APPENDIX

to

Approximations for Heavily-Loaded $G/GI/n+GI$ Queues

by

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A Overview

This appendix supplements the main paper by providing additional numerical examples evaluating the performance of the Gaussian approximations DGA, TGA and TGA-G.

First, in §B, we give more examples providing examples of the good performance. Then, in §C, we give more examples that further expose the limitations of the approximations. In §C.1 we give examples showing that the abandonment rates cannot be too much larger than 1; otherwise the system ceases to be OL. In §C.2 we give more examples showing that the performance of underloaded models is not good except for a few performance measures. Here we consider models with smaller scale. In §C.3 we give examples showing the degraded performance of the approximations when the traffic intensity ρ is too close to 1. We fix $n = 100$ and let ρ decrease toward 1.

B More Good examples

We first give additional positive examples for Markov models in §§B.1–B.2 and non-Markov models in §B.3.

B.1 More Examples of Markov $M/M/n+M$ Models

In the main paper, we compare the exact solution with DGA's and TGA's in heavily loaded regime ($\rho = 1.2$). Table 18 provides results for $\rho = 1.5$. In this very highly loaded system, the TGA and DGA approximations are nearly identifical, but we see some small differences in the cases with $\theta > 1$.

Table 18: A comparison of the TGA and DGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100+M(\theta^{-1})$ model with $\lambda = 100\rho$ and $\rho = 1.5$ for six values of θ, $0.10 \le \theta \le 4.00$.

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Sim.	DGA	TGA	Sim.	DGA	TGA	Sim.	DGA	TGA
E[X] rel. err.	5.99E+2 ±9.19E-1	6.00E+2 0%	same	3.00E+2 ±4.01E-1	3.00E+2 0%	same	2.00E+2 ±2.01E-1	2.00E+2 0%	same
Var(X) rel. err.	1.49E+3 ±1.10E+3	1.50E+3 1%	same	6.00E+2 ±2.41E+2	6.00E+2 0%	same	3.00E+2 ±8.10E+1	3.00E+2 0%	same
E[Q] rel. err.	4.99E+2 ±9.19E-1	5.00E+2 0%	5.00E+2 0%	2.00E+2 ±4.01E-1	2.00E+2 0%	2.00E+2 0%	1.00E+2 ±2.01E-1	9.99E+1 0%	9.99E+1 0%
Var(Q) rel. err.	1.49E+3 ±9.19E+2	1.50E+3 1%	1.50E+3 1%	6.00E+2 ±1.61E+2	6.00E+2 0%	6.00E+2 0%	3.00E+2 ±4.08E+1	3.00E+2 0%	3.00E+2 0%
E[W] rel. err.	4.05E+0 ±7.53E-3	4.06E+0 0%	4.06E+0 0%	1.63E+0 ±3.28E-3	1.62E+0 0%	1.62E+0 0%	8.17E-1 ±1.64E-3	8.11E-1 1%	8.11E-1 1%
Var(W) rel. err.	9.98E-2 ±7.53E-3	1.00E-1 0%	1.00E-1 0%	4.01E-2 ±3.28E-3	4.00E-2 0%	4.00E-2 0%	2.00E-2 ±1.64E-3	2.00E-2 0%	2.00E-2 0%
PoD rel. err.	1.00E+0 ±0.00E+0	1.00E+0 0%	same	1.00E+0 ±0.00E+0	1.00E+0 0%	same	1.00E+0 ±0.00E+0	1.00E+0 0%	same
PoA rel. err.	3.33E-1 ±1.34E-3	3.33E-1 0%	same	3.33E-1 ±1.37E-3	3.33E-1 0%	same	3.34E-1 ±1.33E-3	3.33E-1 0%	same
	$\theta = 2$			$\theta = 4$			$\theta = 10$		
Perf.	Sim.	DGA	TGA	Sim.	DGA	TGA	Sim.	DGA	TGA
E[X] rel. err.	1.25E+2 ±5.05E-2	1.25E+2 0%	same	1.12E+2 ±2.64E-2	1.12E+2 0%	same	1.05E+2 ±1.36E-2	1.05E+2 0%	same
Var(X) rel. err.	7.53E+1 ±1.27E+1	7.49E+1 1%	same	3.79E+1 ±5.96E+0	3.74E+1 1%	same	1.71E+1 ±2.84E+0	1.49E+1 13%	same
E[Q] rel. err.	2.50E+1 ±5.05E-2	2.49E+1 0%	2.49E+1 0%	1.25E+1 ±2.60E-2	1.23E+1 2%	1.24E+1 1%	5.03E+0 ±1.17E-2	4.90E+0 3%	5.08E+0 1%
Var(Q) rel. err.	7.52E+1 ±2.63E+0	7.49E+1 0%	7.46E+1 1%	3.68E+1 ±7.08E-1	3.74E+1 2%	3.59E+1 2%	1.35E+1 ±1.49E-1	1.49E+1 10%	1.24E+1 9%
E[W] rel. err.	2.08E-1 ±4.13E-4	2.03E-1 2%	2.03E-1 2%	1.06E-1 ±2.14E-4	1.01E-1 5%	1.01E-1 5%	4.55E-2 ±1.01E-4	4.10E-2 10%	4.25E-2 7%
Var(W) rel. err.	5.07E-3 ±4.13E-4	5.00E-3 1%	4.98E-3 2%	2.52E-3 ±2.14E-4	2.50E-3 1%	2.40E-3 5%	9.87E-4 ±1.01E-4	1.00E-3 2%	8.44E-4 14%
PoD rel. err.	9.98E-1 ±1.27E-4	9.98E-1 0%	same	9.79E-1 ±4.53E-4	9.78E-1 0%	same	8.62E-1 ±1.03E-3	9.02E-1 5%	same
PoA rel. err.	3.32E-1 ±1.35E-3	3.34E-1 0%	same	3.34E-1 ±1.34E-3	3.32E-1 0%	same	3.35E-1 ±1.31E-3	3.36E-1 0%	same

B.2 Low Abandonment Rates

We next consider the Markovian $M/M/n+M$ queueing system, with the arrival rate, number of servers and service rate fixed at $(\lambda, n, \mu) = (105, 100, 1)$, but decreasing abandonment rate. As studied in [40], the queueing system tends to heavily overloaded when abandonment rates decrease. The Theorem 4 in [40] states that in an $M/M/n/r+M$ model, a scaled process of number in system converges to a OU process as $s/\theta \to 0$. To reshow the results, we give the Markovian $M/M/n+M$ queueing system, with the arrival rate, number of servers and service rate fixed at $(\lambda, n, \mu) = (105, 100, 1)$ but decreasing abandonment rate θ from $0.05, 0.02$ to 0.01. Table 19 shows that our TGAs continue to work effectively for smaller abandonment rates. Notice that little difference between DGAs and TGAs are presented; it is because when $\theta \to 0$, improvements brought about by truncation become less effective as the queue tends to ED regime.

Table 19: A comparison of the TGA and DGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $\lambda = 100\rho$ and $\rho = 1.2$ for three low abandonment rates $\theta < 0.1$.

	$\theta = 0.05$			$\theta = 0.02$			$\theta = 0.01$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	2.01E+2	2.00E+2	same	3.50E+2	3.50E+2	same	5.99E+2	6.00E+2	same
rel. err.		0%			0%			0%	
Var(X)	2.00E+3	2.10E+3	same	5.22E+3	5.25E+3	same	1.03E+4	1.05E+4	same
rel. err.		5%			1%			2%	
E[Q]	1.01E+2	1.00E+2	1.00E+2	2.50E+2	2.50E+2	2.50E+2	4.99E+2	5.00E+2	5.00E+2
rel. err.		1%	1%		0%	0%		0%	0%
Var(Q)	1.99E+3	2.10E+3	2.05E+3	5.22E+3	5.25E+3	5.25E+3	1.03E+4	1.05E+4	1.05E+4
rel. err.		5%	3%		1%	1%		2%	2%
E[W]	9.90E-1	9.76E-1	9.78E-1	2.44E+0	2.44E+0	2.44E+0	4.88E+0	4.88E+0	4.88E+0
rel. err.		1%	1%		0%	0%		0%	0%
Var(W)	1.90E-1	2.00E-1	1.95E-1	4.97E-1	5.00E-1	5.00E-1	9.85E-1	1.00E+0	1.00E+0
rel. err.		5%	3%		1%	0%		2%	2%
PoD	9.93E-1	9.85E-1	same	1.00E+0	1.00E+0	same	1.00E+0	1.00E+0	same
rel. err.		1%			0%			0%	
PoA	4.81E-2	4.75E-2	same	4.76E-2	4.75E-2	same	4.76E-2	4.76E-2	same
rel. err.		1%			0%			0%	

B.3 More Examples of $GI/GI/n + GI$ Models

We now consider examples with various combinations of high and low variabilities for the interarrival, service and patience times. We use simply phase-type (PH) distributions to achieve both high and low variabilities: Erlang-n (E_n) for low variabilities (with SCV $1/n$) and H_2 for high variabilities (with SCV greater than 1). Other parameters remain the same as those in Table 2. Table 20 shows that TGA-G works well except when the SCV of service time is high (e.g. $c_s^2 = 4$).

Table 20: A comparison of the TGA-G approximations to simulation estimates in the $PH/PH/n+PH$ models model with $\lambda = 100\rho$ and $\rho = 1.0.5$ for for values of the service scv c_s^2, $0.25 \leq \theta \leq 4.00$.

Perf. Meas.	SCV		c_s^2					
			0.25			0.5		
	c_λ^2	c_{ab}^2	Sim	CI	TGA-GA	Sim	CI	TGA-GA
E[Q]	0.5	0.5	3.40E+1	±2.15E-1	3.57E+1	3.34E+1	±2.50E-1	3.57E+1
		2	8.67E+0	±6.46E-2	8.42E+0	9.00E+0	±7.46E-2	8.67E+0
	2	0.5	3.37E+1	±3.14E-1	3.63E+1	3.37E+1	±3.23E-1	3.64E+1
		2	1.07E+1	±9.32E-2	1.01E+1	1.09E+1	±1.02E-1	1.03E+1
Var(Q)	0.5	0.5	1.84E+2	±1.44E+1	2.14E+2	2.30E+2	±1.66E+1	2.51E+2
		2	5.14E+1	±1.48E+0	5.55E+1	6.10E+1	±1.84E+0	6.36E+1
	2	0.5	4.30E+2	±2.29E+1	4.89E+2	4.55E+2	±2.38E+1	5.18E+2
		2	1.12E+2	±3.01E+0	1.12E+2	1.21E+2	±3.44E+0	1.19E+2
E[W]	0.5	0.5	3.33E-1	±2.04E-3	3.46E-1	3.28E-1	±2.40E-3	3.47E-1
		2	8.86E-2	±6.32E-4	8.30E-2	9.21E-2	±7.41E-4	8.55E-2
	2	0.5	3.26E-1	±2.90E-3	3.51E-1	3.26E-1	±3.00E-3	3.52E-1
		2	1.06E-1	±8.77E-4	9.84E-2	1.07E-1	±9.71E-4	1.00E-1
Var(W)	0.5	0.5	1.66E-2	±2.04E-3	1.97E-2	2.13E-2	±2.40E-3	2.34E-2
		2	4.96E-3	±6.32E-4	5.32E-3	6.01E-3	±7.41E-4	6.13E-3
	2	0.5	3.68E-2	±2.90E-3	4.32E-2	3.94E-2	±3.00E-3	4.62E-2
		2	1.00E-2	±8.77E-4	1.05E-2	1.10E-2	±9.71E-4	1.12E-2
PoD	0.5	0.5	9.90E-1	±8.51E-4	9.93E-1	9.79E-1	±1.37E-3	9.87E-1
		2	8.18E-1	±2.62E-3	7.97E-1	7.96E-1	±2.89E-3	7.76E-1
	2	0.5	9.30E-1	±2.49E-3	9.43E-1	9.22E-1	±2.66E-3	9.36E-1
		2	7.28E-1	±3.18E-3	7.04E-1	7.19E-1	±3.31E-3	6.97E-1
PoA	0.5	0.5	4.82E-2	±7.19E-4	4.78E-2	4.81E-2	±7.62E-4	4.79E-2
		2	5.54E-2	±7.12E-4	5.36E-2	5.84E-2	±7.76E-4	5.52E-2
	2	0.5	5.16E-2	±8.66E-4	4.90E-2	5.26E-2	±8.83E-4	4.93E-2
		2	6.54E-2	±8.09E-4	6.32E-2	6.66E-2	±8.67E-4	6.44E-2
Perf. Meas.	SCV		c_s^2					
			2			4		
	c_λ^2	c_{ab}^2	Sim	CI	TGA-GA	Sim	CI	TGA-GA
E[Q]	0.5	0.5	3.27E+1	±3.38E-1	3.59E+1	3.27E+1	±4.14E-1	3.62E+1
		2	9.73E+0	±1.03E-1	9.44E+0	1.03E+1	±1.36E-1	1.00E+1
	2	0.5	3.35E+1	±3.76E-1	3.68E+1	3.39E+1	±4.38E-1	3.71E+1
		2	1.13E+1	±1.29E-1	1.09E+1	1.13E+1	±1.52E-1	1.14E+1
Var(Q)	0.5	0.5	4.03E+2	±2.31E+1	3.70E+2	5.13E+2	±2.97E+1	4.60E+2
		2	9.28E+1	±3.04E+0	8.96E+1	1.14E+2	±4.38E+0	1.10E+2
	2	0.5	5.84E+2	±2.91E+1	6.15E+2	6.74E+2	±3.50E+1	6.91E+2
		2	1.47E+2	±4.69E+0	1.43E+2	1.61E+2	±5.75E+0	1.62E+2
E[W]	0.5	0.5	3.23E-1	±3.34E-3	3.49E-1	3.25E-1	±4.14E-3	3.52E-1
		2	1.01E-1	±1.06E-3	9.33E-2	1.07E-1	±1.43E-3	9.91E-2
	2	0.5	3.26E-1	±3.61E-3	3.56E-1	3.31E-1	±4.28E-3	3.60E-1
		2	1.13E-1	±1.28E-3	1.07E-1	1.14E-1	±1.54E-3	1.11E-1
Var(W)	0.5	0.5	3.91E-2	±3.34E-3	3.53E-2	5.10E-2	±4.14E-3	4.42E-2
		2	9.72E-3	±1.06E-3	8.74E-3	1.22E-2	±1.43E-3	1.08E-2
	2	0.5	5.31E-2	±3.61E-3	5.60E-2	6.30E-2	±4.28E-3	6.37E-2
		2	1.42E-2	±1.28E-3	1.36E-2	1.60E-2	±1.54E-3	1.55E-2
PoD	0.5	0.5	9.25E-1	±3.13E-3	9.63E-1	8.89E-1	±4.13E-3	9.41E-1
		2	7.32E-1	±3.74E-3	7.27E-1	7.13E-1	±4.75E-3	7.02E-1
	2	0.5	8.80E-1	±3.66E-3	9.12E-1	8.57E-1	±4.36E-3	8.94E-1
		2	6.87E-1	±4.03E-3	6.77E-1	6.69E-1	±4.64E-3	6.64E-1
PoA	0.5	0.5	5.17E-2	±9.27E-4	4.84E-2	5.46E-2	±1.11E-3	4.91E-2
		2	6.28E-2	±8.88E-4	6.01E-2	6.58E-2	±1.03E-3	6.37E-2
	2	0.5	5.49E-2	±1.03E-3	5.03E-2	5.81E-2	±1.13E-3	5.12E-2
		2	6.80E-2	±9.75E-4	6.83E-2	6.92E-2	±1.10E-3	7.13E-2

C More Examples Revealing Limitations of the Approximations

C.1 High Abandonment Rates

In the most simple $M/M/n+M$ model, both DGA and TGA can not correctly estimate the key performances when abandonment rate $\theta = 4, 10$. Moreover, the estimate of means (and probabilities) deteriorate faster than that of variances, see Table 21 for details.

Table 21: A comparison of the TGA and DGA approximations to exact values for the Markovian $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $n = 100, \rho = 1.05$ and $\theta = 4, 10$

	$\theta = 4$			$\theta = 10$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	96.6	100	-	95.0	100	-
rel. err		4%			5%	
Var(X)	56.1	25.1	-	44.9	10.0	-
rel. err		55%			78%	
E[Q]	1.47	0.100	2.05	0.665	0	1.26
rel. err		83%	39%		100%	90%
Var(Q)	7.77	25.1	8.76	2.47	10.0	3.41
rel. err		223%	13%		303%	38%
E[W]	1.66E-2	2.40E-3	2.11E-2	8.20E-3	1.00E-3	1.31E-2
rel. err		86%	27%		88%	61%
Var(W)	8.80E-4	2.50E-3	9.00E-4	3.10E-4	1.00E-3	3.50E-4
rel. err		182%	2%		219%	13%
PoD	0.379	-	0.508	0.282	-	0.500
rel. err			34%			77%
PoA	5.83E-2	9.50E-3	8.12E-2	6.58E-2	9.95E-2	1.23E-1
rel. err		84%	39%		85%	87%

C.2 Smaller UL Systems

To suppplement the UL results shown in §8.3, we now show the experimental results for smaller UL systems. We consider systems with smaller arrival rates and numbers of servers $n = 20, 10, 5, 3$ and 1. For different n, we choose different values for the traffic intensity, following (25), with the QoS factor fixed at $\beta = 0.5$. Just as in Table 16, Table 22 shows good performance for X_n and B_n for $0.5 \leq \theta \leq 2.0$, but poor performance otherwise.

C.3 Critically Loaded Systems with $\rho > 1$

So far, we have concentrated on $G/GI/n+GI$ models with $(\rho, n) = (1.05, 100)$, or models with smaller scale n but the same QoS factor $\beta = \sqrt{n}(1-\rho) = -0.5$. We now consider systems with lighter loading, closer to the critical loading at $\rho = 1.000$. These include systems with $n = 100$ but $1.00 < \rho < 1.05$ and systems with smaller n but $-0.5 < \beta < 0.0$.

Table 23 provides results for $M(\lambda^{-1})/M/n+M(\theta^{-1})$ (Erlang-A) models with parameter triples $(\lambda, \theta, n) = (n\rho, 0.5, 100)$ and traffic intensity ρ ranges among $\{1.01, 1.005, 1.001\}$. Table 23 shows

Table 22: A comparison of the TGA and DGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/n+M(2)$ with $n = 50, 20, 10, 5, 3$ and 1, $\rho = 1-\beta/\sqrt{n}$, $\lambda = n\rho$ and $\theta = 0.5$.

	$n=50,\rho=0.93$			$n=20,\rho=0.88$			$n=10,\rho=0.84$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	4.75E+1	4.65E+1	same	1.84E+1	1.78E+1	same	8.84E+0	8.42E+0	same
rel. err.		2%			3%			5%	
Var(X)	5.90E+1	4.65E+1	same	2.26E+1	1.78E+1	same	1.08E+1	8.42E+0	same
rel. err.		21%			22%			22%	
E[B]	4.55E+1	4.65E+1	4.52E+1	1.72E+1	1.78E+1	1.70E+1	8.00E+0	8.42E+0	7.89E+0
rel. err.		2%	1%		4%	1%		5%	1%
Var(B)	2.48E+1	4.65E+1	2.61E+1	9.39E+0	1.78E+1	1.01E+1	4.39E+0	8.42E+0	4.79E+0
rel. err.		87%	5%		89%	7%		92%	9%
E[Q]	2.00E+0	0.00E+0	1.31E+0	1.23E+0	0.00E+0	7.95E-1	8.34E-1	0.00E+0	5.35E-1
rel. err.		100%	35%		100%	35%		100%	36%
Var(Q)	1.60E+1	0.00E+0	7.69E+0	6.27E+0	0.00E+0	2.88E+0	3.06E+0	0.00E+0	1.33E+0
rel. err.		100%	52%		100%	54%		100%	56%
E[V]	4.54E-2	0.00E+0	2.62E-2	7.51E-2	0.00E+0	3.97E-2	1.12E-1	0.00E+0	5.35E-2
rel. err.		100%	42%		100%	47%		100%	52%
Var(V)	7.48E-3	0.00E+0	3.60E-3	2.02E-2	0.00E+0	9.19E-3	4.42E-2	0.00E+0	1.87E-2
rel. err.		100%	52%		100%	54%		100%	58%
PoD	3.67E-1	0.00E+0	6.69E-1	3.73E-1	0.00E+0	6.61E-1	3.80E-1	0.00E+0	6.52E-1
rel. err.		100%	82%		100%	77%		100%	72%
PoA	2.16E-2	NaN	1.30E-2	3.45E-2	NaN	1.97E-2	4.95E-2	NaN	2.64E-2
rel. err.		NaN	40%		NaN	43%		NaN	47%

	$n=5,\rho=0.78$			$n=3,\rho=0.71$			$n=1,\rho=0.5$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	4.16E+0	3.88E+0	same	2.33E+0	2.13E+0	same	5.82E-1	5.00E-1	same
rel. err.		7%			9%			14%	
Var(X)	5.00E+0	3.88E+0	same	2.77E+0	2.13E+0	same	6.61E-1	5.00E-1	same
rel. err.		22%			23%			24%	
E[B]	3.61E+0	3.88E+0	3.55E+0	1.94E+0	2.13E+0	1.93E+0	4.18E-1	5.00E-1	5.00E-1
rel. err.		8%	2%		10%	0%		20%	20%
Var(B)	1.99E+0	3.88E+0	2.11E+0	1.08E+0	2.13E+0	1.03E+0	2.43E-1	5.00E-1	1.60E-1
rel. err.		95%	6%		98%	4%		106%	34%
E[Q]	5.53E-1	0.00E+0	3.50E-1	3.97E-1	0.00E+0	2.49E-1	1.64E-1	0.00E+0	9.98E-2
rel. err.		100%	37%		100%	37%		100%	39%
Var(Q)	1.47E+0	0.00E+0	5.93E-1	8.45E-1	0.00E+0	3.12E-1	2.27E-1	0.00E+0	6.00E-2
rel. err.		100%	60%		100%	63%		100%	74%
E[V]	1.70E-1	0.00E+0	7.01E-2	2.36E-1	0.00E+0	8.31E-2	5.20E-1	0.00E+0	9.98E-2
rel. err.		100%	59%		100%	65%		100%	81%
Var(V)	1.01E-1	0.00E+0	3.77E-2	1.93E-1	0.00E+0	6.24E-2	9.27E-1	0.00E+0	1.60E-1
rel. err.		100%	63%		100%	68%		100%	83%
PoD	3.88E-1	0.00E+0	6.41E-1	3.96E-1	0.00E+0	6.30E-1	4.18E-1	0.00E+0	5.99E-1
rel. err.		100%	65%		100%	59%		100%	43%
PoA	7.12E-2	NaN	3.44E-2	9.31E-2	NaN	4.07E-2	1.64E-1	NaN	4.87E-2
rel. err.		NaN	52%		NaN	56%		NaN	70%

Table 23: A comparison of the TGA and DGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/n + M(\theta^{-1})$ model with $(n, \theta, \lambda) = (100, 0.5, 100\rho)$ and $\rho \to 1$

	$\rho = 1.01$			$\rho = 1.005$			$\rho = 1.001$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	1.05E+2	1.02E+2	same	1.04E+2	1.01E+2	same	1.03E+2	1.00E+2	same
rel. err.		3%			3%			3%	
Var(X)	1.55E+2	2.02E+2	same	1.52E+2	2.01E+2	same	1.49E+2	2.00E+2	same
rel. err.		30%			32%			34%	
E[Q]	7.63E+0	1.91E+0	6.67E+0	7.11E+0	9.02E-1	6.12E+0	6.70E+0	1.00E-1	5.69E+0
rel. err.		75%	13%		87%	14%		99%	15%
Var(Q)	8.68E+1	2.02E+2	7.99E+1	8.12E+1	2.01E+2	7.36E+1	7.68E+1	2.00E+2	6.88E+1
rel. err.		133%	8%		147%	9%		161%	10%
E[V]	7.92E-2	2.00E-2	6.70E-2	7.40E-2	1.00E-2	6.16E-2	7.01E-2	2.00E-3	5.74E-2
rel. err.		75%	15%		86%	17%		97%	18%
Var(V)	8.90E-3	2.00E-2	7.98E-3	8.38E-3	2.00E-2	7.39E-3	7.97E-3	2.00E-2	6.93E-3
rel. err.		125%	10%		139%	12%		151%	13%
PoD	6.42E-1	5.56E-1	same	6.20E-1	5.28E-1	same	6.01E-1	5.06E-1	same
rel. err.		13%	same		15%	same		16%	same
PoA	3.78E-2	3.20E-2	same	3.54E-2	2.94E-2	same	3.35E-2	2.75E-2	same
rel. err.		15%	same		17%	same		18%	same

that the TGA performs well in the extreme cases ($\rho = 1.001$) and surprisingly does not degenerate even the system become (critically) underloaded, when the abandonment rate $\theta = 0.5$. However, as θ decreases in light traffic models, the performance of DGA and TGA degenerates, see results of $\theta \leq 0.25$ in Tables 27 and 28. More experiments for ligher loading models with different queues are presented in Tables 24-28 showing performance of TGA for the $M/M/n + M$ model with abandonment rate $0.1 \leq \theta \leq 4$ and different traffic intensity 1.1, 1.03, 1.02, 1.01 and 1.001. As we have observed before, the performance of TGA is very good if θ is sufficiently close to 1, because in the case $\theta = 1$, the number in system coincides with the number of busy servers in the infinite-server $M/M/\infty$ model, which has exactly a Poisson distribution.

Table 24: A comparison of the TGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $(\lambda, \rho) = (110, 1.10)$ and $0.1 \le \theta \le 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	2.00E+2	2.00E+2	0%	1.40E+2	1.40E+2	0%	1.20E+2	1.20E+2	0%
Var(X)	1.10E+3	1.10E+3	0%	4.23E+2	4.40E+2	4%	2.08E+2	2.20E+2	6%
E[Q]	1.00E+2	1.00E+2	0%	4.04E+1	4.02E+1	1%	2.08E+1	2.06E+1	1%
Var(Q)	1.09E+3	1.10E+3	0%	4.13E+2	4.18E+2	1%	1.87E+2	1.88E+2	0%
E[V]	9.58E-1	9.53E-1	1%	3.90E-1	3.83E-1	2%	2.03E-1	1.96E-1	3%
Var(V)	9.96E-2	9.98E-2	0%	3.77E-2	3.80E-2	1%	1.72E-2	1.71E-2	1%
PoD	9.99E-1	9.99E-1	0%	9.80E-1	9.72E-1	1%	9.29E-1	9.11E-1	2%
PoA	9.09E-2	9.05E-2	1%	9.19E-2	9.03E-2	2%	9.46E-2	9.16E-2	3%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.10E+2	1.10E+2	0%	1.04E+2	1.05E+2	1%	1.01E+2	1.02E+2	2%
Var(X)	1.10E+2	1.10E+2	0%	6.53E+1	5.50E+1	16%	4.41E+1	2.75E+1	38%
E[Q]	1.09E+1	1.09E+1	0%	5.77E+0	6.10E+0	6%	3.06E+0	3.57E+0	17%
Var(Q)	8.26E+1	8.08E+1	2%	3.57E+1	3.45E+1	4%	1.51E+1	1.49E+1	1%
E[V]	1.08E-1	1.04E-1	4%	5.88E-2	5.82E-2	1%	3.23E-2	3.41E-2	5%
Var(V)	7.72E-3	7.35E-3	5%	3.43E-3	3.14E-3	9%	1.52E-3	1.36E-3	11%
PoD	8.42E-1	8.30E-1	1%	7.31E-1	7.50E-1	3%	6.12E-1	6.83E-1	12%
PoA	9.92E-2	9.59E-2	3%	1.05E-1	1.05E-1	0%	1.11E-1	1.18E-1	6%

Table 25: A comparison of the TGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $(\lambda, \rho) = (103, 1.03)$ and $0.1 \le \theta \le 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.36E+2	1.30E+2	4%	1.16E+2	1.12E+2	3%	1.08E+2	1.06E+2	2%
Var(X)	7.63E+2	1.03E+3	35%	3.04E+2	4.12E+2	36%	1.68E+2	2.06E+2	23%
E[Q]	3.63E+1	3.30E+1	9%	1.73E+1	1.55E+1	11%	9.97E+0	9.21E+0	8%
Var(Q)	7.10E+2	7.50E+2	6%	2.45E+2	2.44E+2	0%	1.10E+2	1.07E+2	3%
E[V]	3.63E-1	3.25E-1	10%	1.75E-1	1.52E-1	13%	1.02E-1	9.08E-2	11%
Var(V)	6.93E-2	7.28E-2	5%	2.41E-2	2.37E-2	2%	1.10E-2	1.04E-2	6%
PoD	9.13E-1	8.25E-1	10%	8.18E-1	7.23E-1	12%	7.28E-1	6.62E-1	9%
PoA	3.53E-2	3.17E-2	10%	4.20E-2	3.67E-2	13%	4.84E-2	4.32E-2	11%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.03E+2	1.03E+2	0%	9.98E+1	1.01E+2	2%	9.77E+1	1.01E+2	3%
Var(X)	1.03E+2	1.03E+2	0%	7.02E+1	5.15E+1	27%	5.29E+1	2.57E+1	51%
E[Q]	5.70E+0	5.72E+0	0%	3.22E+0	3.67E+0	14%	1.78E+0	2.42E+0	36%
Var(Q)	4.94E+1	4.78E+1	3%	2.17E+1	2.20E+1	1%	9.33E+0	1.03E+1	11%
E[V]	5.94E-2	5.64E-2	5%	3.44E-2	3.62E-2	5%	1.98E-2	2.39E-2	20%
Var(V)	5.04E-3	4.65E-3	8%	2.29E-3	2.14E-3	7%	1.03E-3	1.00E-3	3%
PoD	6.29E-1	6.16E-1	2%	5.29E-1	5.83E-1	10%	4.34E-1	5.59E-1	29%
PoA	5.54E-2	5.27E-2	5%	6.25E-2	6.61E-2	6%	6.92E-2	8.41E-2	22%

Table 26: A comparison of the TGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $(\lambda, \rho) = (102, 1.02)$ and $0.1 \le \theta \le 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.29E+2	1.20E+2	7%	1.13E+2	1.08E+2	5%	1.06E+2	1.04E+2	2%
Var(X)	6.70E+2	1.02E+3	52%	2.82E+2	4.08E+2	45%	1.62E+2	2.04E+2	26%
E[Q]	2.97E+1	2.52E+1	15%	1.48E+1	1.27E+1	15%	8.75E+0	7.91E+0	10%
Var(Q)	6.03E+2	6.19E+2	3%	2.14E+2	2.07E+2	3%	9.84E+1	9.34E+1	5%
E[V]	2.98E-1	2.49E-1	16%	1.51E-1	1.26E-1	17%	9.01E-2	7.84E-2	13%
Var(V)	5.95E-2	6.07E-2	2%	2.14E-2	2.03E-2	5%	9.96E-3	9.16E-3	8%
PoD	8.72E-1	7.34E-1	16%	7.74E-1	6.54E-1	16%	6.86E-1	6.10E-1	11%
PoA	2.91E-2	2.43E-2	16%	3.64E-2	3.03E-2	17%	4.29E-2	3.74E-2	13%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.02E+2	1.02E+2	0%	9.91E+1	1.01E+2	2%	9.71E+1	1.00E+2	3%
Var(X)	1.02E+2	1.02E+2	0%	7.11E+1	5.10E+1	28%	5.45E+1	2.55E+1	53%
E[Q]	5.09E+0	5.10E+0	0%	2.91E+0	3.37E+0	16%	1.62E+0	2.27E+0	40%
Var(Q)	4.46E+1	4.31E+1	3%	1.98E+1	2.03E+1	3%	8.54E+0	9.71E+0	14%
E[V]	5.34E-2	5.06E-2	5%	3.13E-2	3.34E-2	7%	1.82E-2	2.25E-2	24%
Var(V)	4.61E-3	4.23E-3	8%	2.11E-3	1.99E-3	6%	9.58E-4	9.54E-4	0%
PoD	5.92E-1	5.78E-1	2%	4.96E-1	5.56E-1	12%	4.06E-1	5.40E-1	33%
PoA	4.99E-2	4.74E-2	5%	5.70E-2	6.11E-2	7%	6.37E-2	7.97E-2	25%

Table 27: A comparison of the TGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ model with $(\lambda, \rho) = (101, 1.01)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.23E+2	1.10E+2	10%	1.10E+2	1.04E+2	6%	1.05E+2	1.02E+2	3%
Var(X)	5.78E+2	1.01E+3	75%	2.60E+2	4.04E+2	55%	1.55E+2	2.02E+2	30%
E[Q]	2.40E+1	1.83E+1	24%	1.26E+1	1.02E+1	19%	7.63E+0	6.72E+0	12%
Var(Q)	4.96E+2	4.78E+2	4%	1.85E+2	1.71E+2	8%	8.68E+1	8.05E+1	7%
E[V]	2.43E-1	1.82E-1	25%	1.29E-1	1.01E-1	22%	7.92E-2	6.69E-2	15%
Var(V)	4.96E-2	4.73E-2	5%	1.87E-2	1.69E-2	9%	8.90E-3	7.97E-3	10%
PoD	8.23E-1	6.23E-1	24%	7.27E-1	5.79E-1	20%	6.42E-1	5.56E-1	13%
PoA	2.37E-2	1.78E-2	25%	3.12E-2	2.45E-2	22%	3.78E-2	3.20E-2	15%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.01E+2	1.01E+2	0%	9.84E+1	1.00E+2	2%	9.66E+1	1.00E+2	4%
Var(X)	1.01E+2	1.01E+2	0%	7.21E+1	5.05E+1	30%	5.61E+1	2.52E+1	55%
E[Q]	4.52E+0	4.53E+0	0%	2.61E+0	3.09E+0	18%	1.47E+0	2.13E+0	45%
Var(Q)	3.99E+1	3.85E+1	4%	1.79E+1	1.86E+1	4%	7.77E+0	9.09E+0	17%
E[V]	4.78E-2	4.51E-2	6%	2.84E-2	3.08E-2	8%	1.66E-2	2.12E-2	27%
Var(V)	4.18E-3	3.82E-3	9%	1.94E-3	1.85E-3	5%	8.85E-4	9.03E-4	2%
PoD	5.53E-1	5.40E-1	2%	4.63E-1	5.28E-1	14%	3.79E-1	5.20E-1	37%
PoA	4.47E-2	4.23E-2	5%	5.17E-2	5.64E-2	9%	5.83E-2	7.51E-2	29%

Table 28: A comparison of the TGA approximations to exact numerical values in the $M(\lambda^{-1})/M(1)/100+M(\theta^{-1})$ model with $(\lambda,\rho)=(100.1,1.001)$ and $0.1\le\theta\le 4$

	$\theta=0.1$			$\theta=0.25$			$\theta=0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.18E+2	1.01E+2	14%	1.08E+2	1.00E+2	7%	1.03E+2	1.00E+2	3%
Var(X)	5.00E+2	1.00E+3	100%	2.41E+2	4.00E+2	66%	1.49E+2	2.00E+2	34%
E[Q]	1.96E+1	1.31E+1	33%	1.08E+1	8.18E+0	24%	6.70E+0	5.74E+0	14%
Var(Q)	4.08E+2	3.54E+2	13%	1.60E+2	1.40E+2	13%	7.68E+1	6.93E+1	10%
E[V]	2.00E-1	1.31E-1	34%	1.12E-1	8.18E-2	27%	7.01E-2	5.74E-2	18%
Var(V)	4.12E-2	3.54E-2	14%	1.63E-2	1.40E-2	15%	7.97E-3	6.93E-3	13%
PoD	7.72E-1	5.13E-1	34%	6.81E-1	5.08E-1	25%	6.01E-1	5.06E-1	16%
PoA	1.96E-2	1.29E-2	34%	2.70E-2	1.98E-2	27%	3.35E-2	2.75E-2	18%

	$\theta=1$			$\theta=2$			$\theta=4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.00E+2	1.00E+2	0%	9.77E+1	1.00E+2	2%	9.61E+1	1.00E+2	4%
Var(X)	1.00E+2	1.00E+2	0%	7.30E+1	5.00E+1	31%	5.76E+1	2.50E+1	57%
E[Q]	4.04E+0	4.04E+0	0%	2.36E+0	2.84E+0	20%	1.34E+0	2.00E+0	49%
Var(Q)	3.59E+1	3.45E+1	4%	1.62E+1	1.72E+1	6%	7.10E+0	8.55E+0	20%
E[V]	4.30E-2	4.04E-2	6%	2.59E-2	2.85E-2	10%	1.53E-2	2.00E-2	31%
Var(V)	3.81E-3	3.45E-3	9%	1.78E-3	1.72E-3	4%	8.19E-4	8.56E-4	5%
PoD	5.17E-1	5.04E-1	3%	4.33E-1	5.03E-1	16%	3.54E-1	5.02E-1	42%
PoA	4.03E-2	3.80E-2	6%	4.72E-2	5.22E-2	11%	5.36E-2	7.11E-2	33%