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ORDINARY CLT AND WLLN VERSIONS OF $L = \lambda W$*†

PETER W. GLYNN‡ AND WARD WHITT§

The familiar queueing principle expressed by the formula $L = \lambda W$ (Little's law) can be interpreted as a relation among strong laws of large numbers (SLLNs). Here we prove central-limit-theorem (CLT) and weak-law-of-large-numbers (WLLN) versions of $L = \lambda W$. For example, if the sequence of ordered pairs of interarrival times and waiting times is strictly stationary and satisfies a joint CLT, then the queue-length process also obeys a CLT with a related limiting distribution. In a previous paper we proved a functional-central-limit-theorem version of $L = \lambda W$, without stationarity, by very different arguments. The two papers highlight the differences between establishing ordinary limit theorems and their functional-limit-theorem counterparts.

1. Introduction and summary. In [7] we established a functional-central-limit-theorem (FCLT) version of the fundamental queueing formula $L = \lambda W$ [12], [14]. We showed that the time-average of the queue-length process obeys a FCLT if the customer-average waiting time obeys a FCLT jointly with the customer-average interarrival time, and we described the limits. This was accomplished using the continuous mapping theorem and related arguments in the setting of weak convergence on the function space $D[0, \infty)$, as in [1] and [16]. Since FCLTs tend to hold in all the standard situations in which ordinary central limit theorems (CLTs) hold, the FCLTs in [7] seem quite satisfactory for practical purposes (e.g., applications to queueing parameter estimation; see [7]–[9]). Nevertheless, it is natural to ask if it is possible to establish corresponding relations among the associated CLTs; here we show that it is possible, provided that we add the extra condition of stationarity.

In addition to the extending the queueing relation $L = \lambda W$, the results here have general probabilistic interest. In particular, we establish new asymptotic results for random sums and inverse processes. This paper complements [16], in which it is shown that functional limit theorems are preserved under various mappings on the function spaces $D[0, \infty)$ and $D[0, \infty) \times D[0, \infty)$ such as composition and inverse. Here similar results are obtained for ordinary limit theorems. For example, for inverse processes, §3 here is an analog of §7 of [16].

The importance of this paper given [7] hinges on the relation between ordinary limit theorems (CLTs, SLLNs and WLLNs) and their functional-limit-theorem counterparts (FCLTs, FSLLNs and FWLLNs). Consequently, we also address this issue here (§2 and 6). We show that a CLT plus stationarity need not imply a SLLN or a FWLLN (Example 1 in §6). As a consequence, a WLLN need not imply a FWLLN, and a CLT need not imply a FCLT. On the other hand we show that SLLNs and FSLLNs are equivalent (Theorem 4). We also show that a FCLT need not imply a

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‡Stanford University.
§AT & T Bell Laboratories.

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SLLN (Example 2 in §6). As a consequence, we obtain the well-known result that a WLLN need not imply a SLLN. However, as an important tool for establishing our queueing results, we show that a WLLN does imply a SLLN under the extra conditions of stationarity, nonnegativity and finite mean (Theorem 5). Obviously much of this background has been discovered before, e.g., Theorem 4, but the importance makes a brief explicit treatment worthwhile.

As in [7], we use the standard $L = \lambda W$ framework involving the sequence of ordered pairs of random variables $\{(A_k, D_k): k \geqslant 1\}$, where $0 \leqslant A_k \leqslant A_{k+1}$ and $A_k \leqslant D_k$ for all k. This framework is obviously very general, so that there are many applications. In queueing, we interpret A_k and D_k as the arrival and departure epochs of the kth arriving customer, where arrival and departure are understood to be with respect to the system under consideration. For example, if we are interested in the waiting time before beginning service, then the relevant system is the waiting room or queue, not counting the servers, and the departure epochs D_k refer to the instants customers leave the queue and begin service.

Let the associated interarrival times be $U_k = A_k - A_{k-1}$ for $k \geqslant 2$ and $U_1 = A_1$. Let the queue length at time t, $Q(t)$, be the number of k with $A_k \leqslant t \leqslant D_k$ and let the waiting time of the kth customer be $W_k = D_k - A_k$. Let $N(t)$ and $O(t)$ count the number of arrivals and departures, respectively, in the interval $[0, t]$. Let $\Rightarrow$ denote convergence in distribution, i.e., weak convergence [1]. We omit "as $t \to \infty$" when that is obvious.

Our CLT version of $L = \lambda W$ can be viewed as an analog of Theorem 4 in [7], but this paper can be read independently of [7]. The starting point here is an (ordinary) joint CLT for (U_n, W_n), i.e.,

$$n^{-1/2}\left(\sum_{k=1}^{n} U_k - \lambda^{-1}n,\ \sum_{k=1}^{n} W_k - wn \right) \Rightarrow (U, W), \tag{1.1}$$

where $0 < \lambda < \infty$, $w < \infty$ and (U, W) is an arbitrary random vector in R^2. (Note that $\sum_{k=1}^{n} U_k = A_k$ in (1.1). Also note that we do not assume that the limit (U, W) is normally distributed, although that is what typically occurs [8].) The object is to obtain a CLT for the cumulative process $\int_0^t Q(s)\,ds$ and, if possible, a CLT jointly with other related processes of interest. We obtain such a result here, but unlike [7], we have to add an extra condition. We obtain positive results under the extra condition of stationarity (by which we always mean strict stationarity), a condition which appears in many treatments of $L = \lambda W$; *cf.* [3], [6], [12], [15]. We rely heavily on stationarity, but we have yet to establish that it is necessary. We also exploit the fact that W_k and $U_k = A_k - A_{k-1}$ are nonnegative. Here is our main result.

THEOREM 1. *If $\{(U_k, W_k): k \geqslant 1\}$ is a stationary sequence of nonnegative random vectors satisfying the joint CLT* (1.1), *then $EU_k = \lambda^{-1}$, $EW_k = w$, and*

$$\begin{aligned} t^{-1/2}\Bigg(& A_{[\lambda t]} - t,\ N(t) - \lambda t,\ O(t) - \lambda t,\ \sum_{k=1}^{[\lambda t]} W_k - \lambda w t, \\ & \sum_{k=1}^{[\lambda t]} (W_k - \lambda w U_k),\ \sum_{k=1}^{N(t)} W_k - \lambda w t,\ \sum_{k=1}^{O(t)} W_k - \lambda w t,\ \int_0^t Q(s)\,ds - \lambda w t \Bigg) \\ \Rightarrow\ & (\lambda^{1/2}U, -\lambda^{3/2}U, -\lambda^{3/2}U, \lambda^{1/2}W, \lambda^{1/2}(W - wU), \\ & \lambda^{1/2}(W - wU), \lambda^{1/2}(W - wU), \lambda^{1/2}(W - wU)) \quad \text{in } R^8. \end{aligned} \tag{1.2}$$

We prove Theorem 1 and the other two theorems stated in the introduction in §5. Example 3 in §6 shows that the conditions of Theorem 1 need *not* imply a FCLT version of (1.1), so that Theorem 1 cannot be deduced from [7]. The formula $L = \lambda W$ appears in Theorem 1 in the translation terms. To follow the convention of having random variables represented by capital letters and nonrandom real numbers by lower case letters, we change the notation: we replace W by w and L by q. The translation terms for $\int_0^t Q(s)\,ds$, $N(t)$ and $\Sigma_{k=1}^n W_k$ in (1.2) are then q, λ and w, respectively, where $q = \lambda w$.

In queueing applications, the cumulative process $\int_0^t Q(s)\,ds$ is of primary interest, but the random sum $\Sigma_{k=1}^{N(t)} W_k$ is also of interest outside of queueing. Among the many CLTs for random sums, we know of nothing containing the limit of the sixth component in (1.2); cf. §17 of [1], [13], §5 of [16] and references cited there. Example 4 in §6 shows that this CLT for the random sum is not valid without the stationarity and the nonnegativity of W_k. The CLT for $N(t)$ alone (Theorem 6), which does not require stationarity, is also of general interest.

The limiting random vector (U, W) in (1.1) will typically have a bivariate normal distribution, in which case the limit in (1.2) has a multivariate normal distribution. (The distribution on R^8 of the limit in (1.2) is obviously degenerate.) See Corollary 3.1 and Remarks 3.4 and 3.6 in [7] for descriptions of the variances and covariances plus further discussion. Example 1 of [9] describes the M/M/1 special case.

In the process of proving Theorem 1, we establish several other useful weak convergence results, which we now summarize. To interpret the results, recall that convergence in distribution (weak convergence) to a nonrandom element is equivalent to convergence in probability; see p. 25 of [1]. It is easy to apply Theorem 2 to prove Theorem 1. The rest of this paper is primarily devoted to proving Theorem 2.

THEOREM 2. *Under the assumptions of Theorem* 1,

(a) $t^{-1/2}(N(t) - \lambda t) \Rightarrow -\lambda^{3/2}U$,

(b) $t^{-1/2}(N(t) - O(t)) = t^{-1/2}Q(t) \Rightarrow 0$,

(c) $t^{-1/2}(\Sigma_{k=1}^{N(t)}(U_k - \lambda^{-1}) - \Sigma_{k=1}^{[\lambda t]}(U_k - \lambda^{-1})) \Rightarrow 0$,

(d) $t^{-1/2}(\Sigma_{k=1}^{N(t)}(W_k - \lambda w U_k) - \Sigma_{k=1}^{[\lambda t]}(W_k - \lambda w U_k)) \Rightarrow 0$,

(e) $t^{-1/2}(\Sigma_{k=1}^{N(t)} W_k - \Sigma_{k=1}^{O(t)} W_k) \Rightarrow 0$,

(f) $t^{-1/2}(\int_0^t Q(s)\,ds - \Sigma_{k=1}^{N(t)} W_k) \Rightarrow 0$,

(g) $t^{-1/2}(\Sigma_{k=1}^{N(t)}(W_k - \lambda w U_k) - (\Sigma_{k=1}^{N(t)} W_k - \lambda w t)) = \lambda w t^{-1/2}(t - A_{N(t)}) \Rightarrow 0$,

(h) $t^{-1/2}((A_{[\lambda t]} - t) - (t - \lambda^{-1}N(t))) \Rightarrow 0$.

As in [7], we can also go the other way, starting with a joint CLT for $(N(t), \int_0^t Q(s)\,ds)$, but the situation is not symmetric; see Example 1 in [7]. If $(N(t), \int_0^t Q(s)\,ds)$ has stationary increments and

$$t^{-1/2}\left(N(t) - \lambda t, \int_0^t Q(s)\,ds - qt\right) \Rightarrow (N, Q), \tag{1.3}$$

then, by essentially the same argument,

$$n^{-1/2}\left(\int_0^{A_n} Q(s)\,ds - wn\right) \Rightarrow \lambda^{-1/2}(Q - wN). \tag{1.4}$$

Under the extra condition

$$n^{-1/2}\left(\sum_{k=1}^{n} W_k - \int_0^{A_n} Q(s)\,ds\right) \Rightarrow 0, \tag{1.5}$$

we also obtain (1.2) with $U = -\lambda^{-3/2}N$. (We omit the proof.)

In the same spirit as Theorem 1, we also establish the following weak-law-of-large-numbers (WLLN) version of $L = \lambda W$. Here we do not need the stationarity and the proof is much easier. Previous WLLN-versions of $L = \lambda W$ and the generalization $H = \lambda G$ appear in Theorems 3 and 4 of Brumelle [3]. The statement here has appeal because of its simplicity. The joint convergence in Theorem 3 is equivalent to the converge of the components separately; see Theorem 4.4 of [1].

THEOREM 3. *If $n^{-1}A_n \Rightarrow \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, and $n^{-1}\sum_{k=1}^{n} W_k \Rightarrow w$, then*

$$t^{-1}\left(N(t), \sum_{k=1}^{N(t)} W_k, \int_0^t Q(s)\,ds, \sum_{k=1}^{O(t)} W_k, O(t)\right) \tag{1.6}$$

$$\Rightarrow (\lambda, \lambda w, \lambda w, \lambda w, \lambda) \quad \text{in } R^5.$$

Here is how the rest of this paper is organized. In §2 we give background on the basic limit theorems (LLNs, CLTs and their functional counterparts). We do this to put these theorems in perspective and also to provide some key tools for proving Theorem 2. In §3 we discuss the relation between ordinary limit theorems for partial sums and associated counting (inverse) processes, and prove Theorem 2(a). As a further basis for proving Theorem 2, in §4 we prove a theorem establishing conditions for certain fluctuations of random sums to be asymptotically negligible. We bring everything together in §5 and prove Theorems 1–3. In §6 we give the four examples mentioned above.

An important open problem is whether the condition of stationarity in Theorem 1 is necessary. We conjecture that the condition cannot be simply deleted. However, we have succeeded in extending Theorem 1 to a large class of nonstationary processes (paper in preparation). Theorem 1 here plays a vital role in this extension; we establish an equivalence for CLTs, showing that certain processes obey a CLT if and only if an associated stationary version also does.

We have also written other related papers. We present sufficient conditions for FCLT versions of (1.1), and thus (1.1) itself, in terms of regenerative structure in [8]; we discuss statistical issues related to indirect estimation using $L = \lambda W$ in [9]; we prove an ordinary law-of-the-interated-logarithm (LIL) version of $L = \lambda W$ in [10]; and we generalize $H = \lambda G$ and establish FCLT versions of it in [11].

2. Background on the basic limit theorems.

Let $\{X_n: n \geq 1\}$ be a sequence of real-valued random variables and let $\{S_n: n \geq 0\}$ be the associated sequence of partial sums, defined by $S_n = X_1 + \cdots + X_n$, $n \geq 1$, $S_0 = 0$. We say that the sequence $\{X_n\}$ obeys a WLLN if $n^{-1}S_n \Rightarrow \mu$ for some finite real number μ, and a SLLN if this limit holds w.p.1. We say that the sequence $\{X_n\}$ obeys a CLT if $n^{-1/2}(S_n - n\mu) \Rightarrow Z$ for some proper (finite w.p.1) random variable Z. Usually Z has a normal distribution, but we do not require it. (No conditions relating to finite moments, stationarity or independence have been imposed on $\{X_n\}$.) For the CLT, we could also consider normalizations other than $n^{-1/2}$, but we do not. Both the CLT and the SLLN imply the WLLN, but neither the CLT nor the SLLN implies the other; we give examples in §6.

We now discuss functional limit theorems in $D \equiv D[0, \infty)$. Let the space D be endowed with the usual Skorohod J_1 topology, which reduces to uniform convergence on compact subsets for continuous limit functions; see [1] and [16]. Let $\mathbf{X}_n$ and $\mathbf{S}_n$ be

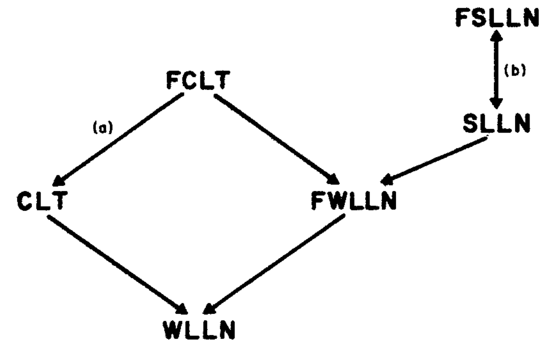

FIGURE 1. Relations among the limit theorems.

Notes: (a) Requires continuity of projection map.
(b) SLLN → FSLLN covered by Theorem 4.

random functions in D defined by

$$\mathbf{X}_n(t) = n^{-1}S_{[nt]} \quad \text{and} \quad \mathbf{S}_n(t) = n^{-1/2}(S_{[nt]} - \mu nt), \qquad t \geqslant 0, \tag{2.1}$$

where $[x]$ is the greatest integer less than or equal to x. Let e be the identity map on $[0, \infty)$, defined by $e(t) = t$, $t \geqslant 0$. The sequence $\{X_n\}$ obeys a FWLLN if $\mathbf{X}_n \Rightarrow \mu e$ in D and a FSLLN if this limit holds w.p.1. (We could initially allow a more general limit for $\mathbf{X}_n$ in the FWLLN or the FSLLN, but the limit will necessarily be μe provided that $\{X_n\}$ obeys a WLLN, because $n^{-1}S_n \Rightarrow \mu$ implies that $\mathbf{X}_n(t) \Rightarrow \mu t$ for each t as $n \to \infty$.) By the definition of the topology on D, the FSLLN is equivalent to

$$\lim_{n\to\infty} \sup_{0 \leqslant t \leqslant T} \left\{ |n^{-1}S_{[nt]} - \mu t| \right\} = 0 \quad \text{w.p.1} \quad \text{for all } T > 0. \tag{2.2}$$

The sequence $\{X_n\}$ obeys a FCLT if $\mathbf{S}_n \Rightarrow \mathbf{S}$ in D for $\mathbf{S}_n$ in (2.1) and some random element $\mathbf{S}$ in D. If $\mathbf{S}(t)$ is continuous at $t = 1$ w.p.1., then the FCLT implies the CLT and Z is distributed as $\mathbf{S}(1)$. Figure 1 describes the relations among these limit theorems; there is no implication where there is no arrow. (Implications extend by transitivity of course.) Three examples suffice to establish all nonimplications: (1) SLLN $\nrightarrow$ CLT, (2) CLT $\nrightarrow$ FWLLN and (3) FCLT $\nrightarrow$ SLLN. It is trivial that a SLLN does not imply a CLT; e.g., just let $S_n = n^{3/4}$. The two nontrivial examples are given in §6. All positive implications in Figure 1 are immediate except for one. We verify it now.

THEOREM 4. *The SLLN and the FSLLN are equivalent.*

PROOF. The implication FSLLN → SLLN is immediate using the continuous mapping theorem with the projection map. To go the other way, suppose that the SLLN holds: $n^{-1}S_n \to \mu$ w.p.1. Let $\epsilon > 0$ and T be given. By the SLLN, there is a $t_0(\epsilon)$ such that $\sup_{t \geqslant t_0(\epsilon)} |t^{-1}S_{[t]} - \mu| < \epsilon/2T$, so that

$$\sup_{\substack{n,t \\ t > t_0(\epsilon)/n}} \left\{ \left| (nt)^{-1} S_{[nt]} - \mu \right| \right\} < \epsilon/2T \quad \text{and} \tag{2.3}$$

$$\sup_{\substack{n,t \\ T \geqslant t > t_0(\epsilon)/n}} \left\{ \left| n^{-1} S_{[nt]} - t\mu \right| \right\} < \epsilon/2.$$

However, we can also treat $t \leqslant t_0(\epsilon)/n$ by bounding as

$$\sup_{\substack{n,t \\ t \leqslant t_0(\epsilon)/n}} \left\{ \left| n^{-1} S_{[nt]} - t\mu \right| \right\} \leqslant n^{-1} \left(\sup_{t \leqslant t_0} \left\{ \left| S_{[t]} \right| \right\} + \mu t_0(\epsilon) \right)$$

which converges to 0 as $n \to \infty$ w.p.1. Given ϵ and $t_0(\epsilon)$, choose $n_0(\epsilon)$ so that

$$n^{-1} \left(\sup_{t \leqslant t_0} \left\{ \left| S_{[t]} \right| + \mu t_0(\epsilon) \right) \right. < \epsilon/2 \tag{2.4}$$

for $n \geqslant n_0(\epsilon)$. From (2.3) and (2.4),

$$\sup_{\substack{n \\ n \geqslant n_0(\epsilon)}} \sup_{\substack{t \\ 0 \leqslant t \leqslant T}} \left\{ \left| n^{-1} S_{[nt]} - \mu t \right| \right\}$$

$$\leqslant \sup_{\substack{n \\ n \geqslant n_0(\epsilon)}} \sup_{\substack{t \\ t \leqslant t_0(\epsilon)}} \left\{ \left| n^{-1} S_{[nt]} - \mu t \right| \right\} + \sup_{\substack{n,t \\ T \geqslant t > t_0(\epsilon)}} \left\{ \left| n^{-1} S_{[nt]} - \mu t \right| \right\} = \epsilon. \quad \blacksquare$$

It is significant that the analog of Theorem 4 for the WLLN is not true. Since the CLT does not imply the FWLLN (Example 1), neither does the WLLN.

In Theorem 1, we start with the CLT in (1.1). We prove Theorem 1 by exploiting the FSLLN, but since even a FCLT does not imply a SLLN (Example 2), we obviously need something extra to get the FSLLN. We get the desired FSLLN from the CLT by combining Theorem 4 with the following result, after adding two extra conditions: stationarity and nonnegativity.

THEOREM 5. *If a stationary sequence of nonnegative random variables $\{X_n\}$ obeys a WLLN, then $EX_n = \mu < \infty$ and it obeys a SLLN.*

PROOF. We apply Birkhoff's ergodic theorem (Chapter 6 of [2]) twice, first to prove that $EX_n < \infty$ and second to establish convergence w.p.1. Let $X_k^m = \min\{X_k, m\}$. Since $0 \leqslant X_k^m \leqslant m$, $EX_k^m < \infty$. Since $\{X_t^m \colon k \geqslant 1\}$ is also stationary, we can apply Birkhoff's ergodic theorem to get $n^{-1}\sum_{k=1}^n X_k^m \to E(X_1^m \mid I^m)$ w.p.1 as $n \to \infty$, where I^m is the invariant σ-field for $\{X_k^m\}$. Since $n^{-1}\sum_{k=1}^n X_k^m \leqslant n^{-1}\sum_{k=1}^n X_k$ for all n and $n^{-1}\sum_{k=1}^n X_k \Rightarrow \mu$ by the assumed WLLN, $E(X_1^m \mid I^m) \leqslant \mu$ w.p.1 and thus also $E(X_1^m) \leqslant \mu$ for all m. By the monotone convergence theorem, $EX_1 \leqslant \mu$. We now can apply the ergodic theorem again to the original sequence $\{X_n\}$ to get $n^{-1}\sum_{k=1}^n X_k \to E(X_1 \mid I)$ w.p.1, where I is the invariant σ-field for $\{X_k\}$, but the assumed WLLN implies that $E(X_1 \mid I) = \mu$ w.p.1, which in turn implies that $EX_1 = \mu$. $\blacksquare$

3. **Inverse processes.** The processes $\{A_n \colon n \geqslant 1\}$ and $\{N(t) \colon t \geqslant 0\}$ are inverse processes in the sense that $A_n \leqslant t$ if and only if $N(t) \geqslant n$. As a consequence, under mild regularity conditions, we have a limit theorem for $N(t)$ if and only if the corresponding limit theorem holds for A_n. For example, this equivalence is elementary for the WLLN and SLLN (e.g., see the proof of Theorem 3 in §5). This equivalence for FCLTs is discussed in §7 of [16] and applied in [7]. Here we establish the equivalence for ordinary CLTs. No stationarity is assumed here. Part of the interest lies in allowing limits without continuous cdf's.

THEOREM 6. *Let A be a proper random variable and assume that $0 < \lambda < \infty$. Then*

$$n^{-1/2}\left(A_n - \lambda^{-1} n\right) \Rightarrow \quad \textit{if and only if} \quad t^{-1/2}\left(N(t) - \lambda t\right) \Rightarrow -\lambda^{3/2} A.$$

PROOF. By the basic inverse relation,

$$
\begin{aligned}
P\left(n^{-1/2}\left(A_n - n\lambda^{-1}\right) \leqslant x\right) &= P\left(A_n \leqslant n\lambda^{-1} + xn^{1/2}\right) \\
&= P\left(N(t_n) \geqslant n\right) \quad \text{for } t_n = n\lambda^{-1} + xn^{1/2} \\
&= P\left(\lambda^{-3/2}t_n^{-1/2}\left(N(t_n) - \lambda t_n\right) \geqslant \lambda^{-3/2}t_n^{-1/2}(n - \lambda t_n)\right) \\
&= P\left(\lambda^{-3/2}t_n^{-1/2}\left(N(t_n) - \lambda t_n\right) \geqslant x_n\right), \quad \text{where}
\end{aligned}
$$

$$
x_n \equiv \lambda^{-3/2}t_n^{-1/2}(n - \lambda t_n) = \lambda^{-3/2}\left(n\lambda^{-1} + xn^{1/2}\right)^{-1/2}\left(\lambda x n^{1/2}\right) \to x \tag{3.1}
$$

as $n \to \infty$.

First suppose that A has a continuous cdf and the limit for A_n holds. Then $\lambda^{-3/2}t_n^{-1/2}(N(t_n) - \lambda t_n) \Rightarrow -A$ as $n \to \infty$. (Suppose that F_n and F are cdf's with F being continuous. It is not difficult to see that $F_n(x) \to F(x)$ as $n \to \infty$ provided that $F_n(x_n) \to F(x)$ for some sequence $\{x_n\}$ with $x_n \to x$.) Since it is always possible to choose $n(t)$ and $x(t)$ as functions of t so that $n(t) \to \infty$, $x(t) \to x$ and (3.1) holds as $t \to \infty$ (let n and t satisfy $n^{-1/2}(t - n\mu) \to x$), we also have $\lambda^{-3/2}t^{-1/2}(N(t) - \lambda t) \Rightarrow -A$ as $t \to \infty$. A similar argument applies in the other direction, assuming that A has a continuous cdf.

Now we treat the general case by letting an arbitrary random variable A be the weak-convergence limit as $\epsilon \to 0$ of random variables A_ϵ with continuous cdf's. In particular, let X be a random variable uniformly distributed on the interval $[0,1]$ that is independent of the original basic sequence $\{A_n\}$ and let $A_n^\epsilon = A_n + \epsilon\sqrt{n}\,X$, $n \geqslant 1$. Obviously $n^{-1/2}(A_n^\epsilon - \lambda^{-1}n) \Rightarrow A + \epsilon X$ where X is independent of A, so that $A + \epsilon X$ has a continuous cdf for each ϵ. Moreover, since $A_n^\epsilon \geqslant A_n$ for all n and ϵ, $N^\epsilon(t) \leqslant N(t)$ for all t and ϵ, where $N^\epsilon(t)$ is the counting process associated with $\{A_n^\epsilon\}$.

To construct a bound on the other side, let $\hat{A}_n^\delta = A_{[n - \delta X \lambda n^{1/2}]}$, $n \geqslant 1$. (The index is positive for all sufficiently large n.) It is easy to see that $n^{-1/2}(\hat{A}_n^\delta - n\lambda^{-1}) \Rightarrow (A - \delta X)$. Moreover, since $\hat{A}_n^\delta \leqslant A_n$ for all n and δ, $\hat{N}^\delta(t) \geqslant N(t)$ for all t and δ.

The bounds imply that

$$
t^{-1/2}\left(N^\epsilon(t) - \lambda t\right) \leqslant t^{-1/2}\left(N(t) - \lambda t\right) \leqslant t^{-1/2}\left(\hat{N}^\delta(t) - \lambda t\right) \tag{3.2}
$$

for all t, ϵ and δ. The first part of the proof implies that

$$
\begin{aligned}
&t^{-1/2}\left(N^\epsilon(t) - \lambda t\right) \Rightarrow -\lambda^{3/2}(A + \epsilon X) \quad \text{and} \\
&t^{-1/2}\left(\hat{N}^\delta(t) - \lambda t\right) \Rightarrow -\lambda^{3/2}(A - \delta X).
\end{aligned} \tag{3.3}
$$

By letting $\epsilon \to 0$ and $\delta \to 0$, we obtain the desired results from (3.2) and (3.3). A similar argument applies in the other direction. ■

4. Fluctuations of random sums. As a basis for proving Theorem 2, we present some preliminary results on the fluctuations of random sums. Again we rely heavily on stationarity. To prove parts (b) and (c), we need the following preliminary result (which does not require stationarity).

LEMMA 1. *If $n^{-1}\Sigma_{k=1}^{n} X_k \to \mu$ w.p.1, then, for any $\alpha > 0$,*

$$\lim_{n\to\infty} n^{-1} \max_{\substack{1\leqslant k\leqslant n\\ 1\leqslant m\leqslant an}} \left\{\left|\sum_{j=k}^{k+m} (X_j - \mu)\right|\right\} = 0 \quad w.p.1,$$

so that

$$\lim_{n\to\infty} n^{-1} \max_{\substack{1\leqslant k\leqslant n\\ 1\leqslant m\leqslant an}} \left\{\left|\sum_{j=k}^{k+m} X_j\right|\right\} \leqslant a|\mu| \quad w.p.1.$$

PROOF. Apply the triangle inequality to get

$$\left|\sum_{j=k}^{k+m} (X_j - \mu)\right| \leqslant \left|\sum_{j=1}^{k+m} (X_j - \mu)\right| + \left|\sum_{j=1}^{k} (X_j - \mu)\right|$$

so that

$$n^{-1} \max_{\substack{1\leqslant k\leqslant n\\ 1\leqslant m\leqslant an}} \left\{\left|\sum_{j=k}^{k+m} (X_j - \mu)\right|\right\} \leqslant 2n^{-1} \max_{1\leqslant k\leqslant n(1+a)} \left\{\left|\sum_{j=1}^{k} (X_j - \mu)\right|\right\}$$

which converges to 0 as $n \to \infty$ by the FSLLN (Theorem 4). The second conclusion is an easy consequence because for any k and m, with $1 \leqslant m \leqslant na$,

$$\left|\sum_{j=k}^{k+m} X_j\right| - an|\mu| \leqslant \left|\sum_{j=k}^{k+m} (X_j - \mu)\right|. \quad \blacksquare$$

THEOREM 7. *Let $\{X_k\colon k \geqslant 1\}$ be a stationary sequence such that $n^{-1}\Sigma_{k=1}^{n} X_k \to \mu$ w.p.1; let Y be a proper random variable, and let $Y(t)$ be an integer-valued process such that*

$$t^{-1/2}(Y(t) - yt) \Rightarrow Y. \tag{4.1}$$

(a) *If $EX_k = 0$, then $t^{-1/2}(\Sigma_{k=1}^{Y(t)} X_k - \Sigma_{k=1}^{[yt]} X_k) \Rightarrow 0$.*
(b) *If $Z(t)$ is any nonnegative process such that $t^{-1/2}(Y(t) - Z(t)) \Rightarrow 0$, then*

$$t^{-1/2}\left(\sum_{k=1}^{Y(t)} X_k - \sum_{k=1}^{[Z(t)]} X_k\right) \Rightarrow 0.$$

(c) *Without additional assumptions,*

$$t^{-1/2}\left|\sum_{k=1}^{Y(t)+[\gamma t^{1/2}]} X_k - \sum_{k=1}^{Y(t)} X_k - \mu\gamma\right|$$

$$\leqslant t^{-1/2} \max_{1\leqslant m\leqslant([\gamma t^{1/2}]+1)} \left\{\left|\sum_{k=Y(t)+1}^{Y(t)+m} (X_k - \mu)\right|\right\} \Rightarrow 0.$$

REMARK. In Theorem 7(b), we do not assume that $EX_k = 0$. If $EX_k = 0$, then (b) follows from (a).

PROOF. (a) Let $C_t(\epsilon)$ be the event of interest, namely,

$$C_t(\epsilon) = \left\{ t^{-1/2} \left| \sum_{k=1}^{Y(t)} X_k - \sum_{k=1}^{[yt]} X_k \right| > \epsilon \right\}.$$

We shall show that for any positive ϵ and δ there exists t_0 such that $P(C_t(\epsilon)) < \delta$ for $t \geqslant t_0$.

Let ϕ be a strictly positive function of η for $\eta > 0$ such that $P(|Y| > \phi(\eta)/2) < \eta/2$ for all $\eta > 0$, which exists because $P(|Y| < \infty) = 1$. Let $B_t(\eta)$ be the event

$$B_t(\eta) = \left\{ t^{-1/2} |Y(t) - [yt]| \leqslant \phi(\eta) \right\}. \tag{4.2}$$

Thus, for any η, $P(B_t^c(\eta)) < \eta$ for all sufficiently large t, wher $B_t^c(\eta)$ is the complement of $B_t(\eta)$. Next

$$C_t(\epsilon) B_t(\eta) \subseteq \left\{ t^{-1/2} \max_{k \leqslant 2([\phi(\eta) t^{1/2}] + 1)} \left| \sum_{j=0}^{k} X_{[yt] - ([\phi(\eta) t^{1/2} + 1]) + j} \right| > \epsilon/2 \right\}$$

so that, by stationarity,

$$P(C_t(\epsilon) B_t(\eta)) \leqslant P\left(t^{-1/2} \max_{k \leqslant 2([\phi(\eta) t^{1/2}] + 1)} \left| \sum_{j=0}^{k} X_j \right| > \epsilon/2 \right)$$

which converges to 0 as $t \to \infty$ for every positive η by the FSLLN (Theorem 4). For given positive ϵ and δ, first choose $\eta < \delta/2$ and then choose t_0 so that $P(C_t(\epsilon) B_t(\eta)) < \delta/2$ for all $t \geqslant t_0$. Then $P(C_t(\epsilon)) \leqslant P(C_t(\epsilon) B_t(\eta)) + P(B_t^c(\eta)) < \delta$ for all $t \geqslant t_0$.

(b) Again let $C_t(\epsilon)$ be the event of interest, here

$$C_t(\epsilon) = \left\{ t^{-1/2} \left| \sum_{k=1}^{Y(t)} X_k - \sum_{k=1}^{[Z(t)]} X_k \right| > \epsilon \right\};$$

let $B_t(\eta)$ be as in (4.2); and let $D_t(\gamma) = \{ t^{-1/2} |Y(t) - [Z(t)]| \leqslant \gamma \}$. Then

$$C_t(\epsilon) B_t(\eta) D_t(\gamma) \subseteq \left\{ t^{-1/2} \max_{k,m} \left\{ \left| \sum_{j=k}^{k+m} X_{[yt]+j} \right| \right\} > \epsilon \right\}$$

where the maximum is over the set $S(\gamma, \eta, t)$ of indices (k, m) defined by

$$S(\gamma, \eta, t) = \left\{ (k, m) \colon 1 \leqslant m \leqslant \gamma t^{1/2} \text{ and } |k| \leqslant \left(\left[\phi(\eta) t^{1/2} \right] + 1 \right) + \gamma t^{1/2} \right\}.$$

By stationarity,

$$P(C_t(\epsilon) B_t(\eta) D_t(\gamma)) \leqslant P\left(t^{-1/2} \max_{k,m} \left| \sum_{j=k}^{k+m} X_j \right| > \epsilon \right),$$

where the maximum is over the set

$$\hat{S}(\gamma, \eta, t) = \{(k, m): 1 \leqslant m \leqslant \gamma t^{1/2} \text{ and } 1 \leqslant k \leqslant 2[(\phi(\eta) + \gamma)t^{1/2} + 1]\}.$$

Now, for positive ϵ and δ given, choose $\eta < \delta/3$ and $\gamma < \epsilon\mu$. Then choose t_0 so that $P(B_t^c(\eta)) < \eta$, $P(D_t^c(\delta)) < \delta/3$ and $P(C_t(\epsilon)B_t(\eta)D_t(\gamma)) < \delta/3$ for all $t \geqslant t_0$, invoking Lemma 1 for the last inequality. Then, for $t \geqslant t_0$,

$$P(C_t(\epsilon)) \leqslant P(C_t(\epsilon)B_t(\eta)D_t(\gamma)) + P(B_t^c(\eta)) + P(D_t^c(\delta)) < \delta.$$

(c) As before, let $C_t(\epsilon)$ be the event of interest, here

$$C_t(\epsilon) = \left\{ t^{-1/2} \max_{1 \leqslant m \leqslant ([\gamma t^{1/2}]+1)} \left| \sum_{k=Y(t)+1}^{Y(t)+m} (X_k - \mu) \right| > \epsilon \right\},$$

and let $B_t(\eta)$ be as in (4.2). Then

$$C_t(\epsilon)B_t(\eta) \subseteq \left\{ t^{-1/2} \max_{k,m} \left| \sum_{j=k}^{k+m} (X_j - \mu) \right| > \epsilon \right\},$$

where the maximum is over the set

$$T(\gamma, \eta, t) = \{(k, m): 1 \leqslant m \leqslant ([yt^{1/2}] + 1) \text{ and } |k - [yt]| \leqslant \phi(\eta)t^{1/2} + 1\}.$$

By stationarity,

$$P(C_t(\epsilon)B_t(\eta)) \leqslant P\left\{ t^{-1/2} \max_{k,m} \left| \sum_{j=k}^{k+m} (X_j - \mu) \right| > \epsilon \right\},$$

where the maximum is over the set

$$\hat{T}(\gamma, \eta, t) = \{(k, m): 1 \leqslant m \leqslant ([\gamma t^{1/2}] + 1) \text{ and } 1 \leqslant k \leqslant 2([\phi(\eta)t^{1/2} + 1])\}.$$

For positive ϵ and δ given, choose $\eta < \delta/2$ and then t_0 so that $P(C_t(\epsilon)B_t(\eta)) < \delta/2$ for all $t \geqslant t_0$, applying Lemma 1. Then, for all $t \geqslant t_0$, $P(C_t(\epsilon)) \leqslant P(C_t(\epsilon)B_t(\eta)) + P(B_t^c(\eta)) < \delta$. ∎

5. Proofs of Theorems 1–3

PROOF OF THEOREM 1. The finite moment conclusion follows from Theorem 5 in §2. The conditions of Theorem 1 plus the continuous mapping theorem, Theorem 5.1 of [1], immediately yield the CLT

$$t^{-1/2}\left(A_{[\lambda t]} - t, \sum_{k=1}^{[\lambda t]} W_k - \lambda w t, \sum_{k=1}^{[\lambda t]} (W_k - \lambda w U_k) \right) \Rightarrow \lambda^{1/2}(U, W, W - \lambda w U) \tag{5.1}$$

in R^3, i.e., components one, four and five in (1.2). Then the convergence-together theorem (Theorems 4.1 and 4.4 of [1]) combines with Theorem 2 to yield the rest: The second component of (1.2) is covered by (5.1) and (h); the third is covered by the second plus (b); the sixth is covered by (5.1) plus (d) and (g); the seventh is covered by the sixth and (e); finally the eighth is covered by the sixth and (f). The only unused parts of Theorem 2, (a) and (d), are used to prove (h). Part (a) is also used to establish one of the conditions in Theorem 7, which is used for many of the other parts of Theorem 2. Part (a) does not require stationarity. ∎

Theorem 2(a) is covered by Theorem 6 in §3 (without stationarity). The most difficult part of Theorem 2 is (b). Let I be the indicator function, i.e., $I(A) \equiv I(A)(x) = 1$ if $x \in A$ and 0 otherwise.

PROOF OF THEOREM 2(b). $t^{-1/2}(N(t) - O(t)) \Rightarrow 0$. We provide the broad outline of the proof here and the supporting details in following lemmas. By Lemma 3, for any $\gamma > 0$,

$$0 \leqslant N(t) - O(t) \leqslant N(t) - N(t - \gamma t^{1/2}) + \sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2}).$$

For given $\epsilon > 0$, choose $\gamma < \lambda^{-1}\epsilon/2$ and t_0 so that, for all $t \geqslant t_0$,

$$P\left(t^{1/2}\left(N(t) - N(t - \gamma t^{1/2})\right) \leqslant \lambda\gamma\right) > 1 - \epsilon/2,$$

which can be done by Lemma 4, and

$$P\left(t^{-1/2} \sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2}) > \epsilon/2\right) < \epsilon/2$$

by Lemma 5. ∎

As a basis for Lemma 3, we need the following.

LEMMA 2. *If $A_n \leqslant t - \gamma t^{1/2}$ for $\gamma > 0$, then $A_n + \gamma A_n^{1/2} \leqslant t$.*

PROOF. Since the function $x + \gamma x^{1/2}$ is strictly increasing in x, the condition implies that

$$A_n + \gamma A_n^{1/2} \leqslant (t - \gamma t^{1/2}) + \gamma(t - \gamma t^{1/2})^{1/2} \leqslant (t - \gamma t^{1/2}) + \gamma t^{1/2} = t. \quad ∎$$

COROLLARY. $\sum_{n=1}^{N(t)} I(A_n + \gamma A_n^{1/2} \leqslant t) \geqslant N(t - \gamma t^{1/2})$.

PROOF. Apply Lemma 2 term by term, using

$$N(t - \gamma t^{1/2}) = \sum_{n=1}^{N(t)} I(A_n > t - \gamma t^{1/2}). \quad ∎$$

LEMMA 3. *For positive t and γ, $N(t) \geqslant O(t) \geqslant N(t - \gamma t^{1/2}) - \sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2})$.*

PROOF. Note that

$$\begin{aligned} O(t) = \sum_{n=1}^{N(t)} I(D_n \leqslant t) &= \sum_{n=1}^{\infty} I(A_n + W_n \leqslant t) \geqslant \sum_{n=1}^{N(t)} I(A_n + W_n \leqslant t, W_n \leqslant \gamma A_n^{1/2}) \\ &\geqslant \sum_{n=1}^{N(t)} I(A_n + \gamma A_n^{1/2} \leqslant t, W_n \leqslant \gamma A_n^{1/2}) \\ &\geqslant \sum_{n=1}^{N(t)} I(A_n + \gamma A_n^{1/2} \leqslant t) - \sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2}) \\ &\geqslant N(t - \gamma t^{1/2}) - \sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2}), \end{aligned}$$

applying the Corollary to Lemma 2 in the last step. ∎

LEMMA 4. *For any* $\gamma > 0$, $\lim_{t\to\infty} P(t^{-1/2}(N(t) - N(t - \gamma t^{1/2})) > \lambda\gamma) = 0$.

PROOF. Note that for $\delta > 0$

$$\left\{N(t) - N\left(t - \gamma t^{1/2}\right) \geqslant \left[(\lambda\gamma + \delta)t^{1/2}\right]\right\}$$

$$\subseteq \left\{A_{N(t-\gamma t^{1/2})+[(\lambda\gamma+\delta)t^{1/2}]} - A_{N(t-\gamma t^{1/2})+1} \leqslant \gamma t^{1/2}\right\} \quad \text{and}$$

$$t^{-1/2}\left|A_{N(t-\gamma t^{1/2})+[(\lambda\gamma+\delta)t^{1/2}]} - A_{N(t-\gamma t^{1/2})+1} - \lambda^{-1}(\lambda\gamma + \delta)\right| \Rightarrow 0$$

by Theorem 7(c), so that

$$\lim_{t\to\infty} P\left(A_{N(t-\gamma t^{1/2})+[(\lambda\gamma+\delta)t^{1/2}]} - A_{N(t-\gamma t^{1/2})+1} \leqslant \gamma t^{1/2}\right) = 0. \quad \blacksquare$$

LEMMA 5. *For any* $\gamma > 0$, $t^{-1/2}\sum_{n=1}^{N(t)} I(W_n > \gamma A_n^{1/2}) \Rightarrow 0$.

PROOF. It suffices to show that $n^{-1/2}\sum_{k=1}^{n} I(W_k > \gamma A_k^{1/2}) \Rightarrow 0$ because, for any $\epsilon > 0$,

$$P\left(t^{-1/2}\sum_{k=1}^{N(t)} I\left(W_k > \gamma A_k^{1/2}\right) > \epsilon\right)$$

$$\leqslant P\left(t^{-1/2}\sum_{k=1}^{N(t)} I\left(W_k > \gamma A_k^{1/2}\right) > \epsilon, N(t) \leqslant 2\lambda t\right) + P\left(N(t) > 2\lambda t\right)$$

$$\leqslant P\left(t^{-1/2}\sum_{k=1}^{[2\lambda t]} I\left(W_k > \gamma A_k^{1/2}\right) > \epsilon\right) + P\left(t^{-1}N(t) > 2\lambda\right)$$

and $P(t^{-1}N(t) > 2\lambda) \Rightarrow 0$ because $N(t)$ satisfies the WLLN with limit λ as a consequence of Theorem 6. Next, for $\delta > 0$,

$$\begin{aligned} n^{-1/2}\sum_{k=1}^{n} I\left(W_k > \gamma A_k^{1/2}\right) &\leqslant n^{-1/2}\sum_{k=1}^{n} I\left(W_k > \gamma A_k^{1/2}, A_k > (\lambda^{-1} - \delta)k\right) \\ &\quad + n^{-1/2}\sum_{k=1}^{n} I\left(A_k \leqslant (\lambda^{-1} - \delta)k\right) \\ &\leqslant n^{-1/2}\sum_{k=1}^{n} I\left(W_k > \gamma(\lambda^{-1} - \delta)^{1/2}k^{1/2}\right) \\ &\quad + n^{-1/2}\sum_{k=1}^{n} I\left(A_k \leqslant (\lambda^{-1} - \delta)k\right). \end{aligned} \tag{5.1}$$

The first term on the right in (5.1) is asymptotically negligible by Lemma 6 below. The second term is asymptotically negligible too because, by the SLLN (Theorem 5),

$n^{-1}A_n \to \lambda^{-1}$ w.p.1, so that

$$\sum_{k=1}^{\infty} I\big(A_k \leqslant (\lambda^{-1} - \delta)k\big) < \infty \quad \text{w.p.1.}$$

and thus $n^{-1}\sum_{k=1}^{\infty} I(A_k \leqslant (\lambda^{-1} - \delta)k) \Rightarrow 0$. ■

LEMMA 6. *For any* $\gamma > 0$, $\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{n} P(W_k > \gamma k^{1/2}) = 0$.

PROOF. Since $n^{-1/2}\sum_{k=1}^{n} P(W_k > \gamma k^{1/2}) = \sum_{k=1}^{\infty} a_{kn} k^{-1/2} P(W_k > \gamma k^{1/2})$ where $a_{kn} = (k/n)^{1/2}$ for $1 \leqslant k \leqslant n$ and 0 otherwise, so that $|a_{kn}| \leqslant 1$ and $a_{kn} \to 0$ as $n \to \infty$ for each k, to establish the desired limit it suffices (as a consequence of the dominated convergence theorem) to show that $\sum_{k=1}^{\infty} k^{-1/2} P(W_k > \gamma k^{1/2}) < \infty$. By stationarity, this is equivalent to $\sum_{k=1}^{\infty} k^{-1/2} P(W_1^2 > \gamma^2 k) < \infty$, which in turn is equivalent to $E(W_1/\gamma) < \infty$ (Example 5, p. 44, of [4]). However, by Theorem 5, $EW_1 < \infty$. ■

PROOF OF THEOREM 2(c). First apply Theorem 5 to show that $\{U_n\}$ obeys the SLLN with $EU_n = \lambda^{-1} < \infty$. Then apply Theorem 2(a) to verify condition (4.1). Finally, apply Theorem 7(a). ■

PROOF OF THEOREM 2(d). Apply Theorem 7(a) again. To verify the conditions of Theorem 7, apply Theorem 2(a) for (4.1) and Theorem 5 to establish that $\{W_k\}$ and $\{U_k\}$ each obey a SLLN with $EW_k = w < \infty$ and $EU_k = \lambda^{-1} < \infty$. Then $\{W_k - \lambda w U_k\}$ is stationary and obeys a SLLN with $E(W_k - \lambda w U_k) = 0$. ■

PROOF OF THEOREM 2(e). Apply Theorem 7(b) after applying Theorem 2(b) and Theorem 5 to establish the conditions there. ■

PROOF OF THEOREM 2(f). Apply Theorem 1 of [7] to get $\sum_{k=1}^{Q(t)} W_k \leqslant \int_0^t Q(s)\,ds \leqslant \sum_{k=1}^{N(t)} W_k$ for all $t \geqslant 0$, and then apply Theorem 2(e). ■

PROOF OF THEOREM 2(g). Note that $t - A_{N(t)} \leqslant A_{N(t)+1} - A_{N(t)}$ and apply Theorem 7(b). Apply Theorem 2(a) and Theorem 5 to establish the conditions there. ■

PROOF OF THEOREM 2(h). Note that

$$\begin{aligned}\big(A_{[\lambda t]} - t\big) - \big(t - \lambda^{-1}N(t)\big) &= \sum_{k=1}^{[\lambda t]} \big(U_k - \lambda^{-1}\big) + \big(\lambda^{-1}[\lambda t] - t\big)\\ &\quad - \sum_{k=1}^{N(t)} \big(U_k - \lambda^{-1}\big) + A_{N(t)} - t,\end{aligned}$$

so that

$$\begin{aligned}&t^{-1/2}\Big|\big(A_{[\lambda t]} - t\big) - \big(t - \lambda^{-1}N(t)\big)\Big|\\ &\leqslant t^{-1/2}\left|\sum_{k=1}^{N(t)} \big(U_k - \lambda^{-1}\big) - \sum_{k=1}^{[\lambda t]} \big(U_k - \lambda^{-1}\big)\right|\\ &\quad + t^{-1/2}\big|\lambda^{-1}[\lambda t] - t\big| + t^{-1/2}\big|A_{N(t)} - t\big|.\end{aligned}$$

The first term goes to 0 by Theorem 2(c), the second trivially, and the third by Theorem 2(g). ■

PROOF OF THEOREM 3. By Theorem 4.4 of [1], it suffices to treat the marginals separately. As indicated in §3, the WLLN for $N(t)$ is elementary: For $\epsilon > 0$,

$$P\big(t^{-1}N(t) \geq \lambda + \epsilon\big) = P\big(A_{[(\lambda+\epsilon)t]} \leq t\big) \to 0 \quad \text{and}$$

$$P\big(t^{-1}N(t) < \lambda - \epsilon\big) = P\big(A_{[(\lambda-\epsilon)t]} > t\big) \to 0$$

as $t \to \infty$. Turning to the second component, suppose that $w < \infty$ and let $\epsilon > 0$ be given. Then

$$\begin{aligned}(5.2)\quad & P\left(\left|t^{-1}\sum_{k=1}^{N(t)} W_k - \lambda w\right| > \epsilon\right)\\ &\leq P\left(\left|t^{-1}\sum_{k=1}^{N(t)} W_k - \lambda w\right| > \epsilon, |N(t) - \lambda t| \leq \eta t\right)\\ &\quad + P\big(|N(t) - \lambda t| > \eta t\big)\\ &\leq P\left(t^{-1}\sum_{k=1}^{[\lambda t + \eta t]+1} W_k > \lambda w + \epsilon\right) + P\left(t^{-1}\sum_{k=1}^{[\lambda t - \eta t]} W_k > \lambda w - \epsilon\right)\\ &\quad + P\big(|N(t) - \lambda t| > \eta t\big).\end{aligned}$$

Choose $\eta = \epsilon/2w$ and let $t \to \infty$. The case $w = \infty$ is an easy modification.

For $\int_0^t Q(s)\,ds$, it suffices to prove that $t^{-1}\sum_{k=1}^{O(t)} W_k \Rightarrow \lambda w$, by the inequality used in the proof of Theorem 2(f). Since the convergence for the random sum $\sum_{k=1}^{N(t)} W_k$ just proved in (5.2) depends on $N(t)$ only through the weak convergence $t^{-1}(N(t) \Rightarrow \lambda$, we can apply that argument again and complete the proof if we can shown that $t^{-1}O(t) \Rightarrow \lambda$. To this end, note that

$$P\big(O(t) < (\lambda - \epsilon)t\big) = P\left(\sum_{k=1}^{N(t)} I(A_k + W_k \leq t) < (\lambda - \epsilon)t\right),$$

$$\begin{aligned} I(A_k + W_k \leq t) &\geq I(A_k + W_k \leq t, W_k \leq \eta A_k) \geq I(A_k + \eta A_k \leq t, W_k \leq \eta A_k)\\ &\geq I\big(A_k \leq t/(1+\eta)\big) - I(W_k > \eta A_k), \quad \text{and}\end{aligned}$$

$$\sum_{k=1}^{N(t)} I\big(A_k \leq t/(1+\eta)\big) = N\big(t/(1+\eta)\big),$$

so that

(5.3)

$$\begin{aligned} P\big(O(t) < (\lambda - \epsilon)t\big) &\leq P\left(N\big(t/(1+\eta)\big) - \sum_{k=1}^{N(t)} I(W_k > \eta A_k) < (\lambda - \epsilon)t\right)\\ &\leq P\big(N(t/(1+\eta)) < t[\lambda - \epsilon/2]\big)\\ &\quad + P\left(\sum_{k=1}^{N(t)} I(W_k > \eta A_k) > t\epsilon/2\right).\end{aligned}$$

Choose η sufficiently small so that $\lambda/(1+\eta) > \lambda - \epsilon/2$ and the first term on the right in (5.3) is asymptotically negligible as $t \to \infty$. We complete the proof by showing that the second term converges to 0 as $t \to \infty$ for any positive ϵ and η. Note that

$$(5.4) \qquad P\left(t^{-1}\sum_{k=1}^{N(t)} I(W_k > \eta A_k) > \epsilon\right) \leqslant P\left(t^{-1}\sum_{k=1}^{[(\lambda+\eta)t]} I(W_k > \eta A_k) > \epsilon\right) + P(N(t) > (\lambda+\eta)t).$$

We have already shown that the second term on the right in (5.4) is asymptotically negligible, so that it suffices to show that $n^{-1}\sum_{k=1}^{n} I(W_k > \eta A_k) \Rightarrow 0$. Since

$$n^{-1}W_n = n^{-1}\sum_{k=1}^{n} W_k - ((n-1)/n)(n-1)^{-1}\sum_{k=1}^{n-1} W_k \Rightarrow w - w = 0,$$

$W_n/A_n \Rightarrow 0$, so that $P(W_n > \eta A_n) \to 0$, which implies that $n^{-1}\sum_{k=1}^{n} P(W_k > \eta A_k) \Rightarrow 0$. ■

6. Examples. We conclude with four examples that help place our results in perspective.

EXAMPLE 1. We show that the CLT does not imply either the SLLN or the FWLLN (and thus also not a FCLT). Let $\{Y_n: n \geqslant 1\}$ be a sequence of i.i.d. nonnegative random variables and let $X_{2n-1} = Y_n$ and $X_{2n} = -Y_n$ for $n \geqslant 1$. Then the associated partial sums are $S_{2n-1} = Y_n$ and $S_{2n} = 0$. Since $n^{-1/2}Y_n \Rightarrow 0$, $\{X_n\}$ obeys the CLT with nonrandom limit, i.e., $P(Z=0)=1$. On the other hand, if $EY = \infty$, then (p. 42 of [4]) $\sum_{n=1}^{\infty} P(Y \geqslant n) = \sum_{n=1}^{\infty} P(Y_n \geqslant n) = \infty$, so that by Borel-Cantelli (p. 76 of [4]), $P(Y_n \geqslant n$ infinitely often$) = 1$ and $n^{-1}S_n$ fails to converge w.p.1.; i.e., the SLLN does not hold. Moreover, it is easy to modify the construction so that the basic sequence $\{X_n\}$ is stationary: just let

$$P(X_{2n-1} = Y_n = -X_{2n} \text{ for all } n) = P(X_{2n-1} = -Y_n = -X_{2n} \text{ for all } n) = 1/2.$$

By Birkhoff's ergodic theorem (Chapter 6 of [2]), then $EY = +\infty$ above is necessary to get nonconvergence of $\{n^{-1}S_n\}$ w.p.1.

Since the CLT holds with $n^{-1/2}S_n \Rightarrow 0$, if the FWLLN held, it must be with $\mu = 0$. To show that the FWLLN need not hold, it suffices to show that we need *not* have $n^{-1}\max_{1\leqslant k\leqslant n}\{S_k\} \Rightarrow 0$. To see this, we specify the distribution of Y_n in more detail, let $P(Y_n = 2^k) = 2^{-k}$ for all positive integers k. Then $P(Y_n \geqslant k) \geqslant 1/k$ for all k, so that, for any $\epsilon > 0$,

$$P\left((2n)^{-1}\max_{k\leqslant 2n}\{S_k\} \geqslant \epsilon\right) = P\left(\max_{k\leqslant n}\{Y_k\} \geqslant 2n\epsilon\right) = 1 - P(Y_1 < 2n\epsilon)^n$$
$$\geqslant 1 - \left(1 - \frac{1}{2n\epsilon}\right)^n \to 1 - e^{-1/2\epsilon} \quad \text{as } n \to \infty,$$

which implies that the FWLLN does not hold. ■

EXAMPLE 2. We now show that a FCLT does not imply a SLLN. Let $\{Y_n: n \geqslant 1\}$ be a sequence of independent random variables, which for most values of n assume the value 0 w.p.1. Let $\{n_k: k \geqslant 1\}$ be a rapidly increasing sequence of indices for which Y_n has a different distribution; in particular, assume that $n_{k+1} > n_k > 1$ and $n_{k+1}/n_k^2 \to \infty$ as $k \to \infty$. Let $Y_n = 0$ w.p.1. for n not in the subsequence $\{n_k\}$ and let

$P(Y_{n_k} = n_k) = k^{-1} = 1 - P(Y_{n_k} = 0)$. Now let the basic sequence $\{X_n\}$ be defined in terms of $\{Y_n\}$ as in Example 1. Since $\Sigma_{k=1}^{\infty} P(Y_{n_k} = n_k) = \infty$, $P(Y_{n_k} = n_k$ infinitely often$) = 1$ by Borel-Cantelli, so that $\{n^{-1}S_n\}$ fails to converge w.p.1. In fact, the set of limit points for $\{n^{-1}S_n\}$ is the two-point set $\{0, 1\}$ w.p.1. On the other hand,

$$n^{-1/2} \max_{1 \leq j \leq n} \{|S_j|\} = n^{1/2} \max_{1 \leq j \leq n/2} \{Y_j\} \leq n_k^{-1/2} \max_{1 \leq j \leq n_k} \{Y_j\} \quad \text{for } n_k \leq n/2 < n_{k+1}$$

$$\leq n_k^{-1/2} n_{k-1} + n_k^{-1/2} Y_{n_k} \Rightarrow 0$$

by the growth condition on $\{n_k\}$ and the distribution of Y_{n_k}. Hence, the FCLT holds: $\mathbf{S}_n \Rightarrow \mathbf{S}$ for $\mathbf{S}_n$ in (2.1) with $\mu = 0$ and $\mathbf{S} = \theta$, where $\theta(t) = 0$, $t \geq 0$. ∎

EXAMPLE 3. We now construct a stationary sequence of *nonnegative* random variables $\{X_n: n \geq 1\}$ obeying a CLT, but not a FCLT. This shows that the conditions of Theorem 1 do not imply a corresponding FCLT, so that Theorem 1 cannot be deduced from [7]. Without loss of generality, we can extend any single-ended stationary sequence $\{X_n: n \geq 1\}$ to a double-ended stationary sequence $\{X_n: -\infty < n < +\infty\}$; see p. 105 of [2]. We do this below. Let $\{\tau_k: k \geq 1\}$ be a sequence of independent random variables with $P(\tau_k = j) = n_k^{-2}$, $1 \leq j \leq n_k^2$, where $\{n_k: k \geq 1\}$ is a rapidly increasing sequence of positive integers, to be specified in more detail below. Let $\{Y_{k,j}: -\infty < j < \infty\}$ be defined by

$$(6.1) \qquad Y_{k,j} = \begin{cases} n_k, & j = mn_k^2 + \tau_k, \\ 0, & mn_k^2 + \tau_k + 1 \leq j \leq mn_k^2 + n_k(2^k - 1), \\ 2^{-k}, & \text{otherwise,} \end{cases}$$

for all integers m and $k \geq 1$. To have (6.1) well defined, we require that $\{n_k\}$ be an increasing sequence satisfying $n_k 2^k < n_k^2$ or, equivalently, $n_k 2^{-k} > 1$ for all k. Since the τ_k variables are independent, $\{Y_{k,j}: -\infty < j < \infty\}$ are independent sequences for different k. For each k, the sequence $\{Y_{k,j}\}$ is made up of deterministic cycles of length n_k^2. The discrete uniform distribution for τ_k provides the proper initialization to make $\{Y_{k,j}: -\infty < j < \infty\}$ defined in (6.1) a double-ended stationary sequence for each k. Note that $\Sigma_{j=1}^{n} Y_{kj} = n2^{-k}$ provided that $Y_{k1} = Y_{kn} = 2^{-k}$, which will occur with high probability for large k.

Let the basic sequence $\{X_n: n \geq 1\}$ be defined by $X_n = \Sigma_{k=1}^{\infty} Y_{k,n}$ for $n \geq 1$. It is easy to see that $\{X_n\}$ is stationary. For the remainder of the construction we require that n_k increase rapidly enough so that

$$(6.2) \qquad \sum_{j=k}^{\infty} n_j^{-1} 2^j \to 0 \quad \text{and} \quad n_k^{-1}\left(\sum_{j=1}^{k-1} n_j\right) \to 0 \quad \text{as } k \to \infty.$$

For any n given, let $k \equiv k(n)$ be such that $n_k^2 \leq n < n_{k+1}^2$. Let $S_n = X_1 + \cdots + X_n$ and note that

$$A_n \equiv \left\{|S_n - n| > \sum_{j=1}^{k(n)-1} n_j\right\} \subseteq \left\{\sum_{j=1}^{n} Y_{k,j} \neq n2^{-k} \text{ for some } k \geq k(n)\right\}$$

$$\subseteq \left\{Y_{k,1} \neq 2^{-k} \text{ or } Y_{k,n} \neq 2^{-k} \text{ for some } k \geq k(n)\right\},$$

so that, by (6.2),

$$P(A_n) \leqslant P\left(\sum_{j=1}^{n} Y_{k,j} \neq n2^{-k} \text{ for some } k \geqslant k(n) \right) \tag{6.3}$$

$$\leqslant \sum_{j=k(n)}^{\infty} P\left(Y_{j,1} \neq 2^{-j} \text{ or } Y_{j,n} \neq 2^{-j} \right)$$

$$\leqslant 2 \sum_{j=k(n)}^{\infty} n_j^{-1} 2^j \to 0 \quad \text{as } n \to \infty.$$

We want to show that $P(B_{n\epsilon}) \to 0$ for each $\epsilon > 0$, where $B_{n\epsilon} = \{n^{-1/2}|S_n - n| > \epsilon\}$. To this end, note that

$$P(B_{n\epsilon}) \leqslant P(A_n) + P(B_{n\epsilon}A_n^c) \leqslant 2 \sum_{j=k(n)}^{\infty} n_j^{-1} 2^j + P(B_{n\epsilon}A_n^c).$$

However, on A_n^c,

$$n^{-1/2}|S_n - n| \leqslant n_{k(n)}^{-1}|S_n - n| \leqslant n_{k(n)}^{-1} \sum_{j=1}^{k(n)-1} n_j \to 0 \quad \text{as } n \to \infty$$

by (6.2). Hence, for any ϵ, there exists an n_0 such that $B_{n\epsilon}A_n^c = \varnothing$ for $n \geqslant n_0$, so that indeed $P(B_{n\epsilon}) \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

We have shown that $n^{-1/2}(S_n - n) \Rightarrow 0$. It follows that $n^{-1/2}(S_{[nt]} - [nt]) \Rightarrow 0$ as $n \to \infty$ for each individual t. If a FCLT held as well, then we would have weak convergence in $C[0,1]$ with the topology of uniform convergence, which would imply that $n^{-1/2} \max\{S_j - j\colon 1 \leqslant j \leqslant n\} \Rightarrow 0$, by the continuous mapping theorem, but we do not, as we now show.

We exploit the fact that $0 \leqslant \Sigma_{j=1}^{n} Y_{k,j} - n2^{-k} \leqslant n_k$ for all n, with equality holding at both bounds at least once in every segment $n_0 \leqslant n \leqslant n_0 + n_k^2$, provided that $Y_{k,1} \neq 0$, by virtue of (6.1). Let C and C_k be the random sets $C = \{i\colon Y_{i,1} \neq 2^{-i}\}$ and $C_k = \{j\colon\ j \neq k, Y_{j,1} \neq 2^{-j}\}$. Let $Z = \Sigma_{j \in C} n_j$ and $Z_k = \Sigma_{j \in C_k} n_j$. Since $C_k \subseteq C$ w.p.1, $Z_k < Z < \infty$ w.p.1. For $k \notin C$, $\Sigma_{j=1}^{n} Y_{k,j} \geqslant n2^{-k}$ for all n, so that $S_n - n + Z \geqslant 0$ for all n. On the other hand, $Y_{k,j} = n_k$ for some n in $\{j\colon 1 \leqslant j \leqslant n_k^2\}$ w.p.1. Hence,

$$\left\{ \max\left\{ S_j - j\colon 1 \leqslant j \leqslant n_k^2 \right\} > n_k/2 \right\} \supseteq \left\{ Z < n_k/4, Y_{k,1} = 2^{-k} \right\}$$

$$= \left\{ Z_k < n_k/4, Y_{k,1} = 2^{-k} \right\}.$$

Since $\{Y_{k,j}\colon -\infty < j < \infty\}$ is independent of Z_k, we can write

$$P\left(\max\left\{ S_j - j\colon 1 \leqslant j \leqslant n_k^2 \right\} > n_k/2 \right) \geqslant P(Z_k < n_k/4) P\left(Y_{k,1} = 2^{-k} \right)$$

$$\geqslant P(Z < n_k/4) P\left(Y_{k,1} = 2^{-k} \right)$$

which converges to 1 as $k \to \infty$. As a consequence,

$$P\left(n_k^{-1} \max\left\{ S_j - j\colon 1 \leqslant j \leqslant n_k^2 \right\} > 1/2 \right) \to 1 \quad \text{as } k \to \infty. \quad \blacksquare$$

REMARK. Since the FCLT does not hold in Example 3, the various mixing conditions that imply the FCLT, such as the ϕ-mixing condition in Theorem 20.1 of [1], must fail. In fact, mixing fails with a vengeance. For example, let $n_k = 2^{2^k}$, so that the conditions of (6.2) are satisfied and we can identify whether or not $Y_{k,j} = n_k$ by looking only at X_j. For the event $E_j = \{Y_{k,j} = n_k\}$, obviously $E_{j+mn_k^2} = E_j$ for all integers m. Hence,

$$|P(E_j \cap E_{j+mn_k^2}) - P(E_j)P(E_{j+mn_k^2})| = P(E_j) - P(E_j)^2 > 0$$

for all m, so that $\phi(n)$ fails to converge to 0 as $n \to \infty$. ∎

EXAMPLE 4. We now show that Theorem 1 does not hold if we drop the nonnegativity and stationarity assumptions. This reveals limitations of the ordinary CLT framework, because the FCLT for the random sum $\Sigma_{k=1}^{N(t)} W_k$ in [7] holds without these conditions. Let $\{N(t), t \geq 0\}$ be a Poisson process with mean 1. Let $W_k = 1$ for all k, except certain special k depending on $N(t)$. In particular, let

$$(6.4) \qquad W_{N(2^n)} = 1 + 2^{n+1}, \quad W_{N(2^n)+1} = 1 - 2^{n+1}, \qquad n \geq 1,$$

provided that $N(2^{n+1}) > N(2^n) + 1$, which occurs all but finitely often by Borel-Cantelli: $P(N(2^{n+1}) \leq N(2^n) + 1) = P(N(2^n) \leq 1)$ and $\Sigma_{n=1}^{\infty} P(N(2^n) \leq 1) < \infty$. In the exceptional case, let $W_{N(2^n)} = W_{N(2^n)+1} = 1$. Let B_N be the random subset of unusual indices, i.e.,

$$B_N = \{k\colon N(2^n) = k \text{ and } N(2^{n+1}) > N(2^n) + 1, \text{ for some } n,\ n = 1, 2, \ldots\}.$$

In Lemma 7 below we show that $\lim_{k \to \infty} P(k \in B_N) = 0$, which implies that the joint CLT (1.1) is valid with the limits U being $N(0,1)$ and $W = 0$. To see this, note that

$$P\left(\sum_{j=1}^{k} W_j - k \neq 0\right) = P(k \in B_N) \to 0 \quad \text{as } k \to \infty.$$

However, $P(\Sigma_{k=1}^{N(2^n)} W_k - 2^n \geq 2^n) \to 1$, so that the CLT for $\Sigma_{k=1}^{N(t)} W_k$, the sixth component in (1.2), fails. Of course, (6.4) also causes the FCLT for $\{W_k\}$ to be invalid. (This is necessary by Theorem 17.1 of [1].) This is easy to see because

$$2^{-n/2} \max_{m \leq 2^n} \left\{\sum_{k=1}^{m} (W_k - k)\right\} \geq 2^{-n/2} 2^n = 2^{n/2} \quad \text{provided } N(2^{n-1}) < 2^n.$$

Hence, $n^{-1/2} \max_{m \leq n} \{\Sigma_{j=1}^{m} (W_j - j)\} \to \infty$ w.p.1 as $n \to \infty$. If the FCLT held, then the limit would have to be 0, by the continuous mapping theorem. ∎

LEMMA 7. *With definition* (6.4), $P(k \in B_N) \to 0$ *as* $k \to \infty$.

PROOF. By the SLLN, for any $\epsilon > 0$, there exists an m such that $|N(2^n) - 2^n| < \epsilon 2^n$ for all $n \geq m$. We will show that for all ϵ sufficiently small, there exists at most one $n \equiv n(k, \epsilon) \geq m$ such that $N(2^n) = k$. To see this, note that we must have $|k - 2^n| < \epsilon 2^n$, which is equivalent to

$$(6.5) \qquad [\log k - \log(1 + \epsilon)]/\log 2 < n < [\log k - \log(1 - \epsilon)]/\log 2,$$

so that it suffices to choose ϵ sufficiently small so that $[-\log(1 - \epsilon) + \log(1 + \epsilon)]/\log 2 < 1$ or, equivalently, so that $(1 + \epsilon)/(1 - \epsilon) < 2$. We suppose that such an ϵ has been selected. Then

$$\begin{aligned}
P(k \in B_N) &\leqslant P(N(2^n) = k \text{ for some } n) \\
&\leqslant P\big(N(2^n) = k, |2^{-n}N(2^n) - 1| < \epsilon \text{ for some } n \geqslant m\big) \\
&\quad + P(N(2^n) = k \text{ for some } n < m) \\
&\quad + P\big(|2^{-n}N(2^n) - 1| \geqslant \epsilon \text{ for some } n \geqslant m\big) \\
&\leqslant P\big(N(2^{n(k)}) = k\big) + \sum_{j=1}^{m} P\big(N(2^j) = k\big) \\
&\quad + P\big(|2^{-n}N(2^n) - 1| > \epsilon \text{ for some } n \geqslant m\big) \\
&\leqslant \sup_{j \geqslant 1} P\big(N(2^{n(k)}) = j\big) + \sum_{j=1}^{m} P\big(N(2^j) = k\big) \\
&\quad + P\big(|2^{-n}N(2^n) - 1| > \epsilon \text{ for some } n \geqslant m\big).
\end{aligned}$$

First let $k \to \infty$ with m fixed to get the first two terms to converge to zero. Then let $m \to \infty$ to get the last term to converge to zero, invoking the SLLN. ∎

References

[1] Billingsley, P. (1968). *Convergence of Probability Measures,* Wiley, New York.
[2] Breiman, L. (1968). *Probability.* Addison-Wesley, Reading, MA.
[3] Brumelle, S. B. (1971). On the Relation between Customer and Time Averages in Queues. *J. Appl. Probab.* **8** 508–520.
[4] Chung, K. L. (1974). *A Course in Probability Theory.* Second Ed., Academic Press, New York.
[5] Feller, W. (1968). *An Introduction to Probability Theory and Its Applications,* Vol. I. Third Ed., Wiley, New York.
[6] Franken, P., König, D., Arndt, U., and Schmidt, V. (1981). *Queues and Point Processes.* Akademie-Verlag, Berlin.
[7] Glynn, P. W. and Whitt, W. (1986). A Central-Limit-Theorem Version of $L = \lambda W$. *Queueing Systems* **1** 191–215.
[8] ______ and ______ (1987). Sufficient Conditions for Functional-Limit-Theorem versions of $L = \lambda W$. *Queueing Systems* **1** 279–287.
[9] ______ and ______ (1988). Indirect Estimation via $L = \lambda W$. *Oper. Res.* (to appear).
[10] ______ and ______ (1988). An LIL Version of $L = \lambda W$. *Math. Oper. Res.* **4** 693–710.
[11] ______ and ______ (1988). Extensions of the Queueing Relations $L = \lambda W$ and $H = \lambda G$. *Oper. Res.* (to appear).
[12] Little, J. D. C. (1961). A Proof of the Queueing Formula: $L = \lambda W$. *Oper. Res.* **9** 383–387.
[13] Serfozo, R. F. (1975). Functional Limit Theorems for Stochastic Processes Based on Embedded Processes. *Adv. in Appl. Probab.* **7** 123–139.
[14] Stidham, S., Jr. (1974). A Last Word on $L = \lambda W$. *Oper. Res.* **22** 417–421.
[15] ______ (1982). Sample-Path Analysis of Queues. In *Applied Probability-Computer Science: The Interface,* R. L. Disney and T. J. Ott (Eds.), Birkhäuser, Boston.
[16] Whitt, W. (1980). Some Useful Functions for Functional Limit Theorems. *Math. Oper. Res.* **5** 67–85.

GLYNN: DEPARTMENT OF OPERATIONS RESEARCH, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305

WHITT: ROOM 2C-178, AT&T BELL LABORATORIES, MURRAY HILL, NEW JERSEY 07974