big] - E[\hat{\gamma}_k(x)|\mathcal{F}_k] \\
&= E[\gamma_{k+1}(x)|\mathcal{F}_k] + \sum_{i=1}^{\infty} E[\gamma_{k+1+i}(x)|\mathcal{F}_k] - \sum_{i=1}^{\infty} E[\gamma_{k+i}(x)|\mathcal{F}_k] \\
&= E[\gamma_{k+1}(x)|\mathcal{F}_k] - E[\gamma_{k+1}(x)|\mathcal{F}_k] = 0,
\end{aligned}$$

and $E[|\tilde{\gamma}_k(x)|] < \infty$ since $E[|\hat{\gamma}_k(x)|] < \infty$ by (2.13).

Define the processes $\tilde{U}_n \equiv \{\tilde{U}_n(t,x) : t, x \geq 0\}$ by

$$\tilde{U}_n(t,x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \tilde{\gamma}_k(x). \tag{4.7}$$

Then it follows that for each $t \geq 0$ and $x \in [0,1]$, (see the proof of Theorem 19.1 in Billingsley (1999))

$$\|\tilde{U}_n(t,x) - \hat{U}_n(t,x)\|_{L^2} = \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (\hat{\gamma}_k(x) - \hat{\gamma}_{k-1}(x)) \right\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

Hence, for each $x \in [0,1]$, κ_n and $n \geq 1$, the process $\{\tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) : t \geq 0\}$ defined by

$$\tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) \equiv \frac{1}{\sqrt{n}} \sum_{k=\lfloor n\kappa_n \rfloor + 1}^{\lfloor n(\kappa_n + t) \rfloor} \tilde{\gamma}_k(x) \tag{4.9}$$

is a locally square integrable martingale with respect to the filtration $\{\mathcal{G}_{\kappa_n + t} : t \geq 0\}$ by Doob's sampling theorem. The difference between $\hat{U}_n(\kappa_n + t, x) - \hat{U}_n(\kappa_n, x)$ and $\tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x)$ is asymptotically negligible as $n \to \infty$ because for $t < \vartheta$ small,

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=\lfloor n\kappa_n \rfloor + 1}^{\lfloor n(\kappa_n + t) \rfloor} (\tilde{\gamma}_k(x) - \gamma_k(x)) \right\|_{L^2} = \left\| \frac{1}{\sqrt{n}} \sum_{k=\lfloor n\kappa_n \rfloor + 1}^{\lfloor n(\kappa_n + t) \rfloor} (\hat{\gamma}_k(x) - \hat{\gamma}_{k-1}(x)) \right\|_{L^2}$$

$$= \quad \left\| \frac{1}{\sqrt{n}} (\hat{\gamma}_{\lfloor n(\kappa_n+t) \rfloor}(x) - \hat{\gamma}_{\lfloor n\kappa_n \rfloor}(x)) \right\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, it suffices to show that

$$\lim_{\vartheta \to 0} \limsup_{n\to\infty} \sup_{\kappa_n} P\left(\sup_{t \le \vartheta} \sup_{x \in [0,1/2]} \left| \tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) \right| \ge \varsigma \right) = 0. \tag{4.10}$$

For each $x \in [0,1]$, κ_n and $n \ge 1$, the process $\{\tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) : t \ge 0\}$ is a locally square integrable martingale with respect to the filtration $\{\mathcal{G}_{\kappa_n + t} : t \ge 0\}$, and then, by Doob's maximal inequality,

$$\begin{aligned} & P\left(\sup_{t \le \vartheta} \sup_{x \in [0,1/2]} \left| \tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) \right| \ge \varsigma \right) \\ \le \quad & \frac{1}{\varsigma^2} E\left[\sup_{x \in [0,1/2]} \left| \tilde{U}_n(\kappa_n + \vartheta, x) - \tilde{U}_n(\kappa_n, x) \right|^2 \right] = \frac{1}{\varsigma^2} E\left[\sup_{x \in [0,1/2]} \left(\frac{1}{\sqrt{n}} \sum_{k=\lfloor n\kappa_n \rfloor + 1}^{\lfloor n(\kappa_n+\vartheta) \rfloor} \tilde{\gamma}_k(x) \right)^2 \right]. \end{aligned}$$

Then, it is obvious that for each fixed n and k, $\{\tilde{\gamma}_k(x) : x \in [0,1]\}$ is a square integrable martingale, and so is $\{\tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) : x \in [0,1]\}$, and thus, by Doob's maximal inequality again,

$$\begin{aligned} & P\left(\sup_{t \le \vartheta} \sup_{x \in [0,1/2]} \left| \tilde{U}_n(\kappa_n + t, x) - \tilde{U}_n(\kappa_n, x) \right| \ge \varsigma \right) \\ \le \quad & \frac{1}{\varsigma^2} E\left[\left(\frac{1}{\sqrt{n}} \sum_{k=\lfloor n\kappa_n \rfloor + 1}^{\lfloor n(\kappa_n+\vartheta) \rfloor} \tilde{\gamma}_k(1/2) \right)^2 \right] \le \frac{1}{\varsigma^2} (\vartheta + 1/n) M_\gamma \end{aligned}$$

where $M_r = \sum_{k=1}^{\infty} E[\tilde{\gamma}_k(1/2)^2] < \infty$. This upper bound goes to zero as $\vartheta \to 0$ and $n \to \infty$ and thus, (4.10) holds. The proof is complete. ∎

4.2 Time-Dependent Splitting

In this section, we obtain an FWLLN and an FCLT for counting processes split from one counting process with a stochastically independent but time-dependent splitting mechanism. This generalizes standard iid splitting of counting processes, as in Theorem 9.5.1 in Whitt (2002).

Let $A \equiv \{A(t) : t \ge 0\}$ be the original counting process with event times $\{\tau_i : i \ge 1\}$ where $\tau_1 < \tau_2 < \cdots$, and $A(t) \equiv \max\{n \ge 0 : \tau_0 + \tau_1 + \cdots + \tau_n \le t\}$ with $\tau_0 \equiv 0$. Suppose that the process A is split into m counting processes $(A_1, ..., A_m)$, where $A_j \equiv \{A_j(t) : t \ge 0\}$ for $j = 1, ..., m$. Let $X_{i,j}$ be a random variable taking values 0 and 1, indicating if the i^{th} event in A is split into the process A_j, that is, $X_{i,j} = 1$ if τ_i becomes an event time for A_j and $X_{i,j} = 0$ otherwise. Then, we can write

$$A_j(t) = \sum_{i=1}^{A(t)} X_{i,j}, \quad t \ge 0. \tag{4.11}$$

Let $\mathbf{F} \equiv \{\mathcal{F}_i : i \geq 0\}$ be the filtration generated by the event times $\{\tau_i\}$: $\mathcal{F}_i \equiv \sigma\{\tau_l : l \leq i\} \vee \mathcal{N}$ with $\mathcal{N}$ being the null set and let $\mathcal{F}_\infty \equiv \sigma\{\tau_i : i \geq 1\} \vee \mathcal{N} = \sigma\{A(t) : t \geq 0\} \vee \mathcal{N}$. We consider a sequence of such counting processes and their split processes indexed by n and let $n \to \infty$.

We consider time-dependent splitting probabilities. Let these splitting probabilities be specified by a vector of functions $p = (p_1, ..., p_m)$ in $D([0, \infty), [0, 1]^m)$, such that the following three conditions hold:

(i) $\sum_{j=1}^m p_j(t) \leq 1$ for each $t \geq 0$.

(ii) p is piecewise-smooth, as specified in Assumption 3. Moreover, we require that

$$\{t_i : i = 1, ..., m\} \cap \{\tau_{n,i} : i \geq 1\} = \emptyset \quad \text{w.p.1} \tag{4.12}$$

for all n.

(iii) the sequence $\{X_{n,i,j} : i \geq 1\}$ for each n and $j = 1, ..., m$ is a sequence of mutually conditionally independent random vectors given $\mathcal{F}_{n,\infty}$, and

$$E[X_{n,i,j} | \mathcal{F}_{n,i}] = p_j(\tau_{n,i}), \quad i \geq 1, \quad j = 1, ..., m. \tag{4.13}$$

The final condition about the discontinuity points in (ii) is automatically satisfied if the stochastic processes A_n are continuous in probability, as in the case of a renewal process, where the time between renewals has a continuous cdf. Condition (iii) can be relaxed to allow the sequence $\{X_{n,i,j} - p_j(\tau_{n,i-1}) : i \geq 1\}$ with $\tau_{n,0} = 0$ to be a $\mathbf{F}_n$-martingale difference sequence (m.d.s.) for each $j = 1, ..., m$, that is,

$$E[X_{n,i,j} - p_j(\tau_{n,i-1}) | \mathcal{F}_{n,i-1}] = 0, \quad i \geq 1. \tag{4.14}$$

Define the following fluid-scaled processes

$$\bar{A}_n = n^{-1} A_n, \quad \bar{A}_{n,j} = n^{-1} A_{n,j}, \quad j = 1, ..., m. \tag{4.15}$$

Theorem 4.1 (*FWLLN for time-dependent splitting processes*) *Suppose that there exists a continuous nondecreasing deterministic function $a \geq 0$ such that*

$$\bar{A}_n \Rightarrow a \quad in \quad D([0, \infty), \mathbb{R}_+) \quad as \quad n \to \infty, \tag{4.16}$$

and a vector of deterministic functions $p = (p_1, ..., p_m) \in D([0, \infty), [0, 1]^m)$ and a sequence of random variables $\{X_{n,i,j} : i \geq 1, j = 1, ..., m\}$ taking values 0 or 1 for each n that satisfy the above conditions $(i) - (iii)$. Then,

$$(\bar{A}_{n,1}, ..., \bar{A}_{n,m}) \Rightarrow (a_1, ..., a_m) \quad in \quad D^m \quad as \quad n \to \infty, \tag{4.17}$$

where

$$a_j(t) \quad = \quad \int_0^t p_j(s)da(s) = a(t)p_j(t) - \int_0^t a(s)dp_j(s), \quad t \geq 0, \quad j = 1, ..., m, \tag{4.18}$$

and the last integral is understood as in (2.7)*. If, in addition,* $a(t)$ *is absolutely continuous with the almost everywhere derivative* $\dot{a}(t)$*, then* $a_j(t)$ *in* (4.18) *can also be expressed as*

$$a_j(t) = \int_0^t p_j(s)\dot{a}(s)ds, \quad t \geq 0, \quad j = 1, ..., m. \tag{4.19}$$

Proof. By (4.11), for each $j = 1, ..., m$ and $t \in [0, T] \setminus \{t_1, ..., t_m\}$,

$$\bar{A}_{n,j}(t) \quad = \quad \frac{1}{n} \sum_{i=1}^{n\bar{A}_n(t)} X_{n,i,j} = \frac{1}{n} \sum_{i=1}^{n\bar{A}_n(t)} \tilde{X}_{n,i,j} + \int_0^t p_j(s)d\bar{A}_n(s) \tag{4.20}$$

where $\tilde{X}_{n,i,j} = X_{n,i,j} - E[X_{n,i,j}|\mathcal{F}_{n,i}]$. It is easy to see that $\{\tilde{X}_{n,ij} : i \geq 1\}$ is a martingale difference sequence with respect to the filtration $\mathbf{F}_n$. Thus, the first term in (4.20) converges to 0 by the FWLLN for m.d.s.'s. (Theorem 2.13 in Hall and Heyde (1980), note that $E[\tilde{X}_{n,i,j}] = 0$ and $E[\tilde{X}^2_{n,i,j}|\mathcal{F}_{n,i}] = p_j(\tau_{n,i})(1 - p_j(\tau_{n,i})) < \infty$ w.p.1 and $\int_0^T p_j(s)(1 - p_j(s))ds < \infty$ for each $T > 0$ so that the m.d.s. is in L^2.) For the second term in (4.20), since $\bar{A}_n(t)$ is increasing with finite variation, we can apply integration by parts,

$$\begin{aligned} &\int_0^t p_j(s)d\bar{A}_n(s) = \bar{A}_n(t)p_j(t) - \int_0^t \bar{A}_n(s)dp_j(s) \\ = \quad &\bar{A}_n(t)p_j(t) - \int_0^t \bar{A}_n(s)\dot{p}_j(s)ds - \sum_{k=1}^m \mathbf{1}(t_k \in [0, t])\bar{A}_n(t_k)(p_j(t_k) - p_j(t_k-)), \end{aligned} \tag{4.21}$$

where the second line follows from the definition of the integral in the last term of the first line as in (2.7). We remark that a key assumption for this integral definition to be valid is that there is no common discontinuity of $\bar{A}_n$ and p_j w.p.1 (condition (ii)).

Consider the mapping $\psi : D[0, T] \to D[0, T]$ defined by

$$\begin{aligned} z(t) \quad &= \quad \psi(x)(t) = \int_0^t p_j(s)dx(s) \\ &= \quad x(t)p_j(t) - \int_0^t x(s)\dot{p}_j(s)ds - \sum_{k=1}^m \mathbf{1}(t_k \in [0, t])x(t_k)(p_j(t_k) - p_j(t_k-)), \end{aligned} \tag{4.22}$$

where x is continuous. Then it is evident that the mapping ψ is continuous in the Skorohod J_1 topology. So by the CMT, we obtain the convergence in (4.17). ∎

Define the diffusion-scaled processes

$$\hat{A}_n(t) = \sqrt{n}(\bar{A}_n(t) - a(t)), \quad \hat{A}_{n,j}(t) = \sqrt{n}(\bar{A}_{n,j}(t) - a_j(t)), \quad j = 1, ..., m, \quad t \geq 0. \tag{4.23}$$

Theorem 4.2 (*FCLT for time-dependent splitting processes*) *Under the assumptions of Theorem 4.1, if in addition, there exists a stochastic process $\hat{A}$ with continuous sample path such that*

$$\hat{A}_n \Rightarrow \hat{A} \quad in \quad D([0,\infty),\mathbb{R}) \quad as \quad n \to \infty, \tag{4.24}$$

jointly with (4.16) *with the Skorohod product topology. Then,*

$$(\hat{A}_{n,1},...,\hat{A}_{n,m}) \Rightarrow (\hat{A}_1,...,\hat{A}_m) \quad in \quad D^m \quad as \quad n \to \infty, \tag{4.25}$$

where the convergence is joint with (4.16) *and* (4.24)*, and for each $j = 1,...,m$,*

$$\begin{aligned} \hat{A}_j(t) &= B_j\left(\int_0^t p_j(s)(1-p_j(s))da(s)\right) + \int_0^t p_j(s)d\hat{A}(s) \\ &= B_j\left(\int_0^t p_j(s)(1-p_j(s))da(s)\right) + p_j(t)\hat{A}(t) - \int_0^t \hat{A}(s)dp_j(s), \end{aligned} \tag{4.26}$$

where $B = (B_1,...,B_m)$ is a standard m-dimensional BM independent of $\hat{A}$, if $\hat{A}(t) \stackrel{\mathrm{d}}{=} c_a B_a(a(t))$ for a standard BM independent of B and c_a^2 being the SCV of interarrival times of A, $\hat{A}_j(t)$ takes the first expression in (4.26) *and the second term*

$$\int_0^t p_j(s)d\hat{A}(s) \stackrel{\mathrm{d}}{=} c_a B_a(a_j(t)) = c_a B_a\left(\int_0^t p_j(s)da(s)\right), \quad t \geq 0, \tag{4.27}$$

with $a_j(t)$ given in (4.18)*, or if $\hat{A}(t)$ is a Gaussian process, $\hat{A}_j(t)$ takes the second expression in* (4.26) *with the last integral understood as in* (2.7).

Proof. Fix $T > 0$. We will prove the limits hold in $D([0,T],\mathbb{R}^m)$, which then easily extends to $D([0,\infty),\mathbb{R}^m)$. By (4.23) and (4.21), we obtain

$$\begin{aligned} \hat{A}_{n,j}(t) &= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n\bar{A}_n(t)} X_{n,i,j} - a_j(t)\right) \\ &= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n\bar{A}_n(t)} [X_{n,i,j} - E[X_{n,i,j}|\mathcal{F}_{n,i}]] + \int_0^t p_j(s)d\bar{A}_n(s) - a_j(t)\right) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n\bar{A}_n(t)} \tilde{X}_{n,i,j} + \int_0^t p_j(s)d\hat{A}_n(s). \end{aligned} \tag{4.28}$$

For the first term in (4.28), we apply the FCLT for m.d.s.'s, Theorem 6 in Rootzén (1980). To check the conditions, first, choose $\zeta_n = A_n(T) + 1$ so that the sequence $\{\zeta_n : n \geq 1\}$ is a sequence of $\mathbf{F}$-stopping times such that $\zeta_n > A_n(T)$ w.p.1 and second, since $|\tilde{X}_{n,i,j}| \leq 1$ w.p.1, it follows that $E[\max_{1\leq i\leq \zeta_n} |\tilde{X}_{n,i,j}/\sqrt{n}|] \to 0$ as $n \to \infty$ for each $j = 1,...,m$. Moreover, by (4.16), we have

$$\left(\sum_{i=1}^{n\bar{A}_n(t)} (\tilde{X}_{n,i,j}/\sqrt{n})^2 : j = 1,...,m\right) \Rightarrow \left(\int_0^t p_j(s)(1-p_j(s))da(s) : j = 1,...,m\right), \tag{4.29}$$

in $D([0,T],\mathbb{R}^m)$ as $n \to \infty$. Thus, by Theorem 6 in Rootzén (1980),

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n\bar{A}_n(t)} \tilde{X}_{n,i,j} : j = 1, ..., m\right) \Rightarrow \left(B_j\left(\int_0^t p_j(s)(1-p_j(s))da(s)\right) : j = 1, ..., m\right) \qquad (4.30)$$

in $D([0,T],\mathbb{R}^m)$ as $n \to \infty$ for standard m-dimensional Brownian motion $B = (B_1, ..., B_m)$.

For the convergence of the second term in (4.28), since $\hat{A}_n$ is the difference of two monotone functions, it is of bounded variation and we can represent the integral in the final term of (4.28) using integration by parts, paralleling (4.21), with $\hat{A}_n$ instead of $\bar{A}_n$. In that framework, we can then apply the CMT with the mapping ψ in (4.22). ■

4.3 Proofs of Characterizations of the Limit Processes

Proof of Proposition 3.2. First, since the process $\hat{K}_1$ is continuous Gaussian, the process $\hat{X}^e_{2,k}$ defined in (3.26) and (3.27) is also continuous Gaussian for each $k \geq 1$, and thus the limit as $k \to \infty$ is also Gaussian. Next, we want to calculate

$$E[(\hat{X}^e_2(t_1,y_1) - \hat{X}^e_2(t_2,y_2))^2] = \lim_{k\to\infty} E[(\hat{X}^e_{2,k}(t_1,y_1) - \hat{X}^e_{2,k}(t_2,y_2))^2] \qquad (4.31)$$

for each $t_1 \leq t_2$ and $y_1 \leq y_2$.

Define for $t_1 \leq t_2$ and $x_1 \leq x_2$,

$$\Delta_{K,1}(t_1,t_2,x_1,x_2) \equiv \hat{K}_1(\bar{a}_1(t_2),x_2) - \hat{K}_1(\bar{a}_1(t_1),x_2) - \hat{K}_1(\bar{a}_1(t_2),x_1) + \hat{K}_1(\bar{a}_1(t_1),x_1). \qquad (4.32)$$

Then, for $t_1 \leq t_2$ and $x_1 \leq x_2$,

$$\begin{aligned}
& E[(\Delta_{K,1}(t_1,t_2,x_1,x_2))^2] \\
= \; & E[\hat{K}_1(\bar{a}_1(t_2),x_2)^2] + E[\hat{K}_1(\bar{a}_1(t_1),x_2)^2] + E[\hat{K}_1(\bar{a}_1(t_2),x_1)^2] + E[\hat{K}_1(\bar{a}_1(t_1),x_1)^2] \\
& -2E[\hat{K}_1(\bar{a}_1(t_2),x_2)\hat{K}_1(\bar{a}_1(t_1),x_2)] - 2E[\hat{K}_1(\bar{a}_1(t_2),x_2)\hat{K}_1(\bar{a}_1(t_2),x_1)] \\
& +2E[\hat{K}_1(\bar{a}_1(t_2),x_2)\hat{K}_1(\bar{a}_1(t_1),x_1)] + 2E[\hat{K}_1(\bar{a}_1(t_1),x_2)\hat{K}_1(\bar{a}_1(t_2),x_1)] \\
& -2E[\hat{K}_1(\bar{a}_1(t_1),x_2)\hat{K}_1(\bar{a}_1(t_1),x_1)] - 2E[\hat{K}_1(\bar{a}_1(t_2),x_1)\hat{K}_1(\bar{a}_1(t_1),x_1)] \\
= \; & \bar{a}_1(t_2)\Gamma_{K,1}(x_2,x_2) + \bar{a}_1(t_1)\Gamma_{K,1}(x_2,x_2) + \bar{a}_1(t_2)\Gamma_{K,1}(x_1,x_1) + \bar{a}_1(t_1)\Gamma_{K,1}(x_1,x_1) \\
& -2\bar{a}_1(t_1)\Gamma_{K,1}(x_2,x_2) - 2\bar{a}_1(t_2)\Gamma_{K,1}(x_2,x_1) + 2\bar{a}_1(t_1)\Gamma_{K,1}(x_2,x_1) + 2\bar{a}_1(t_1)\Gamma_{K,1}(x_2,x_1) \\
& -2\bar{a}_1(t_1)\Gamma_{K,1}(x_2,x_1) - 2\bar{a}_1(t_1)\Gamma_{K,1}(x_1,x_1) \\
= \; & (\bar{a}_1(t_2) - \bar{a}_1(t_1))[\Gamma_{K,1}(x_2,x_2) + \Gamma_{K,1}(x_1,x_1) - 2\Gamma_{K,1}(x_2,x_1)] \\
= \; & (\bar{a}_1(t_2) - \bar{a}_1(t_1))(F_1(x_2) - F_1(x_1))(1 + F_1(x_1) - F_1(x_2))
\end{aligned}$$

$$+(\bar{a}_1(t_2) - \bar{a}_1(t_1))[\Gamma^c_{K,1}(x_2, x_2) + \Gamma^c_{K,1}(x_1, x_1) - 2\Gamma^c_{K,1}(x_2, x_1)] \tag{4.33}$$

and for $t_1 \leq t_2$ and $x_1 \leq x_2$, $t'_1 \leq t'_2$ and $x'_1 \leq x'_2$ and $t_2 < t'_1$,

$$E[\Delta_{K,1}(t_1, t_2, x_1, x_2)\Delta_{K,1}(t'_1, t'_2, x'_1, x'_2)] = 0 \tag{4.34}$$

We choose the same set $\{s^k_i : 0 \leq i \leq k\}$ for $t_1 \leq t_2$ and $y_1 \leq y_2$ so that $t_2 - y_2 = s^k_0 < \cdots < s^k_k = t_2$ for each $k \geq 1$. Without loss of generality, assume that $t_2 - y_2 < t_1 - y_1$. Then, we can write

$$\hat{X}^e_{2,k}(t_1, y_1) - \hat{X}^e_{2,k}(t_2, y_2) = \sum_{i=1}^{k} \Delta_{K,1}(s^k_{i-1}, s^k_i, t_1 - s^k_i, t_2 - s^k_i), \tag{4.35}$$

and by (4.33) and (4.34), we obtain

$$\begin{aligned} E[(\hat{X}^e_{2,k}(t_1, y_1) - \hat{X}^e_{2,k}(t_2, y_2))^2] &= \sum_{i=1}^{k} E[(\Delta_{K,1}(s^k_{i-1}, s^k_i, t_1 - s^k_i, t_2 - s^k_i))^2] \\ = \sum_{i=1}^{k}(\bar{a}_1(s^k_i) - \bar{a}_1(s^k_{i-1}))\Big[&(F_1(t_2 - s^k_i) - F_1(t_1 - s^k_i))(1 + F_1(t_1 - s^k_i) - F_1(t_2 - s^k_i)) \\ &+[\Gamma^c_{K,1}(t_2 - s^k_i, t_2 - s^k_i) + \Gamma^c_{K,1}(t_1 - s^k_i, t_1 - s^k_i) - 2\Gamma^c_{K,1}(t_2 - s^k_i, t_1 - s^k_i)]\Big]. \end{aligned} \tag{4.36}$$

Thus,

$$\begin{aligned} &E[(\hat{X}^e_2(t_1, y_1) - \hat{X}^e_2(t_2, y_2))^2] \\ = &\int_{t_2-y_2}^{t_2} \Big[(F_1(t_2 - u) - F_1(t_1 - u))(1 + F_1(t_1 - u) - F_1(t_2 - u)) \\ &+[\Gamma^c_{K,1}(t_2 - u, t_2 - u) + \Gamma^c_{K,1}(t_1 - u, t_1 - u) - 2\Gamma^c_{K,1}(t_2 - u, t_1 - u)]\Big] d\bar{a}_1(u) \end{aligned} \tag{4.37}$$

for each $t_1 \leq t_2$ and $y_1 \leq y_2$ with $t_2 - y_2 < t_1 - y_1$. The continuity property of $\hat{X}^e_2(t, y)$ in both t and y w.p.1 follows from (4.37) by applying Chebyshev's inequality and the continuity of $\bar{a}_1$. The covariance of $\hat{X}^e_2(t, y)$ follows from a similar argument. The proof is completed. ∎

4.4 Proofs for the FWLLN

Proof of Corollary 3.1. We only need to remark on the long-run average rate for $D_{n,2}$. First,

$$\lim_{t\to\infty} \frac{\bar{d}_2(t)}{t} = \lambda_1 p \lim_{t\to\infty} \int_0^t f_2(t-s)F_1(s)ds,$$

and then,

$$\int_0^t f_2(t-s)F_1(s)ds = \int_0^t F_1(s)dF^c_2(t-s) = F_1(t) - \int_0^t F^c_2(t-s)dF_1(s)$$

$$
\begin{aligned}
&= F_1(t) - \int_0^t \Big(f_1(s) - \int_0^s f_2(s-u) f_1(u)du\Big)ds \\
&= \int_0^t \int_0^s f_2(s-u) f_1(u) du ds \\
&\xrightarrow{t\to\infty} \int_0^\infty \int_0^s f_2(s-u) f_1(u) du ds \\
&= \int_0^\infty \int_u^\infty f_2(s-u) ds f_1(u) du = \int_0^\infty \int_0^\infty f_2(s) ds f_1(u) du = 1.
\end{aligned}
$$

■

4.5 Proofs for the FCLT

Proof of Theorem 3.2 Here we outline the main steps to prove the joint convergence of the processes in (3.28). Once we prove the convergence of $\hat{Q}^e_{n,1}$, the convergence of $\hat{D}_{n,1}$ follows from applying the CMT to the addition mapping, and the convergence of $\hat{A}_{n,2}$ follows from applying the time-dependent splitting of counting processes in Theorem 4.2. Once the convergence of $\hat{Q}^e_{n,2}$ is proven, the convergence of both $\hat{D}_{n,2}$ and $\hat{D}_n$ will follow from again the CMT to the addition mapping. Thus, The main task is to prove the joint convergence of $\hat{Q}^e_{n,1}$ and $\hat{Q}^e_{n,2}$. For both processes, the convergence of $\hat{X}^e_{n,1}$ and $\hat{Y}^e_{n,1}$ follows from applying CMT to the following mapping $\phi : D \times D \to D_D$

$$
\phi(x,z)(t,y) = x(t) - z(y)x(t-y) - \int_{t-y}^t x(s) dz(t-s), \quad t, y \geq 0 \tag{4.38}
$$

where $x, z \in D$. The continuity of the mapping ϕ in the Skorohod J_1 topology follows from a similar argument as in the proof of Lemma 6.1 in Pang and Whitt (2010), and thus, is omitted. Then, it suffices to prove the joint convergence of $\hat{X}^e_{n,2}$ and $\hat{Y}^e_{n,2}$. We will take two steps: tightness (Lemma 4.1) and convergence of f.d.d.'s (Lemma 4.2). ■

Lemma 4.1 (*Tightness*) *Under the assumptions of Theorem 3.2, the processes* $\{(\hat{A}_{n,1}, \hat{X}^e_{n,1}, \hat{X}^e_{n,2}, \hat{D}_{n,1}, \hat{A}_{n,2}, \hat{Y}^e_{n,1}, \hat{Y}^e_{n,2}, \hat{D}_{n,2}) : n \geq 1\}$ *are tight in* $D \times D_D^2 \times D^2 \times D_D^2 \times D$*, and so are the processes* $\{(\hat{A}_{n,1}, \hat{Q}^e_{n,1}, \hat{D}_{n,1}, \hat{A}_{n,2}, \hat{Q}^e_{n,2}, \hat{D}_{n,2}) : n \geq 1\}$.

Proof. The tightness of the processes $\{\hat{A}_{n,1} : n \geq 1\}$ and $\{\hat{X}^e_{n,1} : n \geq 1\}$ follows from the Assumption 1, and the convergence of $\hat{X}^e_{n,1}$ from applying the CMT to the mapping in (4.38).

For the tightness of $\{\hat{X}^e_{n,2} : n \geq 1\}$, we first construct a martingale difference sequence from the sequence $\{\eta_{i,1} : i \geq 1\}$. As in the proof of Theorem 2.1, we follow the idea in the proof of

Theorem 19.1 in Billingsley (1999). Let $\mathbf{F}_1 \equiv \{\mathcal{F}_{k,1} : k \geq 1\}$ be the natural filtration generated by the sequence $\{\eta_{i,1} : i \geq 1\}$, defined by $\mathcal{F}_{k,1} = \sigma\{\eta_{i,1} : i \leq k\} \vee \mathcal{N}$. Define

$$\hat{\gamma}_{k,1}(x) \equiv \sum_{i=1}^{\infty} E[\gamma_{k+i,1}(x)|\mathcal{F}_{k,1}], \quad x \geq 0, \quad k \geq 1, \tag{4.39}$$

and

$$\tilde{\gamma}_{k,1}(x) \equiv \gamma_{k,1}(x) + \hat{\gamma}_{k,1}(x) - \hat{\gamma}_{k-1,1}(x), \quad x \geq 0, \quad k \geq 1, \tag{4.40}$$

where

$$\gamma_{k,1}(x) \equiv \mathbf{1}(\eta_{k,1} \leq x) - F_1(x) = -(\mathbf{1}(\eta_{k,1} > x) - F_1^c(x)), \quad x \geq 0, \quad k \geq 1. \tag{4.41}$$

Then, it is easy to check that for each $x \geq 0$, the sequence $\{\tilde{\gamma}_{k,1}(x) : k \geq 1\}$ is a martingale difference sequence. Define

$$\tilde{K}_{n,1}(t,x) \equiv \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \tilde{\gamma}_{k,1}(x), \quad t, x \geq 0, \tag{4.42}$$

and

$$\tilde{R}_{n,1}(t,x) = \tilde{K}_{n,1}(\bar{A}_{n,1}, x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor n\bar{A}_{n,1}(t) \rfloor} \tilde{\gamma}_{k,1}(x), \quad t, x \geq 0. \tag{4.43}$$

Moreover, define the processes $\tilde{X}^e_{n,2}$ by

$$\tilde{X}^e_{n,2}(t,y) \equiv \int_{t-y}^{t} \int_0^{\infty} \mathbf{1}(s+x>t) d\tilde{R}_{n,1}(s,x), \quad t \geq 0, \quad 0 \leq y \leq t. \tag{4.44}$$

We now show that the difference between $\hat{X}^e_{n,2}$ and $\tilde{X}^e_{n,2}$ becomes negligible as $n \to \infty$. By the definitions of $\hat{X}^e_{n,2}$ and $\tilde{X}^e_{n,2}$, we have

$$\tilde{X}^e_{n,2}(t,y) - \hat{X}^e_{n,2}(t,y) = \int_{t-y}^{t} \int_0^{\infty} \mathbf{1}(s+x>t) d(\tilde{R}_{n,1}(s,x) - \hat{R}_{n,1}(s,x)), \tag{4.45}$$

where

$$\tilde{R}_{n,1}(s,x) - \hat{R}_{n,1}(s,x) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor n\bar{A}_{n,1}(s) \rfloor} (\hat{\gamma}_{k,1}(x) - \hat{\gamma}_{k-1,1}(x)). \tag{4.46}$$

By Assumption 2,

$$E[(\hat{\gamma}_{k,1}(x))^2] = E[(\hat{\gamma}_{k-1,1}(x))^2] < \infty, \quad k \geq 1, \quad x \geq 0, \tag{4.47}$$

and similar to (4.8),

$$\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (\hat{\gamma}_{k,1}(x) - \hat{\gamma}_{k-1,1}(x)) \right\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for} \quad t, x \geq 0. \tag{4.48}$$

By Assumption 1, $\bar{A}_{n,1} \Rightarrow \bar{a}_1$ with $\bar{a}_1$ being a deterministic and continuous function, it follows that

$$E[(\tilde{R}_{n,1}(s,x) - \hat{R}_{n,1}(s,x))^2] \to 0 \quad \text{as} \quad n \to \infty, \quad for \quad s,x \geq 0, \tag{4.49}$$

and thus,

$$E[(\tilde{X}^e_{n,2}(t,y) - \hat{X}^e_{n,2}(t,y))^2] \to 0 \quad \text{as} \quad n \to \infty, \quad for \quad t,y \geq 0. \tag{4.50}$$

Therefore, it suffices to prove the tightness of the processes $\{\tilde{X}^e_{n,2} : n \geq 1\}$ in D_D.

We observe that the processes $\tilde{X}^e_{n,2}$ in (4.44) can be written as

$$\tilde{X}^e_{n,2}(t,y) = \frac{1}{\sqrt{n}} \sum_{i=A_{n,1}(t-y)}^{A_{n,1}(t)} \tilde{\gamma}_{i,1}(t - \tau^n_{i,1}). \tag{4.51}$$

We will apply Theorem 6.2 in Pang and Whitt (2010) to prove the tightness property of $\{\tilde{X}^e_{n,2} : n \geq 1\}$. First, we show the stochastic boundedness of $\tilde{X}^e_{n,2}$. It suffices to show the stochastic boundedness of

$$\check{X}^e_{n,2}(t) = \frac{1}{\sqrt{n}} \sum_{i=0}^{A_{n,1}(t)} \tilde{\gamma}_{i,1}(t - \tau^n_{i,1}) \tag{4.52}$$

since for each t and y, $\tilde{X}^e_{n,2}(t,y) \leq \check{X}^e_{n,2}(t)$. We will show that for any $T > 0$,

$$\lim_{L\to\infty} \lim_{n\to\infty} P\left(\sup_{t\leq T} |\check{X}^e_{n,2}(t)| > L\right) = 0. \tag{4.53}$$

For any constant $\check{L} > 0$, we can write

$$P\left(\sup_{t\leq T} |\check{X}^e_{n,2}(t)| > L\right) \leq P(\bar{A}_{n,1}(T+1) > \check{L}) + P\left(\sup_{t\leq T} |\tilde{K}_{n,1}(\bar{A}_{n,1}(t) \wedge \check{L}, t - \tau^n_{i,1})| > L\right) \tag{4.54}$$

where $\tilde{K}_{n,1}(t,x)$ is defined in (4.42). By the Assumption 1, the sequence of processes $\{\bar{A}_{n,1} : n \geq 1\}$ is tight, and thus

$$\lim_{\check{L}\to\infty} \limsup_{n\to\infty} P(\bar{A}_{n,1}(T+1) > \check{L}) = 0. \tag{4.55}$$

Since $\{\tilde{\gamma}_{k,1}(x) : k \geq 1\}$ in an ergodic martingale difference sequence for each $x \geq 0$, by the Lenglart-Rebolledo inequality (see, e.g., p.30 in Karatzas and Shreve (1991)), for any constant $\tilde{L}$

$$P\left(\sup_{t\leq T} |\tilde{K}_{n,1}(\bar{A}_{n,1}(t) \wedge \check{L}, t - \tau^n_{i,1})| > L\right) \leq \tilde{L}/L + P\left(\langle \tilde{K}_{n,1}(\bar{A}_{n,1}(T) \wedge \check{L}, T - \tau^n_{i,1})\rangle > \tilde{L}\right), \tag{4.56}$$

where

$$\langle \tilde{K}_{n,1}(\bar{A}_{n,1}(T) \wedge \check{L}, T - \tau^n_{i,1})\rangle = \frac{1}{n} \sum_{i=1}^{\lfloor n(\bar{A}_{n,1}(T)\wedge\check{L})\rfloor} E[\tilde{\gamma}_{i,1}(T - \tau^n_{i,1})^2], \tag{4.57}$$

and

$$\frac{1}{n}\sum_{i=1}^{\lfloor n(\bar{A}_{n,1}(t))\rfloor} E[\tilde{\gamma}_{i,1}(t-\tau_{i,1}^n)^2] \Rightarrow \int_0^t E[\tilde{\gamma}_{i,1}(t-s)^2]d\bar{a}_1(s) < \infty \quad \text{as} \quad n\to\infty. \tag{4.58}$$

We can choose $\tilde{L}$ large (but fixed) so that

$$\lim_{L\to\infty}\lim_{n\to\infty} P\left(\sup_{t\le T}|\tilde{K}_{n,1}(\bar{A}_{n,1}(t)\wedge \check{L}, t-\tau_{i,1}^n)| > L\right) = 0, \tag{4.59}$$

and thus (4.53) is proved.

We next show that for any $\varsigma > 0$

$$\lim_{\vartheta\to 0}\limsup_{n\to\infty}\sup_{\kappa_n} P\left(\sup_{t\le\vartheta} d_{J_1}(\tilde{X}_{n,2}(\kappa_n+t,\cdot), \tilde{X}_{n,2}(\kappa_n,\cdot)) > \varsigma\right) = 0, \tag{4.60}$$

where $\{\kappa_n : n\ge 1\}$ is a sequence of uniformly bounded stopping times with respect to the filtration $\mathbf{H}_n \equiv \{\mathcal{H}_n(t) : t\ge 0\}$ and with upper bound κ^*, where

$$\mathcal{H}_n(t) \equiv \sigma\{\eta_{i,1}\le s-\tau_{i,1}^n : 1\le i\le A_{n,1}(t), 0\le s\le t\}\vee\{A_{n,1}(s) : 0\le s\le t\}\vee\mathcal{N} \tag{4.61}$$

and $\mathbf{H}_n$ satisfies the usual conditions. It suffices to show that for any $\varsigma > 0$

$$\lim_{\vartheta\to 0}\limsup_{n\to\infty}\sup_{\kappa_n} P\left(\sup_{t\le\vartheta}\sup_{y\in[0,T\wedge(\kappa_n+t)]}\left|\tilde{X}_{n,2}(\kappa_n+t,y)-\tilde{X}_{n,2}(\kappa_n,y)\right| > \varsigma\right) = 0. \tag{4.62}$$

For each n, κ_n, $y>0$ and $t<\vartheta$ small, by (4.51)

$$\begin{aligned}
&\tilde{X}_{n,2}(\kappa_n+t,y)-\tilde{X}_{n,2}(\kappa_n,y)\\
&= \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n+t-y)}^{A_{n,1}(\kappa_n+t)}\tilde{\gamma}_{i,1}(\kappa_n+t-\tau_{i,1}^n) - \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n-y)}^{A_{n,1}(\kappa_n)}\tilde{\gamma}_{i,1}(\kappa_n-\tau_{i,1}^n)\\
&= \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n+t-y)}^{A_{n,1}(\kappa_n+t)}\tilde{\gamma}_{i,1}(\kappa_n+t-\tau_{i,1}^n) - \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n+t-y)}^{A_{n,1}(\kappa_n)}\tilde{\gamma}_{i,1}(\kappa_n-\tau_{i,1}^n)\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n-y)}^{A_{n,1}(\kappa_n+t-y)}\tilde{\gamma}_{i,1}(\kappa_n-\tau_{i,1}^n)\\
&= \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n)+1}^{A_{n,1}(\kappa_n+t)}\tilde{\gamma}_{i,1}(\kappa_n+t-\tau_{i,1}^n) - \frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n+t-y)}^{A_{n,1}(\kappa_n)}\left[\tilde{\gamma}_{i,1}(\kappa_n-\tau_{i,1}^n)-\tilde{\gamma}_{i,1}(\kappa_n+t-\tau_{i,1}^n)\right]\\
&\quad -\frac{1}{\sqrt{n}}\sum_{i=A_{n,1}(\kappa_n-y)}^{A_{n,1}(\kappa_n+t-y)}\tilde{\gamma}_{i,1}(\kappa_n-\tau_{i,1}^n).
\end{aligned} \tag{4.63}$$

Then, for any $L>0$, we have

$$P\left(\sup_{t\le\vartheta}\sup_{y\in[0,T\wedge(\kappa_n+t)]}\left|\tilde{X}_{n,2}(\kappa_n+t,y)-\tilde{X}_{n,2}(\kappa_n,y)\right| > \varsigma\right)$$

$$
\begin{aligned}
\leq\ & P\left(\bar{A}_{n,1}(\kappa^*+1)>L\right) \\
& +P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n+t-\tau^n_{i,1})\right|>\varsigma\right) \\
& +P\left(\sup_{t\leq\vartheta}\sup_{y\in[0,T\wedge(\kappa_n+t)]}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n+t-y)\wedge L)}^{n(\bar{A}_{n,1}(\kappa_n)\wedge L)}\left[\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})-\tilde{\gamma}_{i,1}(\kappa_n+t-\tau^n_{i,1})\right]\right|>\varsigma\right) \\
& +P\left(\sup_{t\leq\vartheta}\sup_{y\in[0,T\wedge(\kappa_n+t)]}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n-y)\wedge L)}^{n(\bar{A}_{n,1}(\kappa_n+t-y)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})\right|>\varsigma\right). \qquad (4.64)
\end{aligned}
$$

By Assumption 1, we have

$$
\lim_{L\to\infty}\limsup_{n\to\infty}P\left(\bar{A}_{n,1}(\kappa^*+1)>L\right)=0. \qquad (4.65)
$$

For the second term on the right hand side of (4.64),

$$
\begin{aligned}
& P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n+t-\tau^n_{i,1})\right|>\varsigma\right) \\
\leq\ & P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})\right|>\varsigma\right) \\
& +P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}[\tilde{\gamma}_{i,1}(\kappa_n+t-\tau^n_{i,1})-\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})]\right|>\varsigma\right). \qquad (4.66)
\end{aligned}
$$

For each n and $\tilde{\varsigma}>0$, by Lenglart-Rebolledo inequality, the first term on the right hand side of (4.66) satisfies

$$
\begin{aligned}
& P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})\right|>\varsigma\right) \\
\leq\ & \frac{\tilde{\varsigma}}{\varsigma}+P\left(\left\langle\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+\vartheta)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})\right\rangle>\tilde{\varsigma}\right) \\
=\ & \frac{\tilde{\varsigma}}{\varsigma}+P\left(\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+\vartheta)\wedge L)}E[(\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1}))^2]>\tilde{\varsigma}\right) \qquad (4.67)
\end{aligned}
$$

where

$$
\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+\vartheta)\wedge L)}E[(\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1}))^2]\leq\sup_{s,t\leq\Upsilon\wedge T,|s-t|<\vartheta}\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(s)\wedge L)+1}^{n(\bar{A}_{n,1}(t)\wedge L)}E[(\tilde{\gamma}_{i,1}(t-\tau^n_{i,1}))^2] \quad (4.68)
$$

and thus, by (4.58) and choosing $\tilde{\varsigma}$ arbitrarily small, we have that for any $\varsigma>0$,

$$
\lim_{\vartheta\to0}\limsup_{n\to\infty}\sup_{\kappa_n}P\left(\sup_{t\leq\vartheta}\left|\frac{1}{\sqrt{n}}\sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)}\tilde{\gamma}_{i,1}(\kappa_n-\tau^n_{i,1})\right|>\varsigma\right)=0. \qquad (4.69)
$$

Since for each n and i, $\{\tilde{\gamma}_{i,1}(x) : x \geq 0\}$ is a square integrable martingale with respect to the filtration $\mathbf{G} \equiv \{\mathcal{G}(t) : t \geq 0\}$ where $\mathcal{G}(t) = \sigma\{\mathbf{1}(\eta_{i,1} \leq x) : 0 \leq x \leq t, i = 1, 2, ...\}$, then by Doob's maximal inequality, for any $c > 0$,

$$\begin{aligned} & P\left(\sup_{t\leq\vartheta} |\tilde{\gamma}_{i,1}(\kappa_n + t - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(\kappa_n - \tau^n_{i,1})| > c\right] \\ \leq \quad & c^{-2} E\left[(\tilde{\gamma}_{i,1}(\kappa_n + \vartheta - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(\kappa_n - \tau^n_{i,1}))^2\right] \to 0 \quad \text{as} \quad \vartheta \to 0. \end{aligned} \tag{4.70}$$

Moreover,

$$\frac{1}{n} \sum_{i=1}^{\lfloor n(\bar{A}_{n,1}(t)) \rfloor} E[(\tilde{\gamma}_{i,1}(t+\vartheta - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(t - \tau^n_{i,1}))^2] \Rightarrow \int_0^t E[(\tilde{\gamma}_{i,1}(t+\vartheta - s) - \tilde{\gamma}_{i,1}(t-s))^2] d\bar{a}_1(s) < \infty. \tag{4.71}$$

as $n \to \infty$ and the limit in (4.71) goes to 0 as $\vartheta \to 0$. Thus, it follows that for each $\varsigma > 0$,

$$\lim_{\vartheta\to 0} \limsup_{n\to\infty} \sup_{\kappa_n} P\left(\sup_{t\leq\vartheta} \left| \frac{1}{\sqrt{n}} \sum_{i=n(\bar{A}_{n,1}(\kappa_n)\wedge L)+1}^{n(\bar{A}_{n,1}(\kappa_n+t)\wedge L)} [\tilde{\gamma}_{i,1}(\kappa_n + t - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(\kappa_n - \tau^n_{i,1})] \right| > \varsigma \right) = 0 \tag{4.72}$$

For the third term in (4.64), a similar argument applies by observing that for any $y \in [0, T \wedge (\kappa_n + t)]$,

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=n(\bar{A}_{n,1}(\kappa_n+t-y)\wedge L)}^{n(\bar{A}_{n,1}(\kappa_n)\wedge L)} \left|\tilde{\gamma}_{i,1}(\kappa_n - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(\kappa_n + t - \tau^n_{i,1})\right| \\ \leq \quad & \frac{1}{\sqrt{n}} \sum_{i=0}^{n(\bar{A}_{n,1}(\kappa_n)\wedge L)} \left|\tilde{\gamma}_{i,1}(\kappa_n - \tau^n_{i,1}) - \tilde{\gamma}_{i,1}(\kappa_n + t - \tau^n_{i,1})\right|. \end{aligned} \tag{4.73}$$

The last term in (4.64) follows from the same argument as in the first term in (4.66). Thus, (4.62) is proven, and tightness of the processes $\{\tilde{X}^e_{n,2} : n \geq 1\}$ is proven in the space D_D, which implies the tightness of $\{\hat{X}^e_{n,2} : n \geq 1\}$.

By the tightness of $\{\hat{X}^e_{n,1} : n \geq 1\}$ and $\{\hat{X}^e_{n,2} : n \geq 1\}$, we obtain the tightness of $\{\hat{Q}^e_{n,1} : n \geq 1\}$ in D_D, which implies tightness of $\{\hat{D}^e_{n,1} : n \geq 1\}$, and thus tightness of$\{\bar{D}^e_{n,1} : n \geq 1\}$. By the splitting definition of $A_{n,2}$ in (2.3) and their scaled processes $\bar{A}_{n,2}$ in (3.4) and $\hat{A}_{n,2}$ in (3.19), the processes $\{\hat{A}^e_{n,2} : n \geq 1\}$ are tight, which also implies that $\bar{A}_{n,2} \Rightarrow \bar{a}_2$ in D. Then, the proof of tightness of the processes $\{\hat{Y}^e_{n,1} : n \geq 1\}$ and $\{\hat{Y}^e_{n,2} : n \geq 1\}$ follows from a similar argument as for $\{\hat{X}^e_{n,1} : n \geq 1\}$ and $\{\hat{X}^e_{n,2} : n \geq 1\}$. Thus, the processes $\{\hat{Q}^e_{n,1} : n \geq 1\}$ are tight in D_D and $\{\hat{D}^e_{n,2} : n \geq 1\}$ are tight in D. Finally, the joint tightness of all these processes in the product space follows from! tightness of each sequence of processes in their own space (Theorem 11.6.7, Whitt (2002)). This completes the proof. ■

Lemma 4.2 (*Convergence of finite dimensional distributions*) *Under the assumptions of Theorem 3.2, the finite dimensional distributions of the processes* $(\hat{A}_{n,1}, \hat{X}^e_{n,1}, \hat{X}^e_{n,2}, \hat{D}_{n,1}, \hat{A}_{n,2}, \hat{Y}^e_{n,1}, \hat{Y}^e_{n,2}, \hat{D}_{n,2})$ *converge in distribution to those of the processes* $(\hat{A}_1, \hat{X}^e_1, \hat{X}^e_2, \hat{D}_1, \hat{A}_2, \hat{Y}^e_1, \hat{Y}^e_2, \hat{D}_2)$.

Proof. As in the proof of tightness, we mainly focus on the proof for the convergence of the f.d.d.'s of the processes $\hat{X}^e_{n,2}$ to those of $\hat{X}^e_2$.

First, we write the processes $\hat{X}^e_{n,2}$ and $\hat{Y}^e_{n,2}$ defined in (2.26) and (2.28), respectively, as the limit of mean square integrals, as in (3.26) for $\hat{X}^e_2$,

$$\hat{X}^e_{n,2}(t,y) \equiv \text{l.i.m.}_{k\to\infty}\hat{X}^e_{n,2,k}(t,y), \quad \text{and} \quad \hat{Y}^e_{n,2}(t,y) \equiv \text{l.i.m.}_{k\to\infty}\hat{Y}^e_{n,2,k}(t,y), \tag{4.74}$$

where

$$\begin{aligned} \hat{X}^e_{n,2,k}(t,y) &\equiv \int_{t-y}^{t}\int_0^\infty \mathbf{1}_{k,t,y}(s,x)d\hat{R}_{n,1}(s,x) = \sum_{i=1}^{k}\Delta_{\hat{R}_{n,1}}(s^k_{i-1}, s^k_i, t-s^k_i, t) \\ &= \sum_{i=1}^{k}\Delta_{\hat{K}_{n,1}}(\bar{A}_{n,1}(s^k_{i-1}), \bar{A}_{n,1}(s^k_i), t-s^k_i, t) \end{aligned} \tag{4.75}$$

$$\begin{aligned} \hat{Y}^e_{n,2,k}(t,y) &\equiv \int_{t-y}^{t}\int_0^\infty \mathbf{1}_{k,t,y}(s,x)d\hat{R}_{n,2}(s,x) = \sum_{i=1}^{k}\Delta_{\hat{R}_{n,2}}(s^k_{i-1}, s^k_i, t-s^k_i, t) \\ &= \sum_{i=1}^{k}\Delta_{\hat{K}_{n,2}}(\bar{A}_{n,2}(s^k_{i-1}), \bar{A}_{n,2}(s^k_i), t-s^k_i, t) \end{aligned} \tag{4.76}$$

with $\mathbf{1}_{k,t,y}(s,x)$ defined in (3.27) for $t-y = s^k_0 < s^k_1 < \cdots < s^k_k = t$ and $\max_{1\le i\le k}|s^k_i - s^k_{i-1}| \to 0$ as $k\to\infty$, and

$$\Delta_{\hat{R}_{n,l}}(s^k_{i-1}, s^k_i, t-s^k_i, t) = \hat{R}_{n,l}(s^k_i, t) - \hat{R}_{n,l}(s^k_{i-1}, t) - \hat{R}_{n,l}(s^k_i, t-s^k_i) + \hat{R}_{n,l}(s^k_{i-1}, t-s^k_i). \tag{4.77}$$

Similarly, for $\hat{X}^e_2$ and $\hat{Y}^e_2$, we write them as limits of mean square integrals of $\hat{X}^e_{2,k}$ and $\hat{Y}^e_{2,k}$, respectively,

$$\hat{X}^e_{2,k}(t,y) \equiv \int_{t-y}^{t}\int_0^\infty \mathbf{1}_{k,t,y}(s,x)d\hat{K}_1(\bar{a}_1(t),x) = \sum_{i=1}^{k}\Delta_{\hat{K}_1}(\bar{a}_1(s^k_{i-1}), \bar{a}_1(s^k_i), t-s^k_i, t) \tag{4.78}$$

$$\hat{Y}^e_{2,k}(t,y) \equiv \int_{t-y}^{t}\int_0^\infty \mathbf{1}_{k,t,y}(s,x)d\hat{K}_2(\bar{a}_2(t),x) = \sum_{i=1}^{k}\Delta_{\hat{K}_2}(\bar{a}_2(s^k_{i-1}), \bar{a}_2(s^k_i), t-s^k_i, t) \tag{4.79}$$

We prove the convergence of f.d.d.'s of $\hat{X}^e_{n,2}$ and $\hat{Y}^e_{n,2}$ to those of $\hat{X}^e_2$ and $\hat{Y}^e_2$ jointly by using the convergence of $(\hat{K}_{n,1}, \hat{K}_{n,2}) \Rightarrow (\hat{K}_1, \hat{K}_2)$ in D^2_D in (2.19). Define the processes $\check{X}^e_{n,2,k}$ and $\check{Y}^e_{n,2,k}$ by

$$\check{X}^e_{n,2,k}(t,y) \equiv \sum_{i=1}^{k}\Delta_{\hat{K}_{n,1}}(\bar{a}_1(s^k_{i-1}), \bar{a}_1(s^k_i), t-s^k_i, t) \tag{4.80}$$

$$\check{Y}^e_{n,2,k}(t,y) \equiv \sum_{i=1}^{k} \Delta_{\hat{K}_{n,2}}(\bar{a}_2(s^k_{i-1}), \bar{a}_2(s^k_i), t - s^k_i, t) \tag{4.81}$$

Then, by the convergence of $(\hat{K}_{n,1}, \hat{K}_{n,2}) \Rightarrow (\hat{K}_1, \hat{K}_2)$ in D^2_D and the continuity of $\bar{a}_1$ and $\bar{a}_2$, we can conclude that the joint convergence of f.d.d.'s of $(\hat{A}_{n,1}, \hat{X}^e_{n,1}, \check{X}^e_{n,2,k}, \hat{A}_{n,2}, \hat{Y}^e_{n,1}, \check{Y}^e_{n,2,k})$ converge in distribution to those of the processes $(\hat{A}_1, \hat{X}^e_1, \hat{X}^e_{2,k}, \hat{A}_2, \hat{Y}^e_1, \hat{Y}^e_{2,k})$ as $n \to \infty$.

Now it suffices to show that the difference between $(\check{X}^e_{n,2,k}, \check{Y}^e_{n,2,k})$ and $(\hat{X}^e_{n,2,k}, \hat{Y}^e_{n,2,k})$ is asymptotically negligible in probability as $n \to \infty$ for each k, and the difference between $(\hat{X}^e_{n,2,k}, \hat{Y}^e_{n,2,k})$ and $(\hat{X}^e_{n,2}, \hat{Y}^e_{n,2})$ is asymptotically negligible in probability as $n \to \infty$ and $k \to \infty$. Here we only focus on processes in the first service station since the argument is similar for those in the second service station. We will next show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0 \le t \le T, 0 \le y \le t} |\check{X}^e_{n,2,k}(t,y) - \hat{X}^e_{n,2,k}(t,y)| > \epsilon \right) = 0, \quad T > 0, \tag{4.82}$$

and

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(|\hat{X}^e_{n,2,k}(t,y) - \hat{X}^e_{n,2}(t,y)| > \epsilon) = 0, \quad t \ge 0, \quad 0 \le y \le t. \tag{4.83}$$

We obtain (4.82) from the convergence of $\bar{A}_{n,1} \Rightarrow \bar{a}_1$ in (2.4), the continuity of $\bar{a}_1$, and the convergence $\hat{K}_{n,1} \Rightarrow \hat{K}_1$ in (2.19) and the continuity of the generalized Kiefer limit process $\hat{K}_1$. It remains to show (4.83). For that, we define the processes $\tilde{X}^e_{n,2,k}$ for each k and n by

$$\begin{aligned}
\tilde{X}^e_{n,2,k}(t,y) &\equiv \int_{t-y}^{t} \int_0^\infty \mathbf{1}_{k,t,y}(s,x) d\tilde{R}_{n,1}(s,x) = \sum_{i=1}^{k} \Delta_{\tilde{R}_{n,1}}(s^k_{i-1}, s^k_i, t - s^k_i, t) \\
&= \sum_{i=1}^{k} \Delta_{\tilde{K}_{n,1}}(\bar{A}_{n,1}(s^k_{i-1}), \bar{A}_{n,1}(s^k_i), t - s^k_i, t)
\end{aligned} \tag{4.84}$$

where $\tilde{K}_{n,1}$ and $\tilde{R}_{n,1}$ are defined in (4.42) and (4.43), respectively, and the partition of interval $[t-y, t]$ and $\mathbf{1}_{k,t,y}(s,x)$ are the same as in (4.75)-(4.77). (4.48) and (4.49) imply that the processes $\tilde{X}^e_{n,2,k}$ and $\hat{X}^e_{n,2,k}$ are asymptotically negligible as $n \to \infty$ for each k, and moreover, (4.50) implies that $\tilde{X}^e_{n,2}$ in (4.44) and $\hat{X}^e_{n,2}$ are asymptotically negligible as $n \to \infty$. Thus, it suffices to show the following in order to prove (4.83),

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(|\tilde{X}^e_{n,2,k}(t,y) - \tilde{X}^e_{n,2}(t,y)| > \epsilon) = 0, \quad t \ge 0, \quad 0 \le y \le t, \quad \epsilon > 0. \tag{4.85}$$

By (4.84) and (4.44), we have

$$\begin{aligned}
&\tilde{X}^e_{n,2,k}(t,y) - \tilde{X}^e_{n,2}(t,y) \\
= &\int_{t-y}^{t} \int_0^\infty [\mathbf{1}_{k,t,y}(s,x) - \mathbf{1}(s+x>t)] d\tilde{R}_{n,1}(s,x)
\end{aligned}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=A_{n,1}(t-y)}^{A_{n,1}(t)} \tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y), \tag{4.86}$$

where $\tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y)$ is defined by

$$\tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y) = \sum_{j=1}^{k} \mathbf{1}(s^k_{j-1} < \tau^n_{i,1} \le s^k_j) \check{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1}), \tag{4.87}$$

$$\check{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1}) = \check{\gamma}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) + \hat{\check{\gamma}}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) - \hat{\check{\gamma}}_{i-1,1}(\tau^n_{i-1,1}, \eta_{i-1,1}), \tag{4.88}$$

$$\check{\gamma}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) \equiv \mathbf{1}(t - s^k_j < \eta_{i,1} \le t - \tau^n_{i,1}) - (F_1(t - \tau^n_{i,1}) - F_1(t - s^k_j)), \tag{4.89}$$

and

$$\hat{\check{\gamma}}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) \equiv \sum_{m=1}^{\infty} E[\check{\gamma}_{i+m,1}(\tau^n_{i+m,1}, \eta_{i+m,1}) | \mathcal{F}_{i,1}]. \tag{4.90}$$

It is clear that by construction, for each i, n, t, y, k, the sequence $\{\check{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1}) : i \ge 1\}$ is a martingale difference sequence, and so is the sequence $\{\tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y) : i \ge 1\}$. Moreover, $E[\tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y)] = E[\check{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})] = 0$ and $E[(\tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y))^2] < \infty$, and

$$\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} (\hat{\check{\gamma}}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) - \hat{\check{\gamma}}_{i-1,1}(\tau^n_{i-1,1}, \eta_{i-1,1})) \right\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.91}$$

Then, we have that for any $L > 0$ and $\epsilon > 0$,

$$\begin{aligned}
& P(|\tilde{X}^e_{n,2,k}(t,y) - \tilde{X}^e_{n,2}(t,y)| > \epsilon) \\
\le\ & P(\bar{A}_{n,1}(t) > L) + P\left(\left| \frac{1}{\sqrt{n}} \sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)} \tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y) \right| > \epsilon \right) \\
\le\ & P(\bar{A}_{n,1}(t) > L) + \frac{1}{\epsilon^2} E\left[\left\langle \frac{1}{\sqrt{n}} \sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)} \tilde{\beta}^k_{i,1}(\tau^n_{i,1}, \eta_{i,1})(t,y) \right\rangle \right] \\
\le\ & P(\bar{A}_{n,1}(t) > L) \\
& + \frac{1}{\epsilon^2} E\left[\frac{1}{n} \sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)} \sum_{j=1}^{k} \mathbf{1}(s^k_{j-1} < \tau^n_{i,1} \le s^k_j) E\left[\left(\check{\gamma}_{i,1}(\tau^n_{i,1}, \eta_{i,1})\right)^2 \right] \right] \\
& + \frac{1}{\epsilon^2} E\left[\left(\frac{1}{n} \sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)} \sum_{j=1}^{k} \mathbf{1}(s^k_{j-1} < \tau^n_{i,1} \le s^k_j) (\hat{\check{\gamma}}_{i,1}(\tau^n_{i,1}, \eta_{i,1}) - \hat{\check{\gamma}}_{i-1,1}(\tau^n_{i-1,1}, \eta_{i-1,1})) \right)^2 \right]
\end{aligned} \tag{4.92}$$

By Assumption 1, for each $t \ge 0$, the first term in (4.92),

$$\lim_{L\to\infty} \limsup_{n\to\infty} P(\bar{A}_{n,1}(t) > L) = 0. \tag{4.93}$$

For the second term on the right side of (4.92),

$$\begin{aligned} E\big[\big(\check{\gamma}_{i,1}(\tau^n_{i,1},\eta_{i,1})\big)^2\big] &= (F_1(t-\tau^n_{i,1})-F_1(t-s^k_j))[1-(F_1(t-\tau^n_{i,1})-F_1(t-s^k_j))] \\ &\leq F_1(t-\tau^n_{i,1})-F_1(t-s^k_j), \end{aligned} \tag{4.94}$$

which implies that

$$\begin{aligned} & E\left[\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)}\sum_{j=1}^{k}\mathbf{1}(s^k_{j-1}<\tau^n_{i,1}\leq s^k_j)E\big[\big(\check{\gamma}_{i,1}(\tau^n_{i,1},\eta_{i,1})\big)^2\big]\right] \\ \leq\ & E\left[\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)}\sum_{j=1}^{k}\mathbf{1}(s^k_{j-1}<\tau^n_{i,1}\leq s^k_j)(F_1(t-\tau^n_{i,1})-F_1(t-s^k_j))\right] \\ \leq\ & E\left[\frac{1}{n}\sum_{j=1}^{k}(F_1(t-s^k_{j-1})-F_1(t-s^k_j))(n(\bar{A}_{n,1}(s_j)\wedge L)-n(\bar{A}_{n,1}(s_{j-1})\wedge L))\right] \\ \leq\ & E\left[\max_{1\leq j\leq k}((\bar{A}_{n,1}(s_j)\wedge L)-(\bar{A}_{n,1}(s_{j-1})\wedge L))\right] \end{aligned} \tag{4.95}$$

Thus, by Assumption 1 and the continuity of $\bar{a}_1$, and by (4.93), we have

$$\begin{aligned} & \lim_{k\to\infty}\limsup_{n\to\infty}E\left[\frac{1}{n}\sum_{i=n(\bar{A}_{n,1}(t-y)\wedge L)}^{n(\bar{A}_{n,1}(t)\wedge L)}\sum_{j=1}^{k}\mathbf{1}(s^k_{j-1}<\tau^n_{i,1}\leq s^k_j)E\big[\big(\check{\gamma}_{i,1}(\tau^n_{i,1},\eta_{i,1})\big)^2\big]\right] \\ \leq\ & \lim_{k\to\infty}\limsup_{n\to\infty}E\left[\max_{1\leq j\leq k}((\bar{A}_{n,1}(s_j)\wedge L)-(\bar{A}_{n,1}(s_{j-1})\wedge L))\right]=0. \end{aligned} \tag{4.96}$$

For the last term on the right side of (4.92), we apply (4.91). Thus, (4.85) is proven and so is (4.83). The convergence of f.d.d.'s of $\hat{Y}^e_{n,2}$ to those of $\hat{Y}^e_2$ follows from a similar argument. ∎

References

[1] Berkes, I., Philipp, W.: An almost sure invariance principle for empirical distribution function of mixing random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.* **41**, 115–137. (1977)

[2] Berkes, I., Hörmann, S., Schauer, J.: Asymptotic results for the empirical process of stationary sequences. *Stochastic processes and their applications.* **119**, 1298–1324 (2009)

[3] Bickel, P.J., Wichura, M. J.: Convergence criteria for multiparameter stochastic processes and some applications. *Ann. Math. Statist.* 42, 1656–1670 (1971)

[4] Billingsley, P.: *Convergence of Probability Measures.* second ed, Wiley, New York (1999)

[5] Boxma, O.J.: $M/G/\infty$ tandem queues. *Stochastic Processes and their Applications.* **18**,153–164 (1984)

[6] Brown, L., Gans, N., Mandelbaum, A., Sakov, A., Shen, H., Zeltyn, S., Zhao, L.: Statistical analysis of a telephone call center: a queueing-science perspective. *Journal of the American Statistical Association.* Vol. 100, No, 469, 36–50 (2005)

[7] de Véricourt, F., Zhou, Y-P: Managing response time in a call-routing problem with service failure. *Operations Research.* Vol. 53, No.6, 968–981 (2005)

[8] Green, L. V., Kolesar, P. J., Whitt, W.: Coping with time-varying demand when setting staffing requirements for a service system. *Production and Operations Management.* **16** 13–39 (2007)

[9] Hall, P., Heyde, C.C.: *Martingale Limit Theory and its Applications.* Academic Press, Inc. (1980)

[10] Jacobs, P. A. : A cyclic queueing network with dependent exponential service times. *J. Appl. Prob.* **15**, 573–589 (1978)

[11] Jacobs, P. A. : Heavy traffic results for single-server queues with dependent (EARMA) service and interarrival times. *Adv. Appl. Prob.* **12**, 517–529 (1980)

[12] Jiang, L., Giachetti, R.E.; A queueing network model to analyze the impact of parallelization of care on patient cycle time. *Health Care Management Science.* **11** , 248–261 (2008)

[13] Karatzas, I., Shreve, S.E.: *Brownian Motion and Stochastic Calculus.* second ed., Springer, Berlin (1991)

[14] Khoshnevisan D. *Multiparameter Processes: An Introduction to Random Fields.* Springer. (2002)

[15] Khudyakov, P., Feigin, P.D., Mandelbaum, A.: Designing a call center with an IVR (Interactive Voice Response). *Queueing Systems.* **66**, 215–237 (2010)

[16] Krichagina, E. V., Puhalskii, A. A.: A heavy-traffic analysis of a closed queueing system with a GI/∞ service center. *Queueing Systems* 25, 235–280 (1997)

[17] Kurtz, T.G., Protter, P.: Weak limit theorems for stochastic integrals and stochastic differential equations. *The Annals of Probability.* Vol.19, No.3, 1035–1070 (1991)

[18] Liu, L., Templeton, J.G.C.: Autocorrelations in infinite server batch arrival queues. *Queueing Systems.* **14**, 313–337 (1993)

[19] Mandelbaum, A., Massey, W. A., Reiman, M. I.: Strong approximations for Markovian service networks. Queueing Systems. **30**, 149-201 (1998)

[20] Massey, W.A., Whitt, W.: Networks of infinite-server queues with nonstationary Poisson input. *Queueing Systems.* **13**, 183–250 (1993)

[21] Mechata, K.M., Deivamoney Selvam, D.: Covariance structure of infinite-server queues in tandem. *Opsearch.* **21** 172–178 (1984)

[22] Nelson, B.L., Taaffe, M.R.: The $[Ph_t/Ph_t/\infty]^K$ queueing systems: part II–the multiclass network. *INFORMS Journal on Computing.* **16**, 275–283 (2004)

[23] Pang, G., Whitt, W.: Service Interruptions in Large-Scale Service Systems. *Management Science*, **55(9)**, 1499–1512 (2009)

[24] Pang, G., Whitt, W.: Two-parameter heavy-traffic limits for infinite-server queues. *Queueing Systems.* **65** 325–364 (2010)

[25] Pang, G., Whitt, W.: The impact of dependent service times on large-scale service systems. working paper. (2011)

[26] Rootzén, H.: On the functional central limit theorem for martingales, II. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.* 51, 79–93 (1980)

[27] Schmidt, V.: On joint queue-length characteristics in infinite-server tandem queues with heavy traffic. *Adv. Appl. Prob.* **19**, 474–486 (1987)

[28] Straf, M.L. Weak convergence of stochastic processes with several parameters. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* 2, 187–221 (1971)

[29] van Der Vaart, A. W., Wellner, J.: *Weak Convergence and Empirical Processes*, Springer (1996)

[30] Whitt, W.: Comparing batch delays and customer delays. *The Bell System Technical Journal,* Vol. 62, No. 7, 2001–2009 (1983)

[31] Whitt, W.: *Stochastic-Process Limits.* Springer, New York (2002)

[32] Zajic, T.: Rough asymptotics for tandem non-homogeneous $M/G/\infty$ queues via Poissonized empirical processes. *Queueing Systems.* **29**, 161–174 (1998)

5 Appendix: Calculation of $\sigma^2_{Q_2,e}(t,y)$ and $\sigma^2_{D_2}(t)$

By (3.47), we have

$$\begin{aligned} C_{D_1}(t_1,t_2) &= \lambda_1 c_{a,1}^2 \int_0^{t_1\wedge t_2} F_1(t_1-s)F_1(t_2-s)ds \\ &+\lambda_1 \int_0^{t_1\wedge t_2} \Big(F_1^c(t_1\wedge t_2 - s) - F_1^c(t_1-s)F_1^c(t_2-s) + \Gamma_{K,1}^c(t_1-s,t_2-s)\Big)ds, \end{aligned} \tag{5.1}$$

$$C_{A_2}(t_1,t_2) = p^2 C_{D_1}(t_1,t_2) + p(1-p)\lambda_1 \int_0^{t_1\wedge t_2} F_1(s)ds \tag{5.2}$$

$$\begin{aligned} & \sigma^2_{Q_2,e}(t,y) \\ = \;& \sigma^2_{A_2}(t) + (F_2^c(y))^2\sigma^2_{A_2}(t-y) + \int_{t-y}^t\int_{t-y}^t C_{A_2}(u,v)dF_2^c(t-u)dF_2^c(t-v) \\ & -2F_2^c(y)C_{A_2}(t,t-y) - 2\int_{t-y}^t C_{A_2}(t,s)dF_2^c(t-s) \\ & +2F_2^c(y)\int_{t-y}^t C_{A_2}(t-y,s)dF_2^c(t-s) \\ & +p\lambda_1\int_{t-y}^t \Big(F_2^c(t-s) - (F_2^c(t-s))^2 + \Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds \\ = \;& p(1-p)\lambda_1\int_0^t F_1(s)ds + p^2\lambda_1 c_{a,1}^2\int_0^t F_1(s)^2ds + p^2\lambda_1\int_0^t\Big(F_1^c(s) - (F_1^c(s))^2 + \Gamma_{K,1}^c(s,s)\Big)ds \\ & +(F_2^c(y))^2p(1-p)\lambda_1\int_0^{t-y}F_1(s)ds + (F_2^c(y))^2p^2\lambda_1c_{a,1}^2\int_0^{t-y}F_1(s)^2ds \\ & +(F_2^c(y))^2p^2\lambda_1\int_0^{t-y}\Big(F_1^c(s) - (F_1^c(s))^2 + \Gamma_{K,1}^c(s,s)\Big)ds \\ & +\int_{t-y}^t\int_{t-y}^t\Big[p^2\Big(\lambda_1c_{a,1}^2\int_0^{u\wedge v}F_1(u-s)F_1(v-s)ds \\ & +\lambda_1\int_0^{u\wedge v}\Big(F_1^c(u\wedge v-s) - F_1^c(u-s)F_1^c(v-s) + \Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big) \\ & +p(1-p)\lambda_1\int_0^{u\wedge v}F_1(s)ds\Big]dF_2^c(t-u)dF_2^c(t-v) \\ & -2F_2^c(y)\Big[p^2\Big(\lambda_1c_{a,1}^2\int_0^{t-y}F_1(t-s)F_1(t-y-s)ds \\ & +\lambda_1\int_0^{t-y}\Big(F_1^c(t-y-s) - F_1^c(t-s)F_1^c(t-y-s) + \Gamma_{K,1}^c(t-s,t-y-s)\Big)ds\Big) \\ & +p(1-p)\lambda_1\int_0^{t-y}F_1(s)ds\Big] \\ & -2\int_{t-y}^t\Big[p^2\Big(\lambda_1c_{a,1}^2\int_0^s F_1(t-u)F_1(s-u)du \\ & +\lambda_1\int_0^s\Big(F_1^c(s-u) - F_1^c(t-u)F_1^c(s-u) + \Gamma_{K,1}^c(t-u,s-u)\Big)du\Big) \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
& +p(1-p)\lambda_1 \int_0^s F_1(u)du\Big] dF_2^c(t-s) \\
& +2F_2^c(y)\int_{t-y}^t \Big[p^2\Big(\lambda_1 c_{a,1}^2 \int_0^{t-y} F_1(t-y-u)F_1(s-u)du \\
& +\lambda_1 \int_0^{t-y}\Big(F_1^c(t-y-u) - F_1^c(t-y-u)F_1^c(s-u) + \Gamma_{K,1}^c(t-y-u,s-u)\Big)du\Big) \\
& +p(1-p)\lambda_1 \int_0^{t-y} F_1(u)du\Big] dF_2^c(t-s) \\
& +p\lambda_1 \int_{t-y}^t \Big(F_2^c(t-s) - (F_2^c(t-s))^2 + \Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds \\
= \; & p(1-p)\lambda_1 \int_0^t F_1(s)ds + p^2\lambda_1 \int_0^t \Big(F_1(s) + (c_{a,1}^2-1)(F_1(s))^2 + \Gamma_{K,1}^c(s,s)\Big)ds \\
& +(F_2^c(y))^2 p(1-p)\lambda_1 \int_0^{t-y} F_1(s)ds + (F_2^c(y))^2 p^2\lambda_1 \int_0^{t-y}\Big(F_1(s) + (c_{a,1}^2-1)(F_1(s))^2 + \Gamma_{K,1}^c(s,s)\Big)ds \\
& +\int_{t-y}^t\int_{t-y}^t \Big[p^2\Big(\lambda_1 c_{a,1}^2 \int_0^{u\wedge v} F_1(u-s)F_1(v-s)ds \\
& +\lambda_1 \int_0^{u\wedge v}\Big(F_1^c(u\wedge v-s) - F_1^c(u-s)F_1^c(v-s) + \Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big) \\
& +p(1-p)\lambda_1 \int_0^{u\wedge v} F_1(s)ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& -2F_2^c(y)\Big[\lambda_1 p^2 \int_0^{t-y}\Big(F_1(t-s) + (c_{a,1}^2-1)F_1(t-s)F_1(t-y-s) + \Gamma_{K,1}^c(t-s,t-y-s)\Big)ds\Big) \\
& +p(1-p)\lambda_1 \int_0^{t-y} F_1(s)ds\Big] \\
& -2\int_{t-y}^t \Big[\lambda_1 p^2 \int_0^s \Big(F_1(t-u) + (c_{a,1}^2-1)F_1(t-u)F_1(s-u) + \Gamma_{K,1}^c(t-u,s-u)\Big)du\Big) \\
& +p(1-p)\lambda_1 \int_0^s F_1(u)du\Big] dF_2^c(t-s) \\
& +2F_2^c(y)\int_{t-y}^t \Big[p^2\lambda_1 \int_0^{t-y}\Big(F_1(s-u) + (c_{a,1}^2-1)F_1(t-y-u)F_1(s-u) + \Gamma_{K,1}^c(t-y-u,s-u)\Big)du\Big) \\
& +p(1-p)\lambda_1 \int_0^{t-y} F_1(u)du\Big] dF_2^c(t-s) \\
& +p\lambda_1 \int_{t-y}^t \Big(F_2^c(t-s) - (F_2^c(t-s))^2 + \Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds
\end{aligned}
$$

Terms with the coefficient $p(1-p)\lambda_1$:

$$
\begin{aligned}
& \int_0^t F_1(s)ds + (F_2^c(y))^2 \int_0^{t-y} F_1(s)ds + \int_{t-y}^t\int_{t-y}^t \Big[\int_0^{u\wedge v} F_1(s)ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& -2F_2^c(y)\int_0^{t-y} F_1(s)ds - 2\int_{t-y}^t \Big[\int_0^s F_1(u)du\Big] dF_2^c(t-s) + 2F_2^c(y)\int_{t-y}^t \Big[\int_0^{t-y} F_1(u)du\Big] dF_2^c(t-s) \\
= \; & \int_0^t F_1(s)ds + (F_2^c(y))^2 \int_0^{t-y} F_1(s)ds - 2F_2^c(y)\int_0^{t-y} F_1(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t-y}^{t}\Big(\int_{t-y}^{v}\Big[\int_0^u F_1(s)ds\Big]dF_2^c(t-u)\Big)dF_2^c(t-v)+\int_{t-y}^{t}\Big(\int_{v}^{t}\Big[\int_0^v F_1(s)ds\Big]dF_2^c(t-u)\Big)dF_2^c(t-v)\\
&-2\Big(F_2^c(t-s)\Big(\int_0^s F_1(u)du\Big)\Big)\Big|_{s=t-y}^{t}+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds+2F_2^c(y)\Big(\int_0^{t-y}F_1(u)du\Big)(1-F_2^c(y))\\
= \;&-\int_0^t F_1(s)ds-(F_2^c(y))^2\int_0^{t-y}F_1(s)ds+2F_2^c(y)\int_0^{t-y}F_1(s)ds+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds\\
&+\int_{t-y}^{t}\Big(\Big(F_2^c(t-u)\int_0^u F_1(s)ds\Big)\Big|_{u=t-y}^{v}-\int_{t-y}^{v}F_2^c(t-u)F_1(u)du\Big)dF_2^c(t-v)\\
&+\int_{t-y}^{t}\Big[\int_0^v F_1(s)ds\Big](1-F_2^c(t-v))dF_2^c(t-v)\\
= \;&-\int_0^t F_1(s)ds-(F_2^c(y))^2\int_0^{t-y}F_1(s)ds+2F_2^c(y)\int_0^{t-y}F_1(s)ds+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds\\
&+\int_{t-y}^{t}\Big(F_2^c(t-v)\int_0^v F_1(s)ds-F_2^c(y)\int_0^{t-y}F_1(s)ds-\int_{t-y}^{v}F_2^c(t-u)F_1(u)du\Big)dF_2^c(t-v)\\
&+\int_{t-y}^{t}\Big[\int_0^v F_1(s)ds\Big](1-F_2^c(t-v))dF_2^c(t-v)\\
= \;&-\int_0^t F_1(s)ds-(F_2^c(y))^2\int_0^{t-y}F_1(s)ds+2F_2^c(y)\int_0^{t-y}F_1(s)ds+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds\\
&-F_2^c(y)(1-F_2^c(y))\int_0^{t-y}F_1(s)ds-\int_{t-y}^{t}\Big(\int_{t-y}^{v}F_2^c(t-u)F_1(u)du\Big)dF_2^c(t-v)\\
&+\int_{t-y}^{t}\Big[\int_0^v F_1(s)ds\Big]dF_2^c(t-v)\\
= \;&-\int_0^t F_1(s)ds+F_2^c(y)\int_0^{t-y}F_1(s)ds+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds\\
&-\Big(\Big(\int_{t-y}^{v}F_2^c(t-u)F_1(u)du\Big)F_2^c(t-v)\Big)\Big|_{v=t-y}^{t}+\int_{t-y}^{t}F_2^c(t-v)F_2^c(t-v)F_1(v)dv\\
&+\Big(\Big(\int_0^v F_1(s)ds\Big)F_2^c(t-v)\Big)\Big|_{v=t-y}^{t}-\int_{t-y}^{t}F_2^c(t-v)F_1(v)dv\\
= \;&-\int_0^t F_1(s)ds+F_2^c(y)\int_0^{t-y}F_1(s)ds+2\int_{t-y}^{t}F_2^c(t-s)F_1(s)ds\\
&-\int_{t-y}^{t}F_2^c(t-u)F_1(u)du+\int_{t-y}^{t}(F_2^c(t-v))^2F_1(v)dv\\
&+\int_0^t F_1(s)ds-F_2^c(y)\int_0^{t-y}F_1(s)ds-\int_{t-y}^{t}F_2^c(t-v)F_1(v)dv\\
= \;&\int_{t-y}^{t}(F_2^c(t-v))^2F_1(v)dv \qquad (5.4)
\end{aligned}
$$

Terms with the coefficient $p^2\lambda_1$ except the double integrals and without the covariance function Γ^c:

$$
\int_0^t[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds+(F_2^c(y))^2\int_0^{t-y}[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds
$$

$$
\begin{aligned}
& -2F_2^c(y)\int_0^{t-y}[F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s)]ds \\
& -2\int_{t-y}^{t}\Big(\int_0^s[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(s-u)]du\Big)dF_2^c(t-s) \\
& +2F_2^c(y)\int_{t-y}^{t}\Big(\int_0^{t-y}[F_1(s-u)+(c_{a,1}^2-1)F_1(s-u)F_1(t-y-u)]du\Big)dF_2^c(t-s) \\
= \;& \int_0^t[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds+(F_2^c(y))^2\int_0^{t-y}[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds \\
& -2F_2^c(y)\int_0^{t-y}[F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s)]ds \\
& -2\Big(\Big(\int_0^s[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(s-u)]du\Big)F_2^c(t-s)\Big)\Big|_{s=t-y}^{t} \\
& +2\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^s[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(s-u)]du\Big) \\
& +2F_2^c(y)\Big(\Big(\int_0^{t-y}[F_1(s-u)+(c_{a,1}^2-1)F_1(s-u)F_1(t-y-u)]du\Big)F_2^c(t-s)\Big)\Big|_{s=t-y}^{t} \\
& -2F_2^c(y)\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^{t-y}[F_1(s-u)+(c_{a,1}^2-1)F_1(s-u)F_1(t-y-u)]du\Big) \\
= \;& \int_0^t[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds+(F_2^c(y))^2\int_0^{t-y}[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds \\
& -2F_2^c(y)\int_0^{t-y}[F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s)]ds \\
& -2\int_0^t[F_1(t-u)+(c_{a,1}^2-1)(F_1(t-u))^2]du \\
& +2F_2^c(y)\int_0^{t-y}[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(t-y-u)]du \\
& +2\int_{t-y}^{t}F_2^c(t-s)\Big([F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(0)]ds\Big) \\
& +2(c_{a,1}^2-1)\int_{t-y}^{t}F_2^c(t-s)\Big(\int_0^s F_1(t-u)f_1(s-u)du\Big)ds \\
& +2F_2^c(y)\int_0^{t-y}[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(t-y-u)]du \\
& -2(F_2^c(y))^2\int_0^{t-y}[F_1(t-y-u)+(c_{a,1}^2-1)F_1(t-y-u)F_1(t-y-u)]du \\
& -2F_2^c(y)\int_{t-y}^{t}F_2^c(t-s)\Big(\int_0^{t-y}[f_1(s-u)+(c_{a,1}^2-1)f_1(s-u)F_1(t-y-u)]du\Big)ds \\
= \;& -\int_0^t[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds-(F_2^c(y))^2\int_0^{t-y}[F_1(s)+(c_{a,1}^2-1)F_1(s)^2]ds \\
& +2(c_{a,1}^2-1)\int_{t-y}^{t}F_2^c(t-s)\Big(\int_0^s F_1(t-u)f_1(s-u)du\Big)ds \\
& -2F_2^c(y)\int_{t-y}^{t}F_2^c(t-s)\Big(\int_0^{t-y}[f_1(s-u)+(c_{a,1}^2-1)f_1(s-u)F_1(t-y-u)]du\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+2F_2^c(y)\int_0^{t-y}[F_1(t-u)+(c_{a,1}^2-1)F_1(t-u)F_1(t-y-u)]du \\
&+2\int_{t-y}^{t} F_2^c(t-s)F_1(t-s)ds
\end{aligned}
$$

The double integral term with the coefficient $p^2\lambda_1$ and without the covariance function Γ^c:

$$
\begin{aligned}
& \int_{t-y}^{t}\int_{t-y}^{t}\Big(c_{a,1}^2\int_0^{(u\wedge v)} F_1(u-s)F_1(v-s)ds \\
& \quad +\int_0^{(u\wedge v)}(F_1^c((u\wedge v)-s)-F_1^c(u-s)F_1^c(v-s))ds\Big)dF_2^c(t-u)dF_2^c(t-v) \\
= \; & \int_{t-y}^{t}\int_{t-y}^{v}\Big(c_{a,1}^2\int_0^{u} F_1(u-s)F_1(v-s)ds \\
& \quad +\int_0^{u}(F_1^c(u-s)-F_1^c(u-s)F_1^c(v-s))ds\Big)dF_2^c(t-u)dF_2^c(t-v) \\
& +\int_{t-y}^{t}\int_{v}^{t}\Big(c_{a,1}^2\int_0^{v} F_1(u-s)F_1(v-s)ds \\
& \quad +\int_0^{v}(F_1^c(v-s)-F_1^c(u-s)F_1^c(v-s))ds\Big)dF_2^c(t-u)dF_2^c(t-v) \\
= \; & \int_{t-y}^{t}\Big[\Big(\Big(\int_0^{u}(F_1(v-s)+(c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big)F_2^c(t-u)\Big)\Big|_{u=t-y}^{v} \\
& -\int_{t-y}^{v}F_2^c(t-u)d_u\Big(\int_0^{u}(F_1(v-s)+(c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big)\Big]dF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big[\Big(\Big(\int_0^{v}(F_1(u-s)+(c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big)F_2^c(t-u)\Big)\Big|_{u=v}^{t} \\
& -\int_{v}^{t}F_2^c(t-u)d_u\Big(\int_0^{v}(F_1(u-s)+(c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big)\Big]dF_2^c(t-v) \\
= \; & \int_{t-y}^{t}\Big[\Big(F_2^c(t-v)\Big(\int_0^{v}(F_1(s)+(c_{a,1}^2-1)(F_1(s))^2)ds\Big) \\
& -F_2^c(y)\Big(\int_0^{t-y}(F_1(v-s)+(c_{a,1}^2-1)F_1(v-s)F_1(t-y-s))ds\Big) \\
& -\int_{t-y}^{v}F_2^c(t-u)F_1(v-u)du \\
& -\int_{t-y}^{v}\Big(F_2^c(t-u)\Big((c_{a,1}^2-1)\int_0^{u}f_1(u-s)F_1(v-s)ds\Big)\Big)du\Big]dF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big[\int_0^{v}(F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds \\
& -F_2^c(t-v)\int_0^{v}(F_1(v-s)+(c_{a,1}^2-1)F_1(v-s)^2)ds \\
& -\int_{v}^{t}\Big(F_2^c(t-u)\Big(\int_0^{v}(f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big)\Big)du\Big]dF_2^c(t-v) \\
= \; & -F_2^c(y)\int_{t-y}^{t}\Big(\int_0^{t-y}(F_1(v-s)+(c_{a,1}^2-1)F_1(v-s)F_1(t-y-s))ds\Big)dF_2^c(t-v)
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{t-y}^{t}\int_{t-y}^{v} F_2^c(t-u)F_1(v-u)dudF_2^c(t-v) \\
& -(c_{a,1}^2-1)\int_{t-y}^{t}\int_{t-y}^{v}\Big(F_2^c(t-u)\int_0^u f_1(u-s)F_1(v-s)ds\Big)dudF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big[\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big]dF_2^c(t-v) \\
& -\int_{t-y}^{t}\Big[\int_v^t \Big(F_2^c(t-u)\Big(\int_0^v (f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big)\Big)du\Big]dF_2^c(t-v) \\
= \; & -F_2^c(y)\Big[\Big(\int_0^{t-y}(F_1(v-s)+(c_{a,1}^2-1)F_1(v-s)F_1(t-y-s))ds\Big)F_2^c(t-v)\Big]\Big|_{v=t-y}^{t} \\
& +F_2^c(y)\int_{t-y}^{t}F_2^c(t-v)\Big(\int_0^{t-y}(f_1(v-s)+(c_{a,1}^2-1)f_1(v-s)F_1(t-y-s))ds\Big)dv \\
& -\Big[\Big(\int_{t-y}^{v}F_2^c(t-u)F_1(v-u)du\Big)F_2^c(t-v)\Big]\Big|_{v=t-y}^{t} \\
& +\int_{t-y}^{t}F_2^c(t-v)d_v\Big(\int_{t-y}^{v}F_2^c(t-u)F_1(v-u)du\Big) \\
& +\Big[\Big(\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big)F_2^c(t-v)\Big]\Big|_{v=t-y}^{t} \\
& -\int_{t-y}^{t}F_2^c(t-v)d_v\Big[\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big] \\
& -\Big[\Big((c_{a,1}^2-1)\int_{t-y}^{v}\Big(F_2^c(t-u)\Big(\int_0^u f_1(u-s)F_1(v-s)ds\Big)\Big)du\Big)F_2^c(t-v)\Big]\Big|_{v=t-y}^{t} \\
& +(c_{a,1}^2-1)\int_{t-y}^{t}F_2^c(t-v)d_v\Big[\int_{t-y}^{v}\Big(F_2^c(t-u)\int_0^u f_1(u-s)F_1(v-s)ds\Big)du\Big] \\
& -\Big(\Big[\int_v^t\Big(F_2^c(t-u)\Big(\int_0^v [f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s)]ds\Big)\Big)du\Big]F_2^c(t-v)\Big)\Big|_{v=t-y}^{t} \\
& +\int_{t-y}^{t}F_2^c(t-v)d_v\Big[\int_v^t\Big(F_2^c(t-u)\Big(\int_0^v [f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s)]ds\Big)\Big)du\Big] \\
= \; & -F_2^c(y)\int_0^{t-y}(F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s))ds \\
& +(F_2^c(y))^2\int_0^{t-y}(F_1(t-y-s)+(c_{a,1}^2-1)F_1(t-y-s)^2)ds \\
& +F_2^c(y)\int_{t-y}^{t}F_2^c(t-v)\Big(\int_0^{t-y}(f_1(v-s)+(c_{a,1}^2-1)f_1(v-s)F_1(t-y-s))ds\Big)dv \\
& -\int_{t-y}^{t}F_2^c(t-u)F_1(t-u)du \\
& +\int_{t-y}^{t}F_2^c(t-v)\Big(\int_{t-y}^{v}F_2^c(t-u)f_1(v-u)du\Big)dv \\
& +\int_0^t (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)^2)ds \\
& -F_2^c(y)\int_0^{t-y}(F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s))ds
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{t-y}^{t} F_2^c(t-v)F_1(t-v)dv \\
& -(c_{a,1}^2-1)\int_{t-y}^{t} F_2^c(t-v)\Big(\int_0^v F_1(t-s)f_1(v-s)ds\Big)dv \\
& -(c_{a,1}^2-1)\int_{t-y}^{t} F_2^c(t-u)\Big(\int_0^u f_1(u-s)F_1(t-s)ds\Big)du \\
& +(c_{a,1}^2-1)\int_{t-y}^{t} (F_2^c(t-v))^2\Big(\int_0^v f_1(v-s)F_1(v-s)ds\Big)dv \\
& +(c_{a,1}^2-1)\int_{t-y}^{t} F_2^c(t-v)\Big[\int_{t-y}^{v}\Big(F_2^c(t-u)\int_0^u f_1(u-s)f_1(v-s)ds\Big)du\Big]dv \\
& +F_2^c(y)\int_{t-y}^{t}\Big(F_2^c(t-u)\Big(\int_0^{t-y}(f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(t-y-s))ds\Big)\Big)du \\
& -\int_{t-y}^{t}(F_2^c(t-v))^2\Big(\int_0^v(f_1(v-s)+(c_{a,1}^2-1)f_1(v-s)F_1(v-s))ds\Big)dv \\
& +\int_{t-y}^{t}F_2^c(t-v)\Big[\int_v^t\Big(F_2^c(t-u)\Big(f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv \\
= \; & -2F_2^c(y)\int_0^{t-y}(F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-y-s))ds \\
& +(F_2^c(y))^2\int_0^{t-y}(F_1(t-y-s)+(c_{a,1}^2-1)F_1(t-y-s)^2)ds \\
& +2F_2^c(y)\int_{t-y}^{t}F_2^c(t-v)\Big(\int_0^{t-y}(f_1(v-s)+(c_{a,1}^2-1)f_1(v-s)F_1(t-y-s))ds\Big)dv \\
& +\int_0^t(F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)^2)ds \\
& -2\int_{t-y}^{t}F_2^c(t-v)F_1(t-v)dv-\int_{t-y}^{t}(F_2^c(t-v))^2F_1(v)dv \\
& -2(c_{a,1}^2-1)\int_{t-y}^{t}\Big(F_2^c(t-u)\int_0^u f_1(u-s)F_1(t-s)ds\Big)du \\
& +\int_{t-y}^{t}F_2^c(t-v)\Big[\int_{t-y}^{v}\Big(F_2^c(t-u)\Big(f_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv \\
& +\int_{t-y}^{t}F_2^c(t-v)\Big[\int_v^t\Big(F_2^c(t-u)\Big(f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv
\end{aligned}
$$

The terms with the coefficient $p^2\lambda_1$ and with the covariance function Γ^c:

$$
\begin{aligned}
& \int_0^t \Gamma_{K,1}^c(s,s)ds+(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(s,s)ds \\
& +\int_{t-y}^{t}\int_{t-y}^{t}\Big(\int_0^{u\wedge v}\Gamma_{K,1}^c(u-s,v-s)ds\Big)dF_2^c(t-u)dF_2^c(t-v) \\
& -2F_2^c(y)\int_0^{t-y}\Gamma_{K,1}^c(t-s,t-y-s)ds-2\int_{t-y}^{t}\Big(\int_0^s\Gamma_{K,1}^c(t-u,s-u)du\Big)dF_2^c(t-s) \\
& +2F_2^c(y)\int_{t-y}^{t}\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,s-u)du\Big)dF_2^c(t-s)
\end{aligned}
$$

$$
\begin{aligned}
= \;& \int_0^t \Gamma^c_{K,1}(s,s)ds + (F^c_2(y))^2 \int_0^{t-y} \Gamma^c_{K,1}(s,s)ds - 2F^c_2(y) \int_0^{t-y} \Gamma^c_{K,1}(t-s,t-y-s)ds \\
& + \int_{t-y}^t \Big(\int_{t-y}^v \Big[\int_0^u \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& + \int_{t-y}^t \Big(\int_v^t \Big[\int_0^v \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& -2\Big(F^c_2(t-s) \Big(\int_0^s \Gamma^c_{K,1}(t-u,s-u)du \Big) \Big)\Big|_{s=t-y}^t + 2\int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^s \Gamma^c_{K,1}(t-u,s-u)du \Big) \\
& +2F^c_2(y) \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,s-u)du \Big) F^c_2(t-s) \Big|_{s=t-y}^t \\
& -2F^c_2(y) \int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,s-u)du \Big) \\
= \;& \int_0^t \Gamma^c_{K,1}(s,s)ds + (F^c_2(y))^2 \int_0^{t-y} \Gamma^c_{K,1}(s,s)ds - 2F^c_2(y) \int_0^{t-y} \Gamma^c_{K,1}(t-s,t-y-s)ds \\
& + \int_{t-y}^t \Big(\int_{t-y}^v \Big[\int_0^u \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& + \int_{t-y}^t \Big(\int_v^t \Big[\int_0^v \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& -2\int_0^t \Gamma^c_{K,1}(t-u,t-u)du + 2F^c_2(y) \int_0^{t-y} \Gamma^c_{K,1}(t-u,t-y-u)du \\
& +2\int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^s \Gamma^c_{K,1}(t-u,s-u)du \Big) \\
& +2F^c_2(y) \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,t-u)du \Big) - 2(F^c_2(y))^2 \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,t-y-u)du \Big) \\
& -2F^c_2(y) \int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,s-u)du \Big) \\
= \;& -\int_0^t \Gamma^c_{K,1}(s,s)ds - (F^c_2(y))^2 \int_0^{t-y} \Gamma^c_{K,1}(s,s)ds + 2F^c_2(y) \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,t-u)du \Big) \\
& + \int_{t-y}^t \Big(\int_{t-y}^v \Big[\int_0^u \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& + \int_{t-y}^t \Big(\int_v^t \Big[\int_0^v \Gamma^c_{K,1}(u-s,v-s)ds \Big] dF^c_2(t-u) \Big) dF^c_2(t-v) \\
& +2\int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^s \Gamma^c_{K,1}(t-u,s-u)du \Big) \\
& -2F^c_2(y) \int_{t-y}^t F^c_2(t-s) d_s \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,s-u)du \Big) \\
= \;& -\int_0^t \Gamma^c_{K,1}(s,s)ds - (F^c_2(y))^2 \int_0^{t-y} \Gamma^c_{K,1}(s,s)ds + 2F^c_2(y) \Big(\int_0^{t-y} \Gamma^c_{K,1}(t-y-u,t-u)du \Big) \\
& + \int_{t-y}^t \Big(\Big[\int_0^u \Gamma^c_{K,1}(u-s,v-s)ds \Big] F^c_2(t-u) \Big|_{u=t-y}^v
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{t-y}^{v} F_2^c(t-u)d_u\Big[\int_0^u \Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big(\Big[\int_0^v \Gamma_{K,1}^c(u-s,v-s)ds\Big]F_2^c(t-u)\Big|_{u=v}^{t} \\
& -\int_{v}^{t} F_2^c(t-u)d_u\Big[\int_0^v \Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
& +2\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^s \Gamma_{K,1}^c(t-u,s-u)du\Big) \\
& -2F_2^c(y)\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^{t-y} \Gamma_{K,1}^c(t-y-u,s-u)du\Big) \\
= & -\int_0^t \Gamma_{K,1}^c(s,s)ds-(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(s,s)ds+2F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,t-u)du\Big) \\
& +\int_{t-y}^{t}\Big(\Big[\int_0^v \Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)-F_2^c(y)\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,v-s)ds \\
& -\int_{t-y}^{v} F_2^c(t-u)d_u\Big[\int_0^u \Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big(\Big[\int_0^v \Gamma_{K,1}^c(t-s,v-s)ds\Big]-\Big[\int_0^v \Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v) \\
& -\int_{v}^{t} F_2^c(t-u)d_u\Big[\int_0^v \Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
& +2\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^s \Gamma_{K,1}^c(t-u,s-u)du\Big) \\
& -2F_2^c(y)\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^{t-y} \Gamma_{K,1}^c(t-y-u,s-u)du\Big) \\
= & -\int_0^t \Gamma_{K,1}^c(s,s)ds-(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(s,s)ds+2F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,t-u)du\Big) \\
& +2\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^s \Gamma_{K,1}^c(t-u,s-u)du\Big) \\
& -2F_2^c(y)\int_{t-y}^{t} F_2^c(t-s)d_s\Big(\int_0^{t-y} \Gamma_{K,1}^c(t-y-u,s-u)du\Big) \\
& +\int_{t-y}^{t}\Big(\Big[\int_0^v \Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)dF_2^c(t-v) \\
& -F_2^c(y)\int_{t-y}^{t}\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,v-s)ds\Big)dF_2^c(t-v) \\
& -\int_{t-y}^{t}\Big(\int_{t-y}^{v} F_2^c(t-u)d_u\Big[\int_0^u \Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
& +\int_{t-y}^{t}\Big(\int_0^v \Gamma_{K,1}^c(t-s,v-s)ds\Big)dF_2^c(t-v) \\
& -\int_{t-y}^{t}\Big(\Big[\int_0^v \Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)dF_2^c(t-v)
\end{aligned}
$$

$$
\begin{aligned}
& -\int_{t-y}^{t}\Big(\int_{v}^{t}F_2^c(t-u)d_u\Big[\int_0^v\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)dF_2^c(t-v) \\
= \;& -\int_0^t\Gamma_{K,1}^c(s,s)ds-(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(s,s)ds+2F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,t-u)du\Big) \\
& +2\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^s\Gamma_{K,1}^c(t-u,s-u)du\Big) \\
& -2F_2^c(y)\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,s-u)du\Big) \\
& +\Big(\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& -\int_{t-y}^{t}F_2^c(t-v)d_v\Big(\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big) \\
& -F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,v-s)ds\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& +F_2^c(y)\int_{t-y}^{t}F_2^c(t-v)d_v\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,v-s)ds\Big) \\
& -\Big(\int_{t-y}^{v}F_2^c(t-u)d_u\Big[\int_0^u\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& +\int_{t-y}^{t}F_2^c(t-v)d_v\Big(\int_{t-y}^{v}F_2^c(t-u)d_u\Big[\int_0^u\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big) \\
& +\Big(\int_0^v\Gamma_{K,1}^c(t-s,v-s)ds\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& -\int_{t-y}^{t}\Big(\int_0^v\Gamma_{K,1}^c(t-s,v-s)ds\Big)dF_2^c(t-v) \\
& -\Big(\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& +\int_{t-y}^{t}\Big(\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)dF_2^c(t-v) \\
& -\Big(\int_{v}^{t}F_2^c(t-u)d_u\Big[\int_0^v\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)F_2^c(t-v)\Big|_{v=t-y}^{t} \\
& +\int_{t-y}^{t}F_2^c(t-v)d_v\Big(\int_{v}^{t}F_2^c(t-u)d_u\Big[\int_0^v\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big) \\
= \;& -\int_0^t\Gamma_{K,1}^c(s,s)ds-(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(s,s)ds+2F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,t-u)du\Big) \\
& +2\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^s\Gamma_{K,1}^c(t-u,s-u)du\Big) \\
& -2F_2^c(y)\int_{t-y}^{t}F_2^c(t-s)d_s\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-u,s-u)du\Big) \\
& +\int_0^t\Gamma_{K,1}^c(t-s,t-s)ds-(F_2^c(y))^2\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,t-y-s)ds \\
& -\int_{t-y}^{t}F_2^c(t-v)\Big(d_v\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)+\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]f_2(t-v)dv\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -F_2^c(y)\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,t-s)ds\Big)+(F_2^c(y))^2\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,t-y-s)ds\Big)\\
& +F_2^c(y)\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,v-s)ds\Big)\\
& -\Big(\int_{t-y}^t F_2^c(t-u)d_u\Big[\int_0^u\Gamma_{K,1}^c(u-s,t-s)ds\Big]\Big)\\
& +\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_{t-y}^v F_2^c(t-u)d_u\Big[\int_0^u\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)\\
& +\int_0^t\Gamma_{K,1}^c(t-s,t-s)ds-F_2^c(y)\int_0^{t-y}\Gamma_{K,1}^c(t-s,t-y-s)ds\\
& -\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_0^v\Gamma_{K,1}^c(t-s,v-s)ds\Big)\\
& -\Big(\Big[\int_0^t\Gamma_{K,1}^c(t-s,t-s)ds\Big]+(F_2^c(y))^2\Big[\int_0^{t-y}\Gamma_{K,1}^c(t-y-s,t-y-s)ds\Big]\\
& +\int_{t-y}^t F_2^c(t-v)d_v\Big(\Big[\int_0^v\Gamma_{K,1}^c(v-s,v-s)ds\Big]F_2^c(t-v)\Big)\\
& +F_2^c(y)\Big(\int_{t-y}^t F_2^c(t-u)d_u\Big[\int_0^{t-y}\Gamma_{K,1}^c(u-s,t-y-s)ds\Big]\Big)\\
& +\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_v^t F_2^c(t-u)d_u\Big[\int_0^v\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)\\
= & \int_{t-y}^t F_2^c(t-v)d_v\Big(\int_{t-y}^v F_2^c(t-u)d_u\Big[\int_0^u\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)\\
& +\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_v^t F_2^c(t-u)d_u\Big[\int_0^v\Gamma_{K,1}^c(u-s,v-s)ds\Big]\Big)\\
= & \int_{t-y}^t F_2^c(t-v)d_v\Big(\int_{t-y}^v F_2^c(t-u)\Big[\int_0^u d_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)\\
& +\int_{t-y}^t F_2^c(t-v)d_v\Big(\int_v^t F_2^c(t-u)\Big[\int_0^v d_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)\\
= & \int_{t-y}^t F_2^c(t-v)\Big(F_2^c(t-v)\Big[\int_0^v d_u\Gamma_{K,1}^c(v-s,v-s)ds\Big]dv\Big)\\
& +\int_{t-y}^t F_2^c(t-v)\Big(\int_{t-y}^v F_2^c(t-u)\Big[\int_0^u d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)dv\\
& -\int_{t-y}^t F_2^c(t-v)\Big(F_2^c(t-v)\Big[\int_0^v d_u\Gamma_{K,1}^c(v-s,v-s)ds\Big]dv\Big)\\
& +\int_{t-y}^t F_2^c(t-v)\Big(\int_v^t F_2^c(t-u)\Big[\int_0^v d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)dv\\
= & \int_{t-y}^t F_2^c(t-v)\Big(\int_{t-y}^v F_2^c(t-u)\Big[\int_0^u d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)dv\\
& +\int_{t-y}^t F_2^c(t-v)\Big(\int_v^t F_2^c(t-u)\Big[\int_0^v d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]du\Big)dv
\end{aligned}
$$

$$= \int_{t-y}^{t}\int_{t-y}^{t} F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v} d_v d_u \Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv$$

Now combining all the terms, we obtain

$$\begin{aligned}
\sigma^2_{Q_2,e}(t,y) \quad &= \quad p(1-p)\lambda_1 \int_{t-y}^{t} (F_2^c(t-s))^2 F_1(s)ds \\
&\quad +p^2\lambda_1\Big\{-\int_{t-y}^{t}(F_2^c(t-v))^2 F_2(v)dv \\
&\quad +\int_{t-y}^{t} F_2^c(t-v)\Big[\int_{t-y}^{v}\Big(F_2^c(t-u)\Big(f_1(v-u)+(c_a^2-1)\int_0^u f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv \\
&\quad +\int_{t-y}^{t} F_2^c(t-v)\Big[\int_{v}^{t}\Big(F_2^c(t-u)\Big(f_1(u-v)+(c_a^2-1)\int_0^v f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv \\
&\quad +\int_{t-y}^{t}\int_{t-y}^{t} F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v} d_v d_u \Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv\Big\} \\
&\quad +p\lambda_1\int_{t-y}^{t}\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds \\
&= \quad p(1-p)\lambda_1 \int_{t-y}^{t} (F_2^c(t-s))^2 F_1(s)ds \\
&\quad +p^2\lambda_1\Big\{-\int_{t-y}^{t}(F_2^c(t-v))^2 F_2(v)dv \\
&\quad +\int_{t-y}^{t}\int_{t-y}^{t}\Big(F_2^c(t-u\vee v)F_2^c(t-u\wedge v)\Big(f_1(u\vee v-u\wedge v) \\
&\qquad +(c_a^2-1)\int_0^{u\wedge v} f_1(u-s)f_1(v-s)ds\Big)\Big)dudv \\
&\quad +\int_{t-y}^{t}\int_{t-y}^{t} F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v} d_v d_u \Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv\Big\} \\
&\quad +p\lambda_1\int_{t-y}^{t}\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds \\
&= \quad p\lambda_1 \int_{t-y}^{t} (F_2^c(t-s))^2 F_1(s)ds \\
&\quad +p^2\lambda_1\Big\{\int_{t-y}^{t}\int_{t-y}^{t}\Big(F_2^c(t-u\vee v)F_2^c(t-u\wedge v)\Big(f_1(u\vee v-u\wedge v) \\
&\qquad +(c_a^2-1)\int_0^{u\wedge v} f_1(u-s)f_1(v-s)ds\Big)\Big)dudv \\
&\quad +\int_{t-y}^{t}\int_{t-y}^{t} F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v} d_v d_u \Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv\Big\} \\
&\quad +p\lambda_1\int_{t-y}^{t}\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds \\
&= \quad p\lambda_1\int_{t-y}^{t}\Big(F_2^c(t-s)+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+p^2\lambda_1\Big\{\int_{t-y}^{t}\int_{t-y}^{t}\Big(F_2^c(t-u\vee v)F_2^c(t-u\wedge v)\Big(f_1(u\vee v-u\wedge v)\\
&\quad+(c_a^2-1)\int_0^{u\wedge v}f_1(u-s)f_1(v-s)ds\Big)\Big)dudv\\
&+\int_{t-y}^{t}\int_{t-y}^{t}F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v}d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv\Big\}\\
\xrightarrow{t\to\infty}\ &p\lambda_1\int_0^y\Big(F_2^c(s)+\Gamma_{K,2}^c(s,s)\Big)ds\\
&p^2\lambda_1\Big\{\int_0^y\int_0^y\Big(F_2^c(u\vee v)F_2^c(u\wedge v)\Big(f_1(u\vee v-u\wedge v)\\
&\quad+(c_a^2-1)\int_{u\wedge v}^{\infty}f_1(s-u)f_1(s-v)ds\Big)\Big)dudv\\
&+\int_0^y\int_0^y F_2^c(u)F_2^c(v)\Big[\int_{u\wedge v}^{\infty}d_vd_u\Gamma_{K,1}^c(s-u,s-v)ds\Big]dudv\Big\}
\end{aligned}
$$

The limit as $t\to\infty$ is derived as follows:

$$
\begin{aligned}
&\int_{t-y}^{t}F_2^c(t-v)\Big[\int_{t-y}^{v}\Big(F_2^c(t-u)\Big(f_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv\\
=^{(w=t-v)}\ &\int_0^y F_2^c(w)\Big[\int_{t-y}^{t-w}\Big(F_2^c(t-u)\Big(f_1(t-w-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(t-w-s)ds\Big)\Big)du\Big]dw\\
=^{(x=t-u)}\ &\int_0^y F_2^c(w)\Big[\int_w^y\Big(F_2^c(x)\Big(f_1(x-w)+(c_{a,1}^2-1)\int_0^{t-x} f_1(t-x-s)f_1(t-w-s)ds\Big)\Big)dx\Big]dw\\
=^{(z=t-s)}\ &\int_0^y F_2^c(w)\Big[\int_w^y\Big(F_2^c(x)\Big(f_1(x-w)+(c_{a,1}^2-1)\int_x^{t} f_1(z-x)f_1(z-w)dz\Big)\Big)dx\Big]dw\\
\xrightarrow{t\to\infty}\ &\int_0^y F_2^c(w)\Big[\int_w^y\Big(F_2^c(x)\Big(f_1(x-w)+(c_{a,1}^2-1)\int_x^{\infty} f_1(z-x)f_1(z-w)dz\Big)\Big)dx\Big]dw
\end{aligned}
$$

$$
\begin{aligned}
&\int_{t-y}^{t}F_2^c(t-v)\Big[\int_{v}^{t}\Big(F_2^c(t-u)\Big(f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s)ds\Big)\Big)du\Big]dv\Big\}\\
=^{(w=t-v)}\ &\int_0^y F_2^c(w)\Big[\int_{t-w}^{t}\Big(F_2^c(t-u)\Big(f_1(u-(t-w))+(c_{a,1}^2-1)\int_0^{t-w} f_1(u-s)f_1(t-w-s)ds\Big)\Big)du\Big]dw\Big\}\\
=^{(x=t-u)}\ &\int_0^y F_2^c(w)\Big[\int_0^w\Big(F_2^c(x)\Big(f_1(w-x)+(c_{a,1}^2-1)\int_0^{t-w} f_1(t-x-s)f_1(t-w-s)ds\Big)\Big)dx\Big]dw\Big\}\\
=^{(z=t-s)}\ &\int_0^y F_2^c(w)\Big[\int_0^w\Big(F_2^c(x)\Big(f_1(w-x)+(c_{a,1}^2-1)\int_w^{t} f_1(z-x)f_1(z-w)ds\Big)\Big)dx\Big]dw\Big\}
\end{aligned}
$$

Calculation of $\sigma_{D_2}^2$
By (3.48) and (5.2),

$$
\begin{aligned}
\sigma_{D_2}^2(t) &= \int_0^t\int_0^t C_{A_2}(u,v)dF_2^c(t-u)dF_2^c(t-v)\\
&\quad+p\lambda_1\int_0^t\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
= \;& p^2\lambda_1 \int_0^t \int_0^t \Big[c_{a,1}^2 \int_0^{u\wedge v} F_1(u-s)F_1(v-s)ds \\
& + \int_0^{u\wedge v} \Big(F_1^c(u\wedge v-s) - F_1^c(u-s)F_1^c(v-s) \\
& + \Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& + p(1-p)\lambda_1 \int_0^t \int_0^t \Big[\int_0^{u\wedge v} F_1(s)ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& + p\lambda_1 \int_0^t \Big(F_2^c(t-s) - (F_2^c(t-s))^2 + \Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds,
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^t \int_0^t \Big[c_{a,1}^2 \int_0^{u\wedge v} F_1(u-s)F_1(v-s)ds \\
& + \int_0^{u\wedge v} (F_1^c(u\wedge v-s) - F_1^c(u-s)F_1^c(v-s))ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
= \;& \int_0^t \int_0^v \Big[c_{a,1}^2 \int_0^u F_1(u-s)F_1(v-s)ds \\
& + \int_0^u (F_1^c(u-s) - F_1^c(u-s)F_1^c(v-s))ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& + \int_0^t \int_v^t \Big[c_{a,1}^2 \int_0^v F_1(u-s)F_1(v-s)ds \\
& + \int_0^v (F_1^c(v-s) - F_1^c(u-s)F_1^c(v-s))ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
= \;& \int_0^t \int_0^v \Big[\int_0^u (F_1(v-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
& + \int_0^t \int_v^t \Big[\int_0^v (F_1(u-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big] dF_2^c(t-u)dF_2^c(t-v) \\
= \;& \int_0^t \Big[\Big[F_2^c(t-u) \int_0^u (F_1(v-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big]\Big|_{u=0}^{v} \\
& - \int_0^v F_2^c(t-u)d_u\Big[\int_0^u (F_1(v-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big]\Big] dF_2^c(t-v) \\
& + \int_0^t \Big[\Big[F_2^c(t-u) \int_0^v (F_1(u-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big]\Big|_{u=v}^{t} \\
& - \int_v^t F_2^c(t-u)d_u\Big[\int_0^v (F_1(u-s) + (c_{a,1}^2-1)F_1(u-s)F_1(v-s))ds\Big]\Big] dF_2^c(t-v) \\
= \;& \int_0^t \Big[\Big[F_2^c(t-v) \int_0^v (F_1(s) + (c_{a,1}^2-1)F_1(s)^2)ds\Big] \\
& - \int_0^v F_2^c(t-u)\Big(F_1(v-u) + (c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(v-s))ds\Big)du\Big] dF_2^c(t-v) \\
& + \int_0^t \Big[\Big[\int_0^v (F_1(t-s) + (c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big] \\
& - \Big[F_2^c(t-v) \int_0^v (F_1(s) + (c_{a,1}^2-1)F_1(s)^2)ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
& -\int_v^t F_2^c(t-u)\Big[\int_0^v (f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big]du\Big]dF_2^c(t-v) \\
= \;& -\int_0^t \Big[\int_0^v F_2^c(t-u)\Big(F_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(v-s))ds\Big)du\Big]dF_2^c(t-v) \\
& +\int_0^t \Big[\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big]dF_2^c(t-v) \\
& -\int_0^t \Big[\int_v^t F_2^c(t-u)\Big[\int_0^v (f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big]du\Big]dF_2^c(t-v) \\
= \;& -\Big(F_2^c(t-v)\int_0^v F_2^c(t-u)\Big(F_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(v-s))ds\Big)du\Big)\Big|_{v=0}^t \\
& +\int_0^t F_2^c(t-v)d_v\Big[\int_0^v F_2^c(t-u)\Big(F_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(v-s))ds\Big)du\Big] \\
& +\Big[F_2^c(t-v)\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big]\Big|_{v=0}^t \\
& -\int_0^t F_2^c(t-v)d_v\Big[\int_0^v (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(v-s))ds\Big] \\
& -\Big[F_2^c(t-v)\int_v^t F_2^c(t-u)\Big[\int_0^v (f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big]du\Big]\Big|_{v=0}^t \\
& +\int_0^t F_2^c(t-v)d_v\Big[\int_v^t F_2^c(t-u)\Big[\int_0^v (f_1(u-s)+(c_{a,1}^2-1)f_1(u-s)F_1(v-s))ds\Big]du\Big] \\
= \;& -\int_0^t F_2^c(t-u)\Big(F_1(t-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s))ds\Big)du \\
& +(c_{a,1}^2-1)\int_0^t (F_2^c(t-v))^2\Big(\int_0^v f_1(v-s)F_1(v-s)ds\Big)dv \\
& +\int_0^t F_2^c(t-v)\Big[\int_0^v F_2^c(t-u)\Big(f_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(v-s))ds\Big)du\Big]dv \\
& +\int_0^t (F_1(t-s)+(c_{a,1}^2-1)F_1(t-s)F_1(t-s))ds \\
& -\int_0^t F_2^c(t-v)F_1(t-v)dv \\
& -(c_{a,1}^2-1)\int_0^t F_2^c(t-v)\int_0^v F_1(t-s)f_1(v-s)dsdv \\
& -\int_0^t (F_2^c(t-v))^2\Big[\int_0^v (f_1(v-s)+(c_{a,1}^2-1)f_1(v-s)F_1(v-s))ds\Big]dv \\
& +\int_0^t F_2^c(t-v)\Big[\int_v^t F_2^c(t-u)\Big[f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s))ds\Big]du\Big]dv \\
= \;& \int_0^t (F_1(s)+(c_{a,1}^2-1)F_1(s)^2)ds-\int_0^t (F_2^c(t-v))^2F_1(v)dv \\
& -2\int_0^t F_2^c(t-u)\Big(F_1(t-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s))ds\Big)du \\
& +\int_0^t F_2^c(t-v)\Big[\int_0^v F_2^c(t-u)\Big(f_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(v-s))ds\Big)du\Big]dv
\end{aligned}
$$

$$+\int_0^t F_2^c(t-v)\Big[\int_v^t F_2^c(t-u)\Big[f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s))ds\Big]du\Big]dv$$

$$\begin{aligned}
&\int_0^t\int_0^t\Big[\int_0^{u\wedge v}\Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big]dF_2^c(t-u)dF_2^c(t-v)\\
=&\int_0^t\int_0^t F_2^c(t-u)F_2^c(t-v)\Big[\int_0^{u\wedge v}d_vd_u\Gamma_{K,1}^c(u-s,v-s)ds\Big]dudv\\
&+\int_0^t\Gamma_{K,1}^c(s,s)ds-2\int_0^t F_2^c(t-s)\Big(\int_0^s d_s\Gamma_{K,1}^c(t-u,s-u)du\Big)ds
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_0^t\Big[\int_0^{u\wedge v}F_1(s)ds\Big]dF_2^c(t-u)dF_2^c(t-v)\\
=&\int_0^t\int_0^v\Big[\int_0^u F_1(s)ds\Big]dF_2^c(t-u)dF_2^c(t-v)+\int_0^t\int_v^t\Big[\int_0^v F_1(s)ds\Big]dF_2^c(t-u)dF_2^c(t-v)\\
=&\int_0^t\Big[F_2^c(t-v)\int_0^v F_1(s)ds-\int_0^v F_2^c(t-u)F_1(u)du\Big]dF_2^c(t-v)\\
&+\int_0^t\Big[\int_0^v F_1(s)ds\Big](1-F_2^c(t-v))dF_2^c(t-v)\\
=&\int_0^t\Big[\int_0^v F_1(s)ds\Big]dF_2^c(t-v)-\int_0^t\Big[\int_0^v F_2^c(t-u)F_1(u)du\Big]dF_2^c(t-v)\\
=&\int_0^t\Big[\int_0^v F_2(t-u)F_1(u)du\Big]dF_2^c(t-v)\\
=&\int_0^t F_2(t-u)F_1(u)du-\int_0^t F_2^c(t-v)F_2(t-v)F_1(v)dv\\
=&\int_0^t F_2(t-s)^2F_1(s)ds
\end{aligned}$$

Combining the terms for $\sigma_{D_2}^2$, we obtain

$$\begin{aligned}
\sigma_{D_2}^2(t) =\;& p^2\lambda_1\Big[\int_0^t(F_1(s)+(c_{a,1}^2-1)F_1(s)^2)ds-\int_0^t(F_2^c(t-v))^2F_1(v)dv\\
&-2\int_0^t F_2^c(t-u)\Big(F_1(t-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s))ds\Big)du\\
&+\int_0^t F_2^c(t-v)\Big[\int_0^v F_2^c(t-u)\Big(f_1(v-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)f_1(v-s))ds\Big)du\Big]dv\\
&+\int_0^t F_2^c(t-v)\Big[\int_v^t F_2^c(t-u)\Big[f_1(u-v)+(c_{a,1}^2-1)\int_0^v f_1(u-s)f_1(v-s))ds\Big]du\Big]dv\\
&+\int_0^t\int_0^t\Big[\int_0^{u\wedge v}\Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big]f_2(t-u)f_2(t-v)dudv\Big]\\
&+p(1-p)\lambda_1\int_0^t F_2(t-s)^2F_1(s)ds\\
&+p\lambda_1\int_0^t\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds,
\end{aligned}$$

$$
\begin{aligned}
= \; & p^2\lambda_1\Big[\int_0^t (F_1(s)+(c_{a,1}^2-1)F_1(s)^2)ds - \int_0^t (F_2^c(t-v))^2F_1(v)dv \\
& -2\int_0^t F_2^c(t-u)\Big(F_1(t-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s))ds\Big)du \\
& +\int_0^t\int_0^t F_2^c(t-v)F_2^c(t-u)\Big(f_1(u\vee v-u\wedge v)+(c_{a,1}^2-1)\int_0^{u\wedge v} f_1(u-s)f_1(v-s))ds\Big)du\Big]dv \\
& +\int_0^t\int_0^t\Big[\int_0^{u\wedge v}\Gamma_{K,1}^c(u-s,v-s)\Big)ds\Big]f_2(t-u)f_2(t-v)dudv\Big] \\
& +p(1-p)\lambda_1\int_0^t F_2(t-s)^2F_1(s)ds \\
& +p\lambda_1\int_0^t\Big(F_2^c(t-s)-(F_2^c(t-s))^2+\Gamma_{K,2}^c(t-s,t-s)\Big)F_1(s)ds,
\end{aligned}
$$

For the limit as $t\to\infty$,

$$
\begin{aligned}
& \lim_{t\to\infty}\frac{1}{t}\int_0^t (F_1(s)+(c_{a,1}^2-1)F_1(s)^2)ds \\
= \; & \lim_{t\to\infty} F_1(t)+(c_{a,1}^2-1)F_1(t)^2 = 1+(c_{a,1}^2-1)=c_{a,1}^2
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{t\to\infty}\frac{1}{t}\int_0^t (F_2^c(t-v))^2F_1(v)dv = \lim_{t\to\infty}\frac{1}{t}\int_0^t (F_2^c(s))^2F_1(t-s)ds \\
= \; & \lim_{t\to\infty}\int_0^t (F_2^c(s))^2f_1(t-s)ds = \lim_{t\to\infty}\int_0^t (F_2^c(s))^2dF_1^c(t-s) \\
= \; & \lim_{t\to\infty}\Big[(F_2^c(t))^2-F_1^c(t)-\int_0^t F_1^c(t-s)d(F_2^c(s))^2\Big] \\
= \; & \lim_{t\to\infty} 2\int_0^t F_1^c(t-s)F_2^c(s)f_d(s)ds \\
= \; & \lim_{t\to\infty} 2\int_0^t \Big[F_2^c(s)f_d(s)-\int_0^s f_1(s-u)F_2^c(u)f_d(u)du\Big]ds \\
= \; & \lim_{t\to\infty}\Big[-F_2^c(t))^2+F_2^c(0))^2-2\int_0^t\int_0^s f_1(s-u)F_2^c(u)f_d(u)duds\Big] \\
= \; & 1-2\int_0^\infty\int_0^s f_1(s-u)F_2^c(u)f_d(u)duds \\
= \; & 1-2\int_0^\infty\int_u^\infty f_1(s-u)dsF_2^c(u)f_d(u)du \\
= \; & 1-2\int_0^\infty F_2^c(u)f_d(u)du = 1-1=0
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{t\to\infty}\frac{1}{t}\int_0^t F_2^c(t-u)\Big(F_1(t-u)+(c_{a,1}^2-1)\int_0^u f_1(u-s)F_1(t-s)ds\Big)du \\
= \; & \lim_{t\to\infty}\frac{1}{t}\int_0^t F_2^c(v)\Big(F_1(v)+(c_{a,1}^2-1)\int_0^{t-v} f_1(t-v-s)F_1(t-s)ds\Big)dv
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t F_2^c(v)\Big(F_1(v) + (c_{a,1}^2 - 1)\int_0^{t-v} f_1(w)F_1(v+w)dw\Big)dv \\
&= \lim_{t\to\infty} \Big(F_2^c(t)F_1(t) + (c_{a,1}^2 - 1)\int_0^t F_2^c(v)f_1(t-v)F_1(t)dv\Big) \\
&= \lim_{t\to\infty} (c_{a,1}^2 - 1)F_1(t)\int_0^t F_2^c(v)dF_1^c(t-v) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\int_0^t F_2(t-s)^2 F_1(s)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t F_2(v)^2 F_1(t-v)dv \\
&= \lim_{t\to\infty} \int_0^t F_2(v)^2 f_1(t-v)dv = \lim_{t\to\infty} \int_0^t F_2(v)^2 dF_1^c(t-v) \\
&= \lim_{t\to\infty} \Big[F_2(t)^2 - \int_0^t F_1^c(t-v)dF_2(v)^2\Big] \\
&= 1 - \lim_{t\to\infty} \int_0^t \Big(2F_2(s)f_2(s) - \int_0^s f_1(s-v)dF_2(v)^2\Big)ds \\
&= 1 - \lim_{t\to\infty} \Big[F_2(t)^2 - \int_0^t \int_0^s f_1(s-v)dF_2(v)^2 ds\Big] \\
&= \int_0^\infty \int_0^s f_1(s-v)dF_2(v)^2 ds = \int_0^\infty \int_v^\infty f_1(s-v)ds dF_2(v)^2 = 1
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\int_0^t F_2(t-s)F_2^c(t-s)F_1(s)ds \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^t F_2(s)F_2^c(s)F_1(t-s)ds = \lim_{t\to\infty} \int_0^t F_2(s)F_2^c(s)f_1(t-s)ds \\
&= \lim_{t\to\infty} \int_0^t F_2(s)F_2^c(s)dF_1^c(t-s) = \lim_{t\to\infty} \Big[F_2(t)F_2^c(t) - \int_0^t F_1^c(t-s)d(F_2(s)F_2^c(s))\Big] \\
&= -\lim_{t\to\infty} \int_0^t \Big(f_2(s) - 2F_2(s)f_2(s) - \int_0^s f_1(s-u)d(F_2(u)F_2^c(u))\Big)ds = 0
\end{aligned}
$$