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THE RENEWAL-PROCESS STATIONARY-EXCESS OPERATOR

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Abstract

This paper describes the operator mapping a renewal-interval distribution into its associated stationary-excess distribution. This operator is monotone for some kinds of stochastic order, but not for the usual stochastic order determined by the expected value of all non-decreasing functions. Conditions for a renewal-interval distribution to be larger or smaller than its associated stationary-excess distribution for several kinds of stochastic order are determined in terms of familiar notions of ageing. Convergence results are also obtained for successive iterates of the operator, which supplement Harkness and Shantaram (1969), (1972) and van Beek and Braat (1973).

RENEWAL THEORY; STATIONARY POINT PROCESS; PALM THEORY; STOCHASTIC MONOTONICITY; STOCHASTIC ORDER; MONOTONE LIKELIHOOD RATIO; FAILURE RATE; ITERATED OVERSHOOT DISTRIBUTIONS

1. Introduction

For any c.d.f. (cumulative distribution function) F on $[0,\infty)$ with kth moment μ_k, the associated stationary-excess c.d.f. F_e is defined by

$$(1.1)\qquad F_e(t)=\mu_1^{-1}\int_0^t [1-F(s)]ds, \qquad t\geqq 0,$$

and the moments of F_e and F are related by

$$(1.2)\qquad \mu_k(F_e)=\mu_{k+1}(F)/\mu_1(F)(k+1);$$

see p. 64 of Cox (1972). If F is the c.d.f of an interval between points in a renewal process, then F_e is the c.d.f. of the interval to the next point from an arbitrary time in equilibrium. More generally, for stationary point processes, F and F_e are related by the basic one-to-one correspondence associated with the Palm theory; see §7 of Daley and Vere-Jones (1972), Jagers (1973) and Port and Stone (1977). The interval to the first point in a stationary counting process has c.d.f. F_e if and only if an interval in the associated stationary interval sequence has c.d.f. F. As a

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consequence of this Palm correspondence, the stationary-excess distribution often arises in applications.

In this paper we view the stationary-excess distribution as the image of an operator defined on the space of c.d.f.'s and we describe basic properties of this operator. We first consider c.d.f.'s with p.m.f.'s (probability mass functions) $p \equiv \{p_n, n \geqq 1\}$ defined on the positive integers. However, for p.m.f.'s there are actually two different associated stationary-excess distributions. There is the continuous-time version defined by (1.1) and the discrete version to be defined below, in which the associated stationary counting process has support on the integers.

Associated with each p.m.f. p is a *tail sequence* $t(p)$, defined by

$$t(p)_n \equiv \bar{p}_n = \sum_{k=n}^{\infty} p_k, \qquad n \geqq 1, \tag{1.3}$$

a *failure rate sequence* $r(p)$, defined by

$$r(p)_n \equiv r_n = p_n/\bar{p}_n, \qquad n \geqq 1, \tag{1.4}$$

a *moment sequence* $\mu(p)$, defined by

$$\mu(p)_n \equiv \mu_n = \sum_{k=1}^{\infty} k^n p_k, \qquad n \geqq 1, \tag{1.5}$$

and a (discrete) *stationary-excess p.m.f.* $S(p) \equiv p^*$, defined by

$$S(p)_n \equiv p_n^* = t(p)_n/\mu(p)_1 = \bar{p}_n \Big/ \sum_{k=1}^{\infty} \bar{p}_k, \qquad n \geqq 1. \tag{1.6}$$

Let $\bar{p}_n^*$, r_n^* and μ_n^* be the tail probabilities, failure rates and moments, respectively, of $p^* \equiv S(p)$. Obviously $S(p)$ in (1.6) is different from (1.1) because F_e in (1.1) is absolutely continuous with respect to Lebesgue measure on $[0, \infty)$, whereas $S(p)$ in (1.6) is a probability mass function, i.e., the associated c.d.f. is absolutely continuous with respect to the integer counting measure. In this paper we are primarily concerned with the operator S. The word 'operator' is appropriate since p.m.f.'s are mapped into p.m.f.'s. The absolute continuity is important because we use properties of the p.m.f. beyond its c.d.f., e.g., failure rates. With minor modifications, the results also hold for the continuous-time operator in (1.1) applied to absolutely continuous c.d.f.'s F with a density f; see Section 5. For more on discrete life distributions, see Langberg et al. (1980) and Whitt (1983a).

We give definitions and conditions under which (i) $p \leqq S(p)$, (ii) $p^1 \leqq p^2$ implies $S(p^1) \leqq S(p^2)$, and (iii) $\{S^n(p), n \geqq 0\}$ converges. Here the superscript i in p^i indexes the p.m.f. and the superscript n in $S^n(p)$ refers to the nth iterate of the operator S.

We discuss successive iterates of S primarily to gain insight into the effect of applying the operator S once. However, to see how iterates of the stationary-excess operator might arise, consider a sequence of i.i.d. stationary point processes with stationary-excess distribution p^*. Let a new process be formed by observing one stationary point process until a point occurs in it, then observing a second stationary point process until a point occurs in it, and so forth. This new process is a renewal process having p^* as its renewal-interval c.d.f. One could then repeat this procedure. Successive iterates might arise in nested testing or maintenance schemes. Previous work on successive iterates of the continuous stationary-excess operator is contained in Harkness and Shantaram (1969), Shantaram and Harkness (1972), and van Beek and Braat (1972).

A concrete application to queues of the stochastic comparison results here for discrete stationary-excess operators is contained in Whitt (1983a). There it is shown for a large class of queueing systems in which customers arrive in batches that the delay distribution of the last customer in a batch to enter service is a function of the batch-size distribution whereas the delay distribution of an arbitrary customer is the same function of the associated batch-size stationary-excess distribution. Results here help relate the delays experienced by messages and packets in complicated communication systems in which messages are divided into packets for transmission.

Another application to queues is contained in Whitt (1983b), where stochastic comparison results are obtained for the $M/G/s$ queue in light traffic. In light traffic, the conditional delay given that a customer must wait can be expressed simply in terms of the service-time stationary-excess distribution (using (1.1)). Hence, stochastic orderings between two service-time distributions will imply corresponding stochastic orderings for the conditional delays when the operator (1.1) is monotone. The results here for the discrete stationary-excess operator S in (1.6) yield corresponding results for discrete queues with geometric interarrival times.

Considerable insight about the operator S can be obtained by looking at the failure rates. From (1.4) and (1.6) we obtain

$$r_n^* = \sum_{k=n}^{\infty} r_k p_{nk}^*, \qquad n \geqq 1, \tag{1.7}$$

where

$$p_{nk}^* = p_k^*/\bar{p}_n^* = \bar{p}_k \Big/ \sum_{j=n}^{\infty} \bar{p}_j. \tag{1.8}$$

Notice that $\{p_{nk}^*, k \geqq n\}$ is a p.m.f. for each n, so that r_n^* is a weighted average of r_k for $k \geqq n$. Thus we see that S tends to be a smoothing operator which makes $S(p)$ more like the tail of p. Moreover, the weights p_{nk}^* are decreasing in k for each n and strictly positive for all k such that $\bar{p}_k > 0$. We apply (1.7) in Example 3.1 and Theorem 4.3.

Considerable insight about the continuous-time stationary-excess operator in (1.1) can be obtained from the moment relationship in (1.2). For example, on the space of normalized moment-ratio sequences $\{\mu_{k+1}/\mu_k(k+1), k \geqq 0\}$ with $\mu_0 = 1$, the operator in (1.1) acts as a simple shift. As a consequence, the first moment of the nth iterate of the continuous-time stationary-excess operator is just $\mu_{n+1}/\mu_n(n+1)$. Unfortunately, there is not such a simple story with the discrete stationary-excess operator in (1.6). Using the identity

$$(1.9) \qquad \sum_{n=1}^{\infty} n^k p_n = \int_0^{\infty} kt^{k-1}[1-F(t)]dt = \sum_{n=1}^{\infty} t(p)_n(n^k-(n-1)^k),$$

we see that

$$(1.10) \qquad \mu_n = \mu_1 \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \mu^*_{n-j}, \qquad n \geqq 1;$$

e.g., $\mu^*_0 = 1$ and $\mu^*_1 = (\mu_2 + \mu_1)/2\mu_1$. To obtain useful relationships in the discrete case, it is helpful to work with binomial moments, but we do not pursue this here.

The rest of this paper is organized as follows. In Section 2 we define several stochastic order relations and compare them. In Section 3 we use these stochastic order relations to make stochastic comparisons involving the operator S. For example, we show that S is monotone for some definitions of stochastic order, but not for the usual definition characterized by the expectation of all non-decreasing functions. In Section 4 we briefly discuss convergence properties of successive iterates of S. In Sections 2–4 we restrict attention to the discrete case. We conclude in Section 5 by discussing analogues for the continuous-time case based on (1.1).

While we believe that we have provided several interesting new results, some results here are not new; these are presented to provide a comprehensive unified account.

2. Several stochastic order relations

In this section we indicate and investigate several different ways to compare p.m.f.'s. We indicate when one order relation is stronger than another by writing $\leqq_a \rightarrow \leqq_b$; this means that $p^1 \leqq_b p^2$ whenever $p^1 \leqq_a p^2$. One sequence of real numbers $a^1 = \{a^1_n, n \geqq 1\}$ is said to be less than or equal to another a^2 in the *usual sense*, and we write $a^1 \leqq a^2$, if $a^1_n \leqq a^2_n$ for all $n \geqq 1$. We shall also use another order for sequences of non-negative numbers involving ratios. For this definition, we need to be careful about the *support* of such a sequence, defined as

$$s(a)=\{n : a_n > 0\}. \tag{2.1}$$

Let $A \geqq B$ for sets if $x \geqq y$ for all $x \in A$ and $y \in B$.

Definition 2.1. A sequence of non-negative numbers a^1 is less than or equal to another a^2 in the *ratio sense*, and we write $a^1 \leqq_r a^2$, if

$$s(a^2)-s(a^1) \geqq s(a^1) \quad \text{and} \quad s(a^2) \geqq s(a^1)-s(a^2),$$

and

$$a^1_n a^2_m \leqq a^1_m a^2_n$$

whenever $m < n$.

We consider six kinds of stochastic order for p.m.f.'s: monotone likelihood ratio order ($\leqq_{\mathrm{r}}$), failure rate order ($\leqq_{\mathrm{f}}$), ordinary stochastic order ($\leqq_{\mathrm{st}}$), increasing convex order ($\leqq_{\mathrm{ic}}$), moment ratio order ($\leqq_{\mathrm{mr}}$), and moment order ($\leqq_{\mathrm{m}}$). Monotone likelihood ratio order $\leqq_{\mathrm{r}}$ has been defined above; we are now just restricting attention to p.m.f.'s. The other definitions can all be expressed in terms of the operators t and μ defined in (1.3) and (1.5) and the order relations $\leqq$ and $\leqq_{\mathrm{r}}$.

Definition 2.2. Two p.m.f.'s p^1 and p^2 are ordered:
(i) $p^1 \leqq_{\mathrm{f}} p^2$ if $t(p^1) \leqq_{\mathrm{r}} t(p^2)$;
(ii) $p^1 \leqq_{\mathrm{st}} p^2$ if $t(p^1) \leqq t(p^2)$;
(iii) $p^1 \leqq_{\mathrm{ic}} p^2$ if $t(t(p^1)) \leqq t(t(p^2))$;
(iv) $p^1 \leqq_{\mathrm{mr}} p^2$ if $\mu(p^1) \leqq_{\mathrm{r}} \mu(p^2)$;
(v) $p^1 \leqq_{\mathrm{m}} p^2$ if $\mu(p^1) \leqq \mu(p^2)$.

Moment ratio order (iv) seems to be new, but the others are not. Moment ratio order is interesting because of the way the definition parallels failure rate order $\leqq_{\mathrm{f}}$. Also the moment ratios are interesting for the continuous-time operator in (1.1) because the first moment of the nth iterate is $\mu_{n+1}/\mu_n(n+1)$. From (1.2), it is immediate that the continuous-time operator is monotone in $\leqq_{\mathrm{mr}}$; see Theorem 5.1. We now show that $\leqq_{\mathrm{mr}}$ order for p.m.f.'s is related to the other orderings in a useful way.

The following implications hold among these orderings:

$$\begin{array}{ccccc} & & \nearrow \leqq_{\mathrm{mr}} \searrow & & \\ \leqq_{\mathrm{r}} \rightarrow & \leqq_{\mathrm{f}} & & \leqq_{\mathrm{m}} . & \\ & & \searrow \leqq_{\mathrm{st}} \rightarrow \leqq_{\mathrm{ic}} \nearrow & & \end{array} \tag{2.2}$$

We show that $\leqq_{\mathrm{f}} \rightarrow \leqq_{\mathrm{mr}}$, $\leqq_{\mathrm{st}} \not\rightarrow \leqq_{\mathrm{mr}}$, and $\leqq_{\mathrm{mr}} \not\rightarrow \leqq_{\mathrm{ic}}$ below; the other implications are known. Monotone likelihood ratio (failure rate) order is equivalent to uniform conditional stochastic order conditioning on all intervals (semi-infinite intervals of the form $[t, \infty)$); see Keilson and Sumita (1983) and Whitt (1980), (1982). Since $\bar{p}_{n+1}/\bar{p}_n = 1 - r_n$, failure rate order $p^1 \leqq_f p^2$ holds if and only if $r(p^1) \geqq r(p^2)$; see Pinedo and Ross (1980), p. 1251. Stochastic (increasing convex) order holds if and only if $\Sigma_{k=1}^{\infty} a_n p_n^1 \leqq \Sigma_{k=1}^{\infty} a_n p_n^2$ for all non-decreasing (non-decreasing and convex) sequences $\{a_n\}$; see Stoyan (1977), (1983).

To establish $\leqq_f \rightarrow \leqq_{mr}$, we need two lemmas.

Lemma 2.1. If $a^1 \leqq_r a^2$ and $b^1 \leqq_r b^2$ for non-negative sequences, then

$$\left(\sum_{n=1}^{\infty} a_n^1 b_n^2\right)\left(\sum_{n=1}^{\infty} a_n^2 b_n^1\right) \leqq \left(\sum_{n=1}^{\infty} a_n^2 b_n^2\right)\left(\sum_{n=1}^{\infty} a_n^1 b_n^1\right).$$

Proof. Note that

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n^2 b_n^2\right)&\left(\sum_{m=1}^{\infty} a_m^1 b_m^1\right) - \left(\sum_{n=1}^{\infty} a_n^2 b_n^1\right)\left(\sum_{m=1}^{\infty} a_m^1 b_m^2\right) \\ &= \sum_{\substack{m,n=1 \\ m<n}}^{\infty} (a_n^2 b_n^2 a_m^1 b_m^1 + a_m^2 b_m^2 a_n^1 b_n^1 - a_n^2 b_n^1 a_m^1 b_m^2 - a_m^2 b_m^1 a_n^1 b_n^2) \\ &= \sum_{\substack{m,n=1 \\ m<n}}^{\infty} (a_n^2 a_m^1 - a_n^1 a_m^2)(b_n^2 b_m^1 - b_n^1 b_m^2) \geqq 0\end{aligned}$$

because $a_n^2 a_m^1 \geqq a_n^1 a_m^2$ and $b_n^2 b_m^1 \geqq b_n^1 b_m^2$ for $m < n$ by the assumed $\leqq_r$ order.

Lemma 2.2. The function

$$f(x) = (x^{k+1} - (x-1)^{k+1})/(x^k - (x-1)^k)$$

is increasing in x for $x \geqq 1$ and $k \geqq 1$.

Proof. Differentiate $f(x)$ to obtain

$$f'(x) = [k(y_k^2 - y_{k+1}y_{k-1}) + y_k^2]/y_k^2$$

for $y_k = x^k - (x-1)^k$. Then notice that

$$y_k^2 - y_{k+1}y_{k-1} = x^{k-1}(x-1)^{k-1} > 0.$$

Theorem 2.1. $\leqq_f \rightarrow \leqq_{mr}$.

Proof. Using the identity (1.9), we see that $p^1 \leqq_{mr} p^2$ is equivalent to the conclusion of Lemma 2.1 for $a_n^1 = n^k - (n-1)^k$, $a_n^2 = n^{k+1} - (n-1)^{k+1}$ and $b_n^i = \bar{p}_n^i$. Assume that $p^1 \leqq_f p^2$, so that $b^1 \leqq_r b^2$. By Lemma 2.2, $a^1 \leqq_r a^2$, so that the condition of Lemma 2.1 is satisfied.

Example 2.1. To see that $p^1 \leqq_{st} p^2$ does not imply $p^1 \leqq_{mr} p^2$, let $p_1^2 = p_2^2 = p_{10}^2 = p_{10}^1 = 1 - p_1^1 = 1/3$. Then $p^1 \leqq_{st} p^2$, but $\mu_2^1/\mu_1^1 = 102/12 > \mu_2^2/\mu_1^2 = 105/13$.

Example 2.2. To see that $p^1 \leqq_{mr} p^2$ does not imply $p^1 \leqq_{ic} p^2$, let $p_{n_0}^2 = 1 - p_1^2 = 1/2$ and $p_{n_0-1}^1 = 1$ for some $n_0 \geqq 4$. It is easy to check that $p^1 \leqq_{mr} p^2$, but

$$\sum_{k=n_0-2}^{\infty} \bar{p}_k^1 = 2 > 3/2 = \sum_{k=n_0-2}^{\infty} \bar{p}_k^2.$$

3. Stochastic comparisons

In this section we investigate stochastic comparisons involving the stationary-excess operator S using the stochastic order relations defined in Section 2. We first investigate when $p \leqq S(p)$ and then we investigate when S is a monotone operator.

Recall that a p.m.f. p is DFR (IFR), i.e., has decreasing (increasing) failure rate, if r_n is non-increasing (non-decreasing); p is IMRL (DMRL), i.e., has increasing (decreasing) mean residual life, if $\Sigma_{k=n}^{\infty} \bar{p}_k / \bar{p}_n$ is non-decreasing (non-increasing); DFR (IFR) implies IMRL (DMRL), but not conversely; p is IMRL (DMRL) if and only if $S(p)$ is DFR (IFR); see Barlow and Proschan (1975) and Brown (1980). A p.m.f. p is NBUE (NWUE), i.e., new better (worse) than used in expectation if

$$\sum_{k=1}^{\infty} \bar{p}_k \geqq (\leqq) \sum_{k=n}^{\infty} \bar{p}_k / \bar{p}_n, \qquad n \geqq 2.$$

We now give conditions for p to be less than or equal to $S(p) \equiv p^*$; obviously analogous results hold for $\geqq$.

Theorem 3.1. The following characterize when $p \leqq S(p)$:

(i) $p \leqq_r S(p)$ if and only if p is DFR;

(ii) $p \leqq_f S(p)$ if and only if p is IMRL (or $S(p)$ is DFR);

(iii) $p \leqq_{st} S(p)$ if and only if p is NWUE (or $r_n^* \leqq r_1$, $n \geqq 2$);

(iv) $p \leqq_{ic} S(p)$ if and only if p new is worse than $S(p)$ used in expectation, i.e., if

$$\sum_{k=1}^{\infty} \bar{p}_k \leqq \sum_{k=n}^{\infty} \bar{p}_k^* / \bar{p}_n^*, \qquad n \geqq 1;$$

(v) If $S(p)$ is NWUE and $\mu_1 \leqq \mu_1^*$, then $p \leqq_{ic} S(p)$.

Proof. (i) through (iv) follow easily from the definition of $S(p)$. For (v), since $S(p)$ is NWUE,

$$\sum_{k=n}^{\infty} \bar{p}_k^* \geqq (\mu_1^* / \mu_1) \sum_{k=n}^{\infty} \bar{p}_k.$$

Remark 3.1. Theorem 3.1 (iii) is well known and has often been used in queueing; see Whitt (1983a) and references there.

Example 3.1. We show that it is possible for p to be NWUE while $S(p)$ is not. Let $r_n = x$ for $n = 1$ and $n \geqq 3$; let $r_2 = 0$. Then, by (1.7), $r_n^* = x$, $n \geqq 3$, and $0 < r_n^* < x$ for $n = 1, 2$. Since $r_n^* \leqq r_1$ for $n \geqq 1$, p is NWUE by Theorem 3.1 (iii). Let r_n^{**} be the failure rate of $S^2(p)$. By (1.7), $r_n^{**} = x$ for $n \geqq 3$. Thus, $r_n^{**} > r_1^*$, so $S(p)$ is not NWUE by Theorem 3.1 (iii).

Corollary 3.1. S maps the subsets of (i) DFR, (ii) IMRL, (iii) IFR, and (iv) DMRL p.m.f.'s into themselves.

Proof. (i) Consider the subset for which p is DFR. By Theorem 3.1 (i), $p \leqq_r S(p)$. Since $\leqq_r \rightarrow \leqq_f$, $p \leqq_f S(p)$. By Theorem 3.1 (ii), $S(p)$ is DFR. (ii) For $p' \in \{p : S(p) \text{ is DFR}\}$, $S(p')$ is DFR, so that $S(p') \leqq_r S^2(p')$ by Theorem 3.1 (i). Since $\leqq_r \rightarrow \leqq_f$, $S(p') \leqq_f S^2(p')$ too, so that $S^2(p')$ is DFR, and $S(p') \in \{p : S(p) \text{ is DFR}\}$. (iii), (iv) The reasoning in the other cases is the same.

Corollary 3.2. The sequence $\{S^n(p), n \geqq 0\}$ is increasing (decreasing) in $\leqq_r$ if p is DFR (IFR) and in $\leqq_f$ if $S(p)$ is DFR (IFR).

Corollary 3.3. $S(p) = p$ if and only if $r_k = x$ for some constant x and all $k \geqq 1$ (i.e., p is geometric).

Proof. $p = S(p)$ if and only if $p \leqq_r S(p)$ and $p \geqq_r S(p)$ or, by Theorem 3.1, if and only if p is both DFR and IFR.

We now give conditions for S to be monotone, i.e., for $S(p^1) \leqq S(p^2)$ whenever $p^1 \leqq p^2$, usually using the same ordering in both cases. First note that Example 2.1 shows that $p^1 \leqq_{st} p^2$ does not imply that $\mu(S(p^1))_1 \leqq \mu(S(p^2))_1$, let alone $S(p^1) \leqq_{st} S(p^2)$. As a consequence, monotonicity cannot hold when the ordering is $\leqq_{st}$, $\leqq_{ic}$ or $\leqq_m$ in both places. In contrast with the continuous-time case (Theorem 5.1), monotonicity also need not hold with $\leqq_{mr}$ in both places in the discrete case.

However, the following elementary property implies positive results for $\leqq_r$ and $\leqq_f$.

Theorem 3.2. $p_1 \leqq_f p_2$ if and only if $S(p^1) \leqq_r S(p^2)$.

Proof. For any p, $p^*_{n+1}/p^*_n = \bar{p}_{n+1}/\bar{p}_n$.

Corollary 3.4. S is monotone in $\leqq_r$ and $\leqq_f$.

Paralleling Theorem 3.2, we also have the following elementary but important comparison property involving different orderings on the domain and range of S.

Theorem 3.3. Suppose that $\mu_1^1 = \mu_1^2$. Then $p^1 \leqq_{ic} p^2$ if and only if $S(p^1) \leqq_{st} S(p^2)$.

Proof. This is immediate from the definitions.

We have seen that in general S is not monotone in the orderings $\leqq_{ic}$ *and* $\leqq_{st}$. The following example also rules out various weaker results.

Example 3.2. Here we show that there is no nice subset of p.m.f.'s p^1 such that $p^1 \leqq_{st} p^2$ implies that $S(p^1) \leqq_{ic} S(p^2)$. Assume that $p_j^1 > 0$ *for* $j = k_1, k_2$ *and* k_3, where $k_1 < k_2 < k_3$. Let $p^2_{k_1+1} = p^1_{k_1} + p^1_{k_1+1}$, $p^2_{k_1} = 0$ and $p^2_j = p^1_j$ for all other j, so that $p^1 \leqq_{st} p^2$. However, $\mu(p^1)_1 < \mu(p^2)_1$ so that $t(S(p^1))_j > t(S(p^2))_j$ for all $j \geqq k_3$.

4. Convergence of successive iterates

Let p^n be a p.m.f. for each n. We say $p^n \to p$ if $p_k^n \to p_k$, $k \geqq 1$, and $\Sigma_{k=1}^{\infty} p_k = 1$. We say $p^n \overset{m}{\to} p$ if $p^n \to p$ and $\mu_1^n \equiv \Sigma_{k=1}^{\infty} kp_k^n \to \mu_1 \equiv \Sigma_{k=1}^{\infty} kp_k < \infty$. Let Π denote the set of p.m.f.'s with all moments finite. Let Π_d be the subset of decreasing p.m.f.'s.

Theorem 4.1. S is homeomorphism from $(\Pi, \overset{m}{\to})$ to $(\Pi_d, \to)$.

Proof. It is easy to see that S is one-to-one and onto: $S^{-1}(p^*)_n = (p_n^* - p_{n+1}^*)/p_1^*$. Continuity of S and S^{-1} is easy too.

Theorem 4.2. If $S^n(p) \overset{m}{\to} p'$ as $n \to \infty$, then p' is geometric.

Proof. By Theorem 4.1, if $S^n(p) \overset{m}{\to} p'$ then $S^{n+1}(p) \to S(p') = p'$. Apply Corollary 3.3.

Necessary and sufficient conditions in terms of moments for the convergence of $S^n(p)$ unnormalized have been given by Harkness and Shantaram (1969) in the continuous case based on (1.2). We give an interesting sufficient condition for the discrete case in terms of failure rates.

Theorem 4.3. If $r(p)_k \equiv r_k \to x > 0$ as $k \to \infty$, then $S^n(p) \to p'$ as $n \to \infty$ with $r(p')_k = x$, for all $k \geqq 0$. If $r_k \to 0$ as $k \to \infty$, then $S^n(p)$ diverges to ∞ as $n \to \infty$.

Proof. Suppose $r_k \to x$. Then, for any $\varepsilon > 0$, there is a k_0 such that $|r_k - x| < \varepsilon$ for $k \geqq k_0$. Let r^n be the failure rate sequence of $S^n(p)$, $n \geqq 1$. By (1.7), $|r_k^n - x| < \varepsilon$ for all $k \geqq k_0$ and $n \geqq 1$. Now consider $k_0 - m$, $m = 1, 2, \cdots$. By (1.7), there exists an increasing sequence $\{n_m\}$ such that $|r_{k_0-m}^n - x| < \varepsilon(1 + 2^{-m})$ for all $n \geqq n_m$. We use the fact that p_{nk}^* are strictly positive for all k such that $\bar{p}_k > 0$. Hence, $|r_k^n - x| < 2\varepsilon$ for all $n \geqq n_{k_0}$ and all k. Since ε was arbitrary, $r_k^n \to x$ as $n \to \infty$ for each k.

Example 4.1. It is not necessary to have $r_k = r(p)_k$ converge as $k \to \infty$ in order to have $S^n(p)$ converge as $n \to \infty$, and if r_k converges in some more general sense such as Cesáro the limits may not match. Consider the case of $r_{2k+1} = a$ and $r_{2k+2} = b$, $k \geqq 0$, $0 \leqq a$, $b < 1$. Then $p_{2k+1} = a(1-x)^n$, $p_{2k+2} = b(1-a)(1-x)^n$, $\bar{p}_{2k+1} = (1-x)^k$, $\bar{p}_{2k+2} = (1-a)(1-x)^k$, $k \geqq 0$, where $(1-x) = (1-a)(1-b)$. This implies that $p_{2k+1}^* = x(1-x)^k/(2-a)$, $p_{2k+2}^* = (1-a)x(1-x)^k/(2-a)$, $\bar{p}_{2k+1}^* = (1-x)^k$ and $\bar{p}_{2k+2}^* = (2-a-x)(1-x)^k/(2-a)$. This in turn implies that

$$1 - r_{2k+1}^* = \frac{(2-b)(1-a)}{2-a} = \frac{[1+(1-r_2)](1-r_1)}{1+(1-r_1)}, \qquad k \geqq 0,$$

and

$$1 - r_{2k+2}^* = \frac{(2-a)(1-b)}{(2-b)} = \frac{[1+(1-r_1)](1-r_2)}{1+(1-r_2)}, \qquad k \geqq 0.$$

Note that $(1-r_{2k}^*)(1-r_{2k+1}^*)=(1-r_1)(1-r_2)=1-x$. Moreover, if $a<b$, then $r_1<r_1^*<r_2^*<r_2$, and if $a>b$, then $r_1>r_1^*>r_2^*>r_2$. So $S^n(p)$ converges to p' as $n\to\infty$ and the constant failure rate $r(p')_k$ of the limit must satisfy $(1-r(p')_k)^2=1-x$ or

$$r(p')_k=1-\sqrt{(1-a)(1-b)}.$$

Note that $r(p)_k\to(a+b)/2$ as $k\to\infty$ with Cesáro convergence, but $r(p')_k\neq(a+b)/2$ unless $a=b$.

Example 4.2. On the other hand, it is possible to have the sequence $\{S^n(p)\}$ be uniformly tight, i.e., for any $\varepsilon>0$ there exists an m such that $\Sigma_{k=m}^{\infty}S^n(p)_k<\varepsilon$ for all n, so that every subsequence contains a convergent subsequence, while the sequence $\{S^n(p)\}$ itself fails to converge. It suffices to let the failure rate $r(p)_k$ assume only two values a and b, $0<a<b<1$, with $r(p)_k=a$ for $n_{2m-1}<k\leqq n_{2m}$ and $r(p)_k=b$ otherwise, where the successive intervals $n_k-n_{k-1}\gg n_{k-1}$ for each k. By Corollary 3.4, $p^b\leqq_f S^n(p)\leqq_f p^a$ for each n, where p^a and p^b are the geometric p.m.f.'s with failure rates a and b. Hence, $\{S^n(p)\}$ is tight, but by (1.7) for any k the failure rate $r(S^n(p))_k$ continues to oscillate in the interval (a,b), being arbitrarily close to both a and b infinitely often.

We close this section with an example of S operating on mixtures.

Examples 4.3. Let p^i be geometric with mean μ_1^i, $1\leqq i\leqq m$, and $\mu_1^1\leqq\mu_1^2\leqq\cdots\leqq\mu_1^m$. Then the mixture $\Sigma_{i=1}^m\lambda_i p^i$ with $\lambda_i>0$ and $\Sigma_{i=1}^m\lambda_i=1$ is hypergeometric, denoted by H_m. It is easy to see that

$$S^n(p)=\sum_{i=1}^m\left[\lambda_i(\mu_1^i)^n\Big/\sum_{i=1}^m\lambda_i(\mu_1^i)^n\right]p^i\to p^m$$

as $n\to\infty$. Hence, S maps the class H_m into itself. Since hypergeometrics are DFR, $S^n(p)$ increases in $\leqq_r$ to p^m by Theorem 3.1 (i).

5. The continuous-time stationary-excess operator

With minor modifications, all the results in Sections 2–3 hold for the continuous-time stationary-excess operator in (1.1), say $\hat{S}$, applied to c.d.f.'s F with a density f. Formulas (1.3)–(1.8) have obvious analogues, but (1.9) should be replaced by

$$\int_0^\infty x^k f(x)dx=k\int_0^\infty x^{k-1}[1-F(x)]dx. \tag{5.1}$$

Everything in Section 2 extends, but we must modify the proof of Theorem 2.1 as follows.

Proof of continuous-time analogue of Theorem 2.1. Using identity (5.1), we have $f^1 \leqq_{\text{mr}} f^2$ if and only if

$$\int_0^\infty xg^1(x)dx \Big/ \int_0^\infty xg^2(x)dx \leqq \int_0^\infty g^1(x)dx \Big/ \int_0^\infty g^2(x)dx$$

where $g^i(x) = x^{k-1}[1 - F^i(x)]$, or

$$\int_0^\infty dy \int_0^y [xg^1(x)g^2(y) + yg^1(x)g^2(y)]dx \leqq \int_0^\infty dy \int_0^y [xg^1(y)g^2(x) + yg^1(x)g^2(y)]dx.$$

However,

$$yg^1(y)g^2(x) + xg^1(x)g^2(y) \leqq xg^1(y)g^2(x) + yg^1(x)g^2(y)$$

for each x and y with $x < y$ if $f^1 \leqq_{\text{f}} f^2$ because then $g^2(x)/g^1(x)$ is non-decreasing in x. (We apply the analogue of Lemma 2.1.)

Section 3 also extends to $\hat{S}$ with minor changes. First, the NWUE characterization in Theorem 3.1 (iii) should be changed to $r^*(t) \leqq r(0)$ for $t > 0$. For the continuous-time operator, we can also apply (1.2) to get a new monotonicity result.

Theorem 5.1. If $f^1 \leqq_{\text{mr}} f^2$, then $\hat{S}(f^1) \leqq_{\text{mr}} \hat{S}(f^2)$.

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