Chapter 8

Queueing Networks

8.1. Introduction

This chapter contains proofs omitted from Chapter 14 of the book, with the same title. Section 8.9 also contains supplementary material on the existence of a limiting stationary version for a general reflected process. With the exception of Section 8.9, the section and theorem numbering here parallels Chapter 14, so that the proofs should be easy to find.

Here is how this chapter is organized: We start in Section 8.2 by carefully defining the multidimensional reflection map and establishing its basic properties. Since the definition (Definition 8.2.1) is somewhat abstract, a key property is having the reflection map be well defined; i.e., we show that there exists a unique function satisfying the definition (Theorem 8.2.1). We also provide multiple characterizations of the reflection map, one alternative being as the unique fixed point of an appropriate operator (Theorem 8.2.2), while another is a basic complementarity property (Theorem 8.2.3).

A second key property of the multidimensional reflection map is Lipschitz continuity in the uniform norm on $D([0,T],\mathbb{R}^k)$ (Theorem 8.2.5). We also establish continuity of the multidimensional reflection map as a function of the reflection matrix, again in the uniform topology (Theorems 8.2.8 and 14.2.9 in the book). It is easy to see that the Lipschitz property is inherited when the metric on the domain and range is changed to d_{J_1} (Theorem 8.2.7). However, a corresonding direct extension for the SM_1 metric d_s does not hold. Much of the rest of the chapter is devoted to obtaining positive results for the M_1 topologies.

Section 8.3 provides yet another characterization of the multidimensional reflection map via an associated instantaneous reflection map on $\mathbb{R}^k$.

Sections 8.4 and 8.5 are devoted to obtaining the M_1 continuity results.

In Section 8.4 we establish properties of reflection of parametric representations. We are able to extend Lipschitz and continuity results from the uniform norm to the M_1 metrics when we can show that the reflection of a parametric representation can serve as the parametric representation of the reflected function. The results are somewhat complicated, because this property holds only under certain conditions.

In Sections 8.6 and 8.7, respectively, we apply the previous results to obtain heavy-traffic stochastic-process limits for stochastic fluid networks and conventional queueing networks. In the queueing networks we allow service interruptions. When there are heavy-tailed distributions or rare long service interruptions, the M_1 topologies play a critical role.

In Section 8.8 we consider the two-sided regulator and other reflection maps. The two-sided regulator is used to obtain heavy-traffic limits for single queues with finite waiting space, as considered in Section 2.3 and Chapter 5 of the book. With the scaling, the size of the waiting room is allowed to grow in the limit as the traffic intensity increases, but at a rate such that the limit process involves a two-sided regulator (reflection map) instead of the customary one-sided one. Like the one-sided reflection map, the two-sided regulator is continuous on (D^1, M_1). Moreover, the content portion of the two-sided regulator is Lipschitz, but the two regulator portions (corresponding to the two barriers) are only continuous; they are not Lipschitz.

We also give general conditions for other reflection maps to have M_1 continuity and Lipschitz properties. For these, we require that the limit function to be reflected belong to D_1, the subset of functions with discontinuities in only one coordinate at a time.

In Section 8.9 we show that reflected stochastic processes have proper limiting stationary distributions and proper limiting stationary versions (stochastic-process limits for the entire time-shifted processes) under very general conditions. Our main result, Theorem 8.9.1, establishes such limits for stationary ergodic net-input stochastic processes satisfying a natural drift condition (9.7). It is noteworthy that a proper limit can exist even if there is positive drift in some (but not all) coordinates. Theorem 8.9.1 is limited by having a special initial condition: starting out empty. Much of the rest of Section 8.9 is devoted to obtaining corresponding results for other initial conditions. Theorem 8.9.6 establishes convergence for all proper initial contents when the net input process is also a Lévy process with mutually independent coordinate processes. Theorem 8.9.6 covers limit processes obtained in the heavy-traffic limits for the stochastic fluid networks in Section 14.6 of the book.

8.2. The Multidimensional Reflection Map

We start by giving basic definitions and establishing alternative characterizations. Then we establish continuity and Lipschitz properties.

8.2.1. Definition and Characterization

Let $\mathcal{Q}$ be the set of all reflection matrices, i.e., the set of all column-stochastic matrices Q (with $Q^t_{i,j} \geq 0$ and $\sum_{j=1}^k Q^t_{i,j} \leq 1$) such that $Q^n \to 0$ as $n \to \infty$, where Q^n is the n^{th} power of Q.

Definition 8.2.1. (reflection map) *For any* $x \in D^k \equiv D([0,T], \mathbb{R}^k)$ *and any reflection matrix* $Q \in \mathcal{Q}$*, let the* feasible regulator *set be*

$$\Psi(x) \equiv \{w \in D^k_\uparrow : x + (I - Q)w \geq 0\} \tag{2.1}$$

and let the reflection map *be* $R \equiv (\psi, \phi) : D^k \to D^{2k}$ *with* regulator component

$$y \equiv \psi(x) \equiv \inf \Psi(x) \equiv \inf\{w : w \in \Psi(x)\} \ , \tag{2.2}$$

i.e.,

$$y^i(t) \equiv \inf\{w^i(t) \in \mathbb{R} : w \in \Psi(x)\} \quad \textit{for all} \quad i \quad \textit{and} \quad t \ , \tag{2.3}$$

and content component

$$z \equiv \phi(x) \equiv x + (I - Q)y \ . \tag{2.4}$$

It remains to show that the reflection map is well defined by Definition 8.2.1; i.e., we need to know that the feasible regulator set $\Psi(x)$ is nonempty and that its infimum y (which necessarily is well defined and unique for nonempty $\Psi(x)$) is itself an element of $\Psi(x)$, so that $z \in D^k$ and $z \geq 0$.

To show that $\Psi(x)$ in (2.1) is nonempty, we exploit the well known fact that the matrix $I - Q$ has nonnegative inverse.

Lemma 8.2.1. (nonnegative inverse of reflection matrix) *For all* $Q \in \mathcal{Q}$*,* $I - Q$ *is nonsingular with nonnegative inverse*

$$(I - Q)^{-1} = \sum_{n=0}^{\infty} Q^n \ ,$$

where $Q^0 = I$*.*

Proof. Note that

$$(I-Q)(I+Q+\cdots+Q^{n-1}) = I - Q^n \ . \tag{2.5}$$

Since $Q^n \to 0$ as $n \to \infty$, $I - Q^n \to I$ as $n \to \infty$, where I has determinant 1. Hence, for all sufficiently large n, the left and right sides of (2.5) have nonzero determinant. Since the determinant of the product of two matrices is the product of the determinants, the determinant of $I-Q$ must be nonzero, so that $I-Q$ must be nonsingular. Now multiply both sides of (2.5) by this inverse, which we have shown exists, to obtain

$$I + Q + \cdots + Q^{n-1} = (I-Q)^{-1}(I-Q^n) \ .$$

Since the right side tends to the proper limit $(I-Q)^{-1}$ as $n \to \infty$, so does the left. ∎

The key to showing that the infimum belongs to the feasibility set is a basic result about semicontinuous functions. Recall that a real-valued function x on $[0,T]$ is *upper semicontinuous* at a point t in its domain if

$$\limsup_{t_n \to t} x(t_n) \le x(t)$$

for any sequence $\{t_n\}$ with $t_n \in [0,T]$ and $t_n \to t$ as $n \to \infty$. The function x is upper semicontinuous if it is upper semicontinuous at all arguments t in its domain.

Lemma 8.2.2. (preservation of upper semicontinuity) *Suppose that $\{x_s : s \in S\}$ is a set of upper semicontinuous real-valued function on a subinterval of $\mathbb{R}$. Then the infimum $\underline{x} \equiv \inf\{x_s : s \in S\}$ is also upper semicontinuous.*

Proof. For any t and $\epsilon > 0$ given, we need to find δ such that $\underline{x}(t') \le \underline{x}(t) + \epsilon$ whenever $|t'-t| < \delta$. Since $\underline{x}$ is the infimum, for any t and ϵ, we can find $x \in \{x_s : s \in S\}$ such that $x(t) \le \underline{x}(t) + \epsilon/2$. Since x is upper semicontinuous, there exists δ such that $x(t') \le x(t) + \epsilon/2$ for all t' with $|t - t'| < \delta$. As a consequence,

$$\underline{x}(t') \le x(t') \le x(t) + \epsilon/2 \le \underline{x}(t) + \epsilon$$

whenever $|t - t'| < \delta$. ∎

Recall that $x^{\uparrow} \equiv \sup_{0 \le s \le t} x(s)$, $t \ge 0$, for $x \in D^1$. For $x \equiv (x^1, \ldots, x^k) \in D^k$, let $x^{\uparrow} \equiv ((x^1)^{\uparrow}, \ldots, (x^k)^{\uparrow})$.

Theorem 8.2.1. (existence of the reflection map) *For any* $x \in D^k$ *and* $Q \in \mathcal{Q}$,

$$(I-Q)^{-1}[(-x)^{\uparrow} \vee 0] \in \Psi(x) \ , \tag{2.6}$$

so that $\Psi(x) \neq \phi$,

$$y \equiv \psi(x) \in \Psi(x) \subseteq D_{\uparrow}^k \tag{2.7}$$

for y *in (2.2) and*

$$z \equiv \phi(x) = x + (I-Q)y \geq 0 \ . \tag{2.8}$$

Proof. The proof is in the book. ∎

We now characterize the regulator function $y \equiv \psi(x)$ as the unique fixed point of a mapping $\pi \equiv \pi_{x,Q} : D_{\uparrow}^k \to D_{\uparrow}^k$, defined by

$$\pi(w) = (Qw - x)^{\uparrow} \vee 0 \tag{2.9}$$

for $w \in D_{\uparrow}^k$. For this purpose, we use two elementary lemmas.

Lemma 8.2.3. (feasible regulator set characterization) *The feasible regulator set* $\Psi(x)$ *in* (2.1) *can be characterized by*

$$\Psi(x) = \{w \in D_{\uparrow}^k : w \geq \pi(w)\}$$

for π *in* (2.9).

Proof. The proof is in the book. ∎

Lemma 8.2.4. (closed subset of D) *With the uniform topology on* D, *The feasible regulator set* $\Psi(x)$ *is a closed subset of* $D_{\uparrow}^k$, *while* $D_{\uparrow}^k$ *is a closed subset of* D.

Theorem 8.2.2. (fixed-point characterization) *For each* $Q \in \mathcal{Q}$, *the regulator map* $y \equiv \psi(x) \equiv \psi_Q(x) : D^k \to D_{\uparrow}^k$ *can be characterized as the unique fixed point of the map* $\pi \equiv \pi_{x,Q} : D_{\uparrow}^k \to D_{\uparrow}^k$ *defined in* (2.9).

Proof. The proof is in the book. ∎

Theorem 8.2.3. (complementarity characterization) *A function* y *in the feasible regulator set* $\Psi(x)$ *in* (2.1) *is the infimum* $\psi(x)$ *in* (2.2) *if and only if the pair* (y,z) *for* $z \equiv x + (I-Q)y$ *satisfies the complementarity property*

$$\int_0^{\infty} z^i dy^i = 0, \quad 1 \leq i \leq k \ . \tag{2.10}$$

Proof. The proof is in the book. ∎

8.2.2. Continuity and Lipschitz Properties

We now establish continuity and Lipschitz properties of the reflection map as a function of the function x and the reflection matrix Q. We use the *matrix norm*, defined for any $k \times k$ real matrix A by

$$\|A\| \equiv \max_j \sum_{i=1}^{k} |A_{i,j}| \ . \tag{2.11}$$

We use the maximum column sum in (2.11) because we intend to work with the column-substochastic matrices in $\mathcal{Q}$. Note that

$$\|A_1 A_2\| \le \|A_1\| \cdot \|A_2\|$$

for any two $k \times k$ real matrices A_1 and A_2. Also, using the sum (or l_1) norm

$$\|u\| \equiv \sum_{i=1}^{k} |u^i| \tag{2.12}$$

on $\mathbb{R}^k$, we have

$$\|Au\| \le \|A\| \cdot \|u\| \tag{2.13}$$

for each $k \times k$ real matrix A and $u \in \mathbb{R}^k$. Indeed, we can also define the matrix norm by

$$\|A\| \equiv \max\{\|Au\| : u \in \mathbb{R}^n, \|u\| = 1\} \ , \tag{2.14}$$

using the sum norm in (2.12) in both places on the right. Then (2.11) becomes a consequence. Consistent with (2.12), we let

$$\|x\| \equiv \sup_{0 \le t \le T} \|x(t)\| \equiv \sup_{0 \le t \le T} \sum_{i=1}^{k} \|x^i(t)\| \tag{2.15}$$

for $x \in D([0,T], \mathbb{R}^k)$. Combining (2.13) and (2.15), we have

$$\|Ax\| \le \|A\| \cdot \|x\| \tag{2.16}$$

for each $k \times k$ real matrix A and $x \in D([0,T], \mathbb{R}^k)$.

We use the following basic lemma.

Lemma 8.2.5. (reflection matrix norms) *For any $k \times k$ matrix $Q \in \mathcal{Q}$,*

$$\|Q\| \le 1, \quad \|Q^k\| = \gamma < 1 \tag{2.17}$$

and

$$\|(I - Q)^{-1}\| \le \frac{k}{1 - \gamma} \ . \tag{2.18}$$

Proof. The first relation in (2.17) is immediate. Since $Q^n \to 0$ for all Q in $\mathcal{Q}$, the Markov chain associated with Q^t is transient. Since the Markov chain has k states,

$$\sum_{j=1}^{k} (Q^t_{i,j})^k < 1 \ , \tag{2.19}$$

which, with (2.11), implies the second relation in (2.17). Probabilistically, if the probability of eventually exiting the state space $\{1, \ldots, k\}$ of the Markov chain is 1, then the probability of immediately exiting the state space from some state must be positive. Then the probability of reaching that state or the exterior (leaving the state space) in one step must be positive from some other state. Proceeding on by induction, the state space must be exhausted after k steps, so that (2.19) holds. Finally,

$$\left\| \sum_{n=0}^{\infty} Q^n \right\| \le \sum_{n=0}^{\infty} \|Q^n\| \le \sum_{n=0}^{k-1} \|Q^n\| + \gamma \sum_{n=0}^{\infty} \|Q^n\|$$

so that (2.18) holds. ■

We now show that $\pi \equiv \pi_{x,Q}$ in (2.9) is a k-stage contraction map on $D^k_{\uparrow}$. Recall that for $x \in D$, $|x|$ denotes the function $\{|x(t)| : t \ge 0\}$ in D, where $|x(t)| = (|x^1(t)|, \ldots, |x^k(t)|) \in \mathbb{R}^k$. Thus, for $x \in D$, $|x|^{\uparrow} = (|x^1|^{\uparrow}, \ldots, |x^k|^{\uparrow})$, where $|x^i|^{\uparrow}(t) = \sup_{0 \le s \le t} |x^i(s)|$, $0 \le t \le T$.

Lemma 8.2.6. (π is a k-stage contraction) *For any $Q \in \mathcal{Q}$ and $w_1, w_2 \in D^k_{\uparrow}$,*

$$|\pi^n(w_1) - \pi^n(w_2)|^{\uparrow} \le |Q^n(|w_1 - w_2|^{\uparrow})| \quad \text{for} \quad n \ge 1 \ , \tag{2.20}$$

so that

$$\|\pi^n(w_1) - \pi^n(w_2)\| \le \|Q^n\| \cdot \|w_1 - w_2\| \le \|w_1 - w_2\| \tag{2.21}$$

for $n \ge 1$ and

$$\|\pi^n(w_1) - \pi^n(w_2)\| \le \gamma \|w_1 - w_2\| \quad \text{for} \quad n \ge k \ ,$$

where

$$\|Q^k\| \equiv \gamma < 1 \ .$$

Hence

$$\|\pi^n(w) - \psi(x)\| \to 0 \quad \text{as} \quad n \to \infty \ .$$

Proof. The proof is in the book. ∎

We now establish inequalities that imply that the reflection map is a Lipschitz continuous map on $(D, \|\cdot\|)$. We will use the stronger inequalities themselves in Section 8.9.

Theorem 8.2.4. (one-sided bounds) *For any $Q \in \mathcal{Q}$ and $x_1, x_2 \in D$,*

$$-(I-Q)^{-1}\eta_1(x_1 - x_2) \le \psi(x_1) - \psi(x_2) \le (I-Q)^{-1}\eta_1(x_2 - x_1) \quad (2.22)$$

where $\eta_1(x) \equiv (\hat{\eta}_1(x^1), \ldots, \hat{\eta}_1(x^k))$ with $\hat{\eta}_1 : D^1 \to D^1$ defined by

$$\hat{\eta}_1(x^i) \equiv (x^i)^{\uparrow} \vee 0 \ .$$

Proof. The proof is in the book. ∎

As a direct consequence of Theorem 8.2.4, we obtain the desired Lipschitz property.

Theorem 8.2.5. (Lipschitz property with uniform norm) *For any $Q \in \mathcal{Q}$ and $x_1, x_2 \in D$,*

$$\begin{aligned} \|\psi(x_1) - \psi(x_2)\| &\le \|(I-Q)^{-1}\| \cdot \|x_1 - x_2\| \\ &\le \sum_{n=0}^{\infty} \|Q^n\| \cdot \|x_1 - x_2\| \\ &\le \frac{k}{1-\gamma}\|x_1 - x_2\| \ , \end{aligned} \quad (2.23)$$

where $\gamma \equiv \|Q^k\| < 1$, and

$$\begin{aligned} \|\phi(x_1) - \phi(x_2)\| &\le (1 + \|I-Q\| \cdot \|(I-Q)^{-1}\|)\|x_1 - x_2\| \\ &\le \left(1 + \frac{2k}{1-\gamma}\right)\|x_1 - x_2\| \ . \end{aligned} \quad (2.24)$$

Proof. The proof is in the book. ∎

We now summarize some elementary but important properties of the reflection map.

Theorem 8.2.6. (reflection map properties) *The reflection map satisfies the following properties:*

(i) adaptedness*: For any $x \in D$ and $t \in [0, T]$, $R(x)(t)$ depends upon x only via $\{x(s) : 0 \le s \le t\}$.*

(ii) monotonicity*: If $x_1 \le x_2$ in D, then $\psi(x_1) \ge \psi(x_2)$.*

(iii) rescaling*: For each* $x \in D([0,T],\mathbb{R}^k)$*,* $\eta \in \mathbb{R}^k$*,* $\beta > 0$ *and* γ *nondecreasing right-continuous function mapping* $[0,T_1]$ *into* $[0,T]$*,* $\eta + \beta(x \circ \gamma) \in D([0,T_1],\mathbb{R}^k)$ *and*

$$R(\eta + \beta(x \circ \gamma)) = \beta R(\beta^{-1}\eta + x) \circ \gamma \ .$$

(iv) shift*: For all* $x \in D$ *and* $0 < t_1 < t_2 < T$*,*

$$\psi(x)(t_2) = \psi(x)(t_1) + \psi(\phi(x)(t_1) + x(t_1 + \cdot) - x(t_1))(t_2 - t_1)$$

and

$$\phi(x)(t_2) = \phi(\phi(x)(t_1) + x(t_1 + \cdot) - x(t_1))(t_2 - t_1)$$

(v) continuity preservation*: If* $x \in C$*, then* $R(x) \in C$*.*

We can apply Theorems 8.2.5 and 8.2.6 (iii) to deduce that the reflection map inherits the Lipschitz property on (D, J_1) from (D, U). Unfortunately, we will have to work harder to obtain related results for the M_1 topologies.

Theorem 8.2.7. (Lipschitz property with d_{J_1}) *For any* $Q \in \mathcal{Q}$*, there exist constants* K_1 *and* K_2 *(the same as in Theorem 8.2.5) such that*

$$d_{J_1}(\psi(x_1), \psi(x_2)) \le K_1 d_{J_1}(x_1, x_2) \tag{2.25}$$

and

$$d_{J_1}(\phi(x_1), \phi(x_2)) \le K_2 d_{J_1}(x_1, x_2) \tag{2.26}$$

for all $x_1, x_2 \in D$*.*

Proof. The proof is in the book. ▪

We now want to consider the reflection map R as a function of the reflection matrix Q as well as the net input function x. We first consider the maps $\pi \equiv \pi^n_{x,Q}(0)$ in (2.9) and $\psi \equiv \psi_Q$ in (2.2) as functions of Q when Q is a strict contraction in the matrix norm (2.11), i.e., when $\|Q\| < 1$.

Theorem 8.2.8. (stability bounds for different reflection matrices) *Let* $Q_1, Q_2 \in \mathcal{Q}$ *with* $\|Q_1\| = \gamma_1 < 1$ *and* $\|Q_2\| = \gamma_2 < 1$*. For all* $n \ge 1$*,*

$$\|\pi^n_{x,Q_j}(0)\| \le (1 + \gamma_j + \cdots + \gamma_j^{n-1})\|x\| \tag{2.27}$$

and

$$\|\pi^n_{x,Q_1}(0) - \pi^n_{x,Q_2}(0)\| \le (1 + \gamma_2 + \cdots + \gamma_2^{n-1}) \frac{\|x\| \cdot \|Q_1 - Q_2\|}{1 - \gamma_1} \ , \tag{2.28}$$

so that

$$\|\psi_{Q_j}(x)\| \le \frac{\|x\|}{1-\gamma_j} \tag{2.29}$$

and

$$\|\psi_{Q_1}(x) - \psi_{Q_2}(x)\| \le \frac{\|x\| \cdot \|Q_1 - Q_2\|}{(1-\gamma_1)(1-\gamma_2)} \ . \tag{2.30}$$

Proof. First

$$\|\pi^1_{x,Q_j}(0)\| = \|(-x)^{\uparrow} \vee 0\| \le \|x\| \ .$$

Next, by induction,

$$\begin{aligned} \|\pi^{n+1}_{x,Q_j}(0)\| &= \|(Q_j \pi^n_{x,Q_j}(0) - x)^{\uparrow} \vee 0| \\ &\le \|Q_j\| \cdot \|\pi^n_{x,Q_j}(0)\| + \|x\| \\ &\le \gamma_j(1 + \gamma_j + \cdots + \gamma_j^{n-1})\|x\| + \|x\| \\ &\le (1 + \gamma_j + \cdots + \gamma_j^n)\|x\| \, . \end{aligned}$$

Similarly, by induction

$$\begin{aligned} \|\pi^{n+1}_{x,Q_1}(0) - \pi^{n+1}_{x,Q_2}(0)\| &\le \|Q_1 \pi^n_{x,Q_1}(0) - Q_2 \pi_{x,Q_2}(0)\| \\ &\le \|Q_1 \pi^n_{x,Q_1}(0) - Q_2 \pi^n_{x,Q_1}(0)\| + \|Q_2 \pi^n_{x,Q_1}(0) - Q_2 \pi^n_{x,Q_2}(0)\| \\ &\le \|Q_1 - Q_2\| \cdot \|x\|/(1-\gamma_1) + \|Q_2\| \cdot \|\pi^n_{x,Q_1}(0) - \pi^n_{x,Q_2}(0)\| \\ &\le (1 + \gamma_2 + \cdots + \gamma_2^n)\|Q_1 - Q_2\| \cdot \|x\|/(1-\gamma_1) \ . \end{aligned}$$

Finally, since $\|\pi^n_{x,Q}(0) - \psi_Q(x)\| \to 0$ as $n \to \infty$, the final two bounds (2.29) and (2.30) follow. ∎

Nothin more is omitted from Section 14.2 of the book.

8.3. The Instantaneous Reflection Map

Nothing has been deleted from this section in the book.

8.4. Reflections of Parametric Representations

In order to establish continuity and stronger Lipschitz properties of the reflection map R on D with the M_1 topologies, we would like to have $(R(u), r)$ be a parametric representation of $R(x)$ when (u, r) is a parametric representation of x. That is not always true, but we now obtain positive results in that direction.

Theorem 8.4.1. (reflections of parametric representations) *Suppose that* $x \in D$, $(u,r) \in \Pi_s(x)$ *and* $r^{-1}(t) = [s_-(t), s_+(t)]$.

(a) If $t \in Disc(x)^c$, *then*

$$R(u)(s) = R(x)(t) \quad for \quad s_-(t) \le s \le s_+(t) \ .$$

(b) If $t \in Disc(x)$, *then*

$$R(u)(s_-(t)) = R(x)(t-) \quad and \quad R(u)(s_+(t)) = R(x)(t) \ .$$

(c) If $t \in Disc(x)$ *and* $x(t) \ge x(t-)$, *then*

$$\phi(u)(s) = \phi(x)(t-) + \left(\frac{u^j(s) - u^j(s_-(t))}{u^j(s_+(t)) - u^j(s_-(t))} \right) [x(t) - x(t-)]$$

for any j, $1 \le j \le k$, *and*

$$\psi(u)(s) = \psi(x)(t-) = \psi(x)(t) \quad for \quad s_-(t) \le s \le s_+(t) \ ,$$

so that

$$R(u)(s) \in [R(x)(t-), R(x)(t)] \quad for \quad s_-(t) \le s \le s_+(t) \ .$$

(d) If $t \in Disc(x)$ *and* $x(t) \le x(t-)$, *then* $\phi^i(u)$ *and* $\psi^i(u)$ *are monotone in* $[s_-(t), s_+(t)]$ *for each* i, *so that*

$$R(u)(s) \in [[R(x)(t-), R(x)(t)]] \quad for \quad s_-(t) \le s \le s_+(t) \ .$$

We can draw the desired conclusion that $(R(u), r)$ is a parametric representation of $R(x)$ if we can apply parts (c) and (d) of Theorem 8.4.1 to all jumps. Recall that D_+ (D_s) is the subset of D for which condition (c) (condition (c) or (d)) holds at all discontinuity points of x. For $x \in D_s$, the direction of the inequality is allowed to depend upon t.

Theorem 8.4.2. (preservation of parametric representations under reflection) *Suppose that* $x \in D$ *and* $(u,r) \in \Pi_s(x)$.

(a) If $x \in D_+$, *then* $(R(u), r) \in \Pi_s(R(x))$.

(b) If $x \in D_s$, *then* $(R(u), r) \in \Pi_w(R(x))$.

We also have an analog of Theorems 8.4.1 and 8.4.2 for the case $x \in D_s$ and $(u,r) \in \Pi_w(x)$.

Theorem 8.4.3. (preservation of weak parametric representations) *If* $x \in D_s$ *and* $(u,r) \in \Pi_w(x)$, *then* $(R(u), r) \in \Pi_w(R(x))$.

As a basis for proving Theorem 8.4.1, we exploit piecewise-constant approximations.

Lemma 8.4.1. (left and right limits) *For any* $x \in D_c$, $(u,r) \in \Pi_s(x)$ *and* $r^{-1}(t) = [s_-(t), s_+(t)]$,

$$R(u)(s_-(t)) = R(x)(t-) \quad \textit{and} \quad R(u)(s_+(t)) = R(x)(t) \ . \tag{4.1}$$

In order to prove Lemma 8.4.1, we establish several other lemmas. First, the following property of the reflection map applied to a single jump at time t is an easy consequence of the definition of the reflection map. We consider the reflection map applied to the jump in two parts. Given the linear relationship in (2.4), it suffices to focus on only one of ψ or ϕ.

Lemma 8.4.2. (the case of a single jump) *For any* $b_1, b_2 \in \mathbb{R}^k$, $0 < \beta < 1$ *and* $0 < t \le T$,

$$\phi(b_1+b_2 I_{[t,T]})(u) = \phi(\phi(b_1+\beta b_2 I_{[t,T]})(t)+(1-\beta)b_2 I_{[t,T]})(u) \quad \textit{for} \quad t \le u \le T \ .$$

Lemma 8.4.3. (generalization) *For any* $b_1, b_2 \in \mathbb{R}^k$ *and right-continuous nondecreasing nonnegative real-valued function* α *on* $[0,T]$ *with* $\alpha(0) = 0$,

$$\phi(b_1 + \alpha b_2)(t) = \phi(b_1 + \alpha(t) b_2 I_{[0,T]})(t), \quad 0 \le t \le T \ . \tag{4.2}$$

Proof. Represent α as the uniform limit of nondecreasing nonnegative functions α_n in D_c. Then $\|\phi(b_1 + \alpha_n b_2) - \phi(b_1 + \alpha b_2)\| \to 0$ as $n \to \infty$ by the known continuity of ϕ in the uniform metric. Hence it suffices to assume that $\alpha \in D_c$. We then establish (4.2) by recursively considering the successive discontinuity points of α, using Lemma 8.4.2 and Theorem 8.2.6(iv). ■

Proof of Lemma 8.4.1. Any $x \in D_c$ can be represented as

$$x = \sum_{j=0}^{m} b_j I_{[t_j,T]}$$

for $0 = t_0 < t_1 < \cdots < t_m \le T$ and $b_j \in \mathbb{R}^k$ for $0 \le j \le m$. Thus t_j is the j^{th} discontinuity point of x. Let $[s_-(t_j), s_+(t_j)] = r^{-1}(t_j)$ for each j. Since $(u,r) \in \Pi_s(x)$ instead of just $\Pi_w(x)$, u can be expressed as

$$u = \sum_{j=0}^{m} \alpha_j b_j \ ,$$

where $\alpha_0(s) = 1$ for all s and, for $j \geq 1$, $\alpha_j : [0,1] \to [0,1]$ is continuous and nondecreasing with $\alpha_j(s) = 0$, $s \leq s_-(t_j)$ and $\alpha_j(s) = 1$, $s \geq s_+(t_j)$. We can now consider successive intervals $[s_-(t_j), s_+(t_j)]$ recursively exploiting Lemma 8.4.3. First, for any s with $0 \leq s \leq s_-(t_1)$.

$$\phi(u)(s) = \phi(b_0 I_{[0,1]})(s) = \phi(x)(0) = \phi_0(x(0)) \ .$$

Now assume that (4.1) holds for all $j \leq m-1$ and consider $s \in [s_-(t_m), s_+(t_m)]$. By the induction hypothesis, Lemma 8.4.3 and Theorem 8.2.6(iv),

$$\begin{aligned} \phi(u)(s) &= \phi(\phi(x)(t_{m-1}) + \alpha_m b_m I_{[s_-(t_m),1]})(s) \\ &= \phi(\phi(x)(t_{m-1}) + \alpha_m(s) b_m I_{[s_-(t_m),1]})(s) \ , \end{aligned}$$

so that (4.1) holds for t_m. ∎

Proof of Theorem 8.4.1. (a) Since $t \in Disc(x)^c$, $u(s) = x(t)$ for $s_-(t) \leq s \leq s_+(t)$. Given $x \in D$ with $t \in Disc(x)^c$, it is possible to choose $x_n \in D_c$ such that $t \in Disc(x_n)^c$ for all n and $\|x_n - x\| \to 0$, by a slight strengthening of Theorem 6.2.2 in Section 6.2. By characterization (i) of M_1 convergence in Theorem 6.1 in Section V.6, given $(u, r) \in \Pi_s(x)$, we can find $(u_n, r_n) \in \Pi_s(x_n)$ such that

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \quad \text{as} \quad n \to \infty \ .$$

Since R is continuous in the uniform topology, $\|R(u_n) - R(u)\| \to 0$ and $\|R(x_n) - R(x)\| \to 0$ as $n \to \infty$. Let s_n be such that $r_n(s_n) = t$. Since $x_n \in D_c$ and $t \in Disc(x_n)^c$, $R(u_n)(s_n) = R(x_n)(t)$ by Lemma 8.4.1. Since $0 \leq s_n \leq 1$, $\{s_n\}$ has a convergent subsequence $\{s_{n_k}\}$. Let s' be the limit of that convergent subsequence. Since $r_{n_k}(s_{n_k}) = t$ for all n_k, we necessarily have $s' \in [s_-(t), s_+(t)]$. Since $\|R(u_n) - R(u)\| \to 0$, $R(x_{n_k})(t) = R(u_{n_k})(s_{n_k}) \to R(u)(s')$. Since we have already seen that $R(x_n)(t) \to R(x)(t)$, we must have $R(u)(s') = R(x)(t)$. Since $R(u)$ is constant on $[s_-(t), s_+(t)]$, we must have $R(u)(s) = R(x)(t)$ for all s with $s_-(t) \leq s \leq s_+(t)$.

(b) Since R maps D into D and C into C, $R(x)$ is right-continuous with left limits, while $R(u)$ is continuous. Given $t \in Disc(x)$, we can find $t_n \in Disc(x)^c$ with $t_n \uparrow t$. We can apply part (a) to obtain $R(u)(s_+(t_n)) = R(x)(t_n) \to R(x)(t-)$, but $s_+(t_n) \uparrow s_-(t)$, so that $R(u)(s_+(t_n)) \to R(u)(s_-(t))$. Hence, we have established the first claim: $R(u)(s_-(t)) = R(x)(t-)$. Similarly, we can find $t_n \in Disc(x)^c$ with $t_n \downarrow t$. Then we can apply part (a) again to obtain $R(u)(s_-(t_n)) = R(x)(t_n) \to R(x)(t)$. Since $s_-(t_n) \downarrow s_+(t)$, $R(u)(s_-(t_n)) \downarrow R(u)s_+(t))$. Hence $R(x)(t) = R(u)(s_+(t))$ as claimed.

(c) We can apply Lemma 14.3.4 (a) in the book. Since the increment $x(t) - x(t-)$ is nonnegative in each component,

$$z(t) = z(t-) + x(t) - x(t-)$$

and $y(t) = y(t-)$. Similarly,

$$\phi(u)(s) = \phi(u)(s_-(t)) + u(s) - u(s_-(t))$$

and $\psi(u)(s) = \psi(u)(s_-(t))$ for $s_-(t) \leq s \leq s_+(t)$.

(d) We apply Lemma 14.3.4 (b) in the book. Each coordinate $\phi^i(u)$ and $\psi^i(u)$ is monotone in s over $[s_-(t), s_+(t)]$, so that the desired conclusion holds.

Proof of Theorem 8.4.2. (a) We combine parts (a)–(c) of Theorem 8.4.1 to get $(R(u), r)(s) \in \Gamma_{R(x)}$ for all s. Since R maps C into C, $(R(u), r)$ is continuous. Also r is nondecreasing with $r(0) = 0$ and $r(1) = T$ because $(u, r) \in \Pi_s(x)$. Finally, $(R(u), r)$ maps $[0, 1]$ onto $\Gamma_{R(x)}$ and $(R(u), v)$ is nondecreasing with respect to the order on $\Gamma_{R(x)}$ because the increments of $R(u)$ coincide with the increments of u over each discontinuity in x because $x \in D_+$, and (u, r) has these properties.

(b) We incorporate part (d) of Theorem 8.4.1 to get $R(u)$ monotone over $[s_-(t), s_+(t)] = r^{-1}(t)$ for each $t \in Disc(x) = Disc(R(x))$. This allows us to conclude that $(R(u), r) \in \Pi_w(R(x))$. ■

We now turn to the proof of Theorem 8.4.3. For the proof, we find it convenient to use a different class of approximating functions. Let D_l be the subset of all functions in D that (i) have only finitely many jumps and (ii) are continuous and piecewise linear in between jumps with only finitely many changes of slope. Let $D_{s,l} = D_s \cap D_l$.

Analogous to Theorem 6.2.2 in Section 6.2, we have the following result.

Lemma 8.4.4. (approximation of elements of D_s by elements of $D_{s,l}$) *For any $x \in D_s$, there exist $x_n \in D_{s,l}$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.*

Proof. For $x \in D_s$ and $\epsilon > 0$ given, apply Theorem 6.2.2 in Section 6.2 to find $x_1 \in D_c$ (with only finitely many discontinuities) such that $\|x - x_1\| < \epsilon/4$. The function x_1 can have jumps of opposite sign, but the magnitude of the jumps in one of the two directions must be at most $\epsilon/2$. Form the desired function, say x_2, from x_1. Suppose that $\{t_1, \ldots, t_k\} = Disc(x_1)$. Suppose that x_1 has one or more negative jump at t_j, none of which has magnitude exceeding $\epsilon/2$. If x_1 has a negative jump at t_j in coordinate i for some i, then replace x_1^i over $[t_{j-1}, t_j)$ by the linear function connecting $x_1^i(t_{j-1})$ and $x_1^i(t_j)$. Similarly, if x_1 has one or more positive jumps at some t_j with all magnitudes less than $\epsilon/2$, then proceed as above. It is easy to see that $Disc(x_2) \subseteq Disc(x_1)$, $x_2 \in D_{s,l}$ and $\|x - x_2\| < \epsilon$. ∎

We now show that limits of parametric representations are parametric representations when $\|x_n - x\| \to 0$.

Lemma 8.4.5. (limits of parametric representations) *If (i)* $\|x_n - x\| \to 0$ *as* $n \to \infty$*, (ii)* $(u_n, r_n) \in \Pi_z(x_n)$ *for each* n*, where* $z = s$ *or* w*, and (iii)* $\|u_n - u\| \vee \|r_n - r\| \to 0$ *as* $n \to \infty$ *where* u *and* r *are functions mapping* $[0,1]$ *into* $\mathbb{R}^k$ *and* $\mathbb{R}^1$*, respectively, then* $(u, r) \in \Pi_z(x)$ *for the same* z*.*

Proof. Since (u, r) is the uniform limit of the continuous functions (u_n, r_n), (u, r) is itself continuous. Since r is the limit of the nondecreasing functions r_n, r is itself nondecreasing. Since $r_n(0) = 0$ and $r_n(1) = T$ for all n, $r(0) = 0$ and $r(1) = T$. Since r is also nondecreasing and continuous, r maps $[0,1]$ onto $[0,T]$. Pick any s with $0 < s < 1$. Then $r(s) = t$ for some t, $0 \le t \le T$, and $r_n(s) = t_n \to t$ as $n \to \infty$. Suppose that $(u_n, r_n) \in \Pi_s(x_n)$ for all n. That means that

$$u_n(s) = \alpha_n(s)x_n(t_n) + (1 - \alpha_n(s))x_n(t_n-)$$

for all n. Since $0 \le \alpha_n(s) \le 1$, there exists a convergent subsequence $\{\alpha_{n_k}(s)\}$ such that $\alpha_{n_k}(s) \to \alpha(s)$ as $n_k \to \infty$. At least one of the following three cases must prevail: (i) $t_{n_k} > t$ for infinitely many n_k, (ii) $t_{n_k} = t$ for infinitely many n_k and (iii) $t_{n_k} < t$ for infinitely many n_k. In case (i), we can choose a further subsequence $\{n_{k_j}\}$ so that $u_{n_{k_j}}(s) \to x(t)$; in case (ii), we can choose a further subsequence so that $u_{n_{k_j}}(s) \to \alpha(s)x(t) + [1 - \alpha(s)]x(t-)$; in case (iii) we can choose a further subsequence so that $u_{n_{k_j}}(s) \to x(t-)$. Since $u_n(s) \to u(s)$, the limit of the subsequence must be $u(s)$. Hence, $(u(s), r(s)) \in \Gamma_x$ for each s. Since (u, r) is continuous with $r(0) = 0$ and $r(1) = T$, (u, r) maps $[0,1]$ onto Γ_x. Since (u_n, r_n) is monotone as a function from $[0,1]$ to $(\Gamma_{x_n}, \le)$ and $\|u_n - u\| \vee \|r_n - r\| \to 0$, (u, r) is monotone from $[0,1]$ to $(\Gamma_x, \le)$. Hence, $(u, r) \in \Pi_s(x)$. Finally, suppose that $(u_n, r_n) \in \Pi_w(x_n)$ for all n. By the result above applied to the individual coordinates, $(u^i(s), r(s)) \in \Gamma_{x^i}$ and thus $(u^i, r) \in \Pi_s(x^i)$ for each i, which implies that $(u, r) \in \Pi_w(x)$. ∎

Proof of Theorem 8.4.3. For $x \in D_s$, apply Lemma 8.4.4 to find $x_n \in D_{s,l}$ such that $\|x_n - x\| \to 0$. Suppose that $(u,r) \in \Pi_w(x)$. Then it is possible to find u_n such that $(u_n, r) \in \Pi_w(x_n)$ and $\|u_n - u\| \to 0$. To do so, let $u_n(s_-(t)) = x_n(t-)$ and $u_n(s_+(t)) = x_n(t)$, where $[s_-(t), s_+(t)] = r^{-1}(t)$ for each $t \in Disc(x)$. If $t \in Disc(x_n)^c$, let $u_n(s) = u_n(s_+(t))$ for $s_-(t) \leq s \leq s_+(t)$; if $t \in Disc(x_n)$, define u_n so that $\|u_n - u\| \to 0$. Given that $(u_n, r) \in \Pi_w(x_n)$, we can apply mathematical induction over the finitely many time points such that x_n has a jump or a change of slope to show that $(R(u_n), r) \in \Pi_w(R(x_n))$ for each n. We use Lemma 14.3.4 of the book critically at this point to treat the discontinuity points of x_n in $D_{s,l}$. The continuous linear pieces between discontinuities can be treated by applying the rescaling property in Theorem 8.2.6 (iii) with $\beta = 1$ and $\eta = 0$. Finally, we apply Lemma 8.4.5 to deduce that $(R(u), r) \in \Pi_w(R(x))$. For that, we use the fact that $\|R(x_n) - R(x)\| \to 0$ and $\|R(u_n) - R(u)\| \to 0$.

8.5. M_1 Continuity Results

In this section we establish continuity and Lipschitz properties of the reflection map on $D \equiv D^k \equiv D([0,T], \mathbb{R}^k)$ with the M_1 topologies. Our first result establishes continuity of the reflection map R (for an arbitrary reflection matrix Q) as a map from (D, SM_1) to (D, L_1), where L_1 is the topology on D induced by the L_1 norm

$$\|x\|_{L_1} \equiv \int_0^T \|x(t)\| dt \; . \qquad (5.1)$$

Under a further restriction, the map from (D, WM_1) to (D, WM_1) will be continuous.

Recall that D_s is the subset of functions in D without simultaneous jumps of opposite sign in the coordinate functions; i.e., $x \in D_s$ if, for all $t \in (0,T)$, either $x(t) - x(t-) \leq 0$ or $x(t) - x(t-) \geq 0$, with the sign allowed to depend upon t. The subset D_s is a closed subset of D in the J_1 topology and thus a measurable subset of D with the SM_1 and WM_1 topologies (since the Borel σ-fields coincide). The proofs of the main theorems here appear in Section 6.2 of the Internet Supplement.

Theorem 8.5.1. (continuity with the SM_1 topology on the domain) *Suppose that $x_n \to x$ in (D, SM_1).*

(a) Then

$$R(x_n)(t_n) \to R(x)(t) \quad in \quad \mathbb{R}^{2k} \qquad (5.2)$$

for each $t \in Disc(x)^c$ *and sequence* $\{t_n : n \geq 1\}$ *with* $t_n \to t$,

$$\sup_{n \geq 1} \|R(x_n)\| < \infty \ , \tag{5.3}$$

$$R(x_n) \to R(x) \quad in \quad (D, L_1) \tag{5.4}$$

and

$$\psi(x_n) \to \psi(x) \quad in \quad (D, WM_1) \ . \tag{5.5}$$

(b) If in addition $x \in D_s$, *then*

$$\phi(x_n) \to \phi(x) \quad in \quad (D, WM_1) \ , \tag{5.6}$$

so that

$$R(x_n) \to R(x) \quad in \quad (D, WM_1) \ . \tag{5.7}$$

Proof. (a) We first prove (5.2). Since $x_n \to x$ in (D, SM_1), we can find parametric representations $(u, r) \in \Pi_s(x)$ and $(u_n, r_n) \in \Pi_s(x_n)$ for $n \geq 1$ such that

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \ .$$

By Theorem 14.4.1 (a) in the book, $R(u)(s) = R(x)(t)$ for any $s \in [s_-(t), s_+(t)] \equiv r^{-1}(t)$, since $t \in Disc(x)^c$. Moreover, by Corollary 14.3.4 in the book, $t \in Disc(R(x))^c$. For any sequence $\{t_n : n \geq 1\}$ with $t_n \to t$, we can find another sequence $\{t'_n : n \geq 1\}$ such that $t'_n \to t$, $t'_n \in Disc(x_n)^c$ and $\|R(x_n)(t'_n) - R(x_n)(t_n)\| \to 0$ as $n \to \infty$. (Here we exploit the fact that $R(x_n) \in D$ for each n.) Consequently, $R(x_n)(t_n) \to R(x)(t)$ if and only if $R(x_n)(t'_n) \to R(x)(t)$. By Theorem 13.4.1 (a) again, $R(u_n)(s_n) = R(x)(t'_n)$ for any $s_n \in [s_-(t'_n), s_+(t'_n)] = r_n^{-1}(t'_n)$. Since $0 \leq s_n \leq 1$ for all n, any such sequence $\{s_n : n \geq 1\}$ has a convergent subsequence $\{s_{n_k} : k \geq 1\}$. Suppose that $s_{n_k} \to s'$ as $n_k \to \infty$. Since $t'_n \to t$ as $n \to \infty$ and $t'_{n_k} = r_{n_k}(s_{n_k}) \to r(s')$ as $n_k \to \infty$, we must have $s' \in [s_-(t), s_+(t)]$. Then, since $\|R(u_n) - R(u)\| \to 0$,

$$R(x_{n_k})(t'_{n_k}) = R(u_{n_k})(s_{n_k}) \to R(u)(s') = R(x)(t) \ .$$

Since every subsequence of $\{R(x_n)(t'_n) : n \geq 1\}$ must have a convergent subsequence with the same limit, we must have $R(x_n)(t'_n) \to R(x)(t)$ as $n \to \infty$, which we have shown implies that $R(x_n)(t_n) \to R(x)(t)$ as $n \to \infty$, as claimed in (5.2). Next we establish (5.3). For any $x \in D$, $\|x\| \equiv \sup_{0 \leq t \leq T} \|x(t)\| < \infty$. Since $d_s(x_n, x) \to 0$, $\|x_n\| \to \|x\|$ as $n \to \infty$. Hence, it suffices to show that there is a constant K such that

$$\|R(x)\| \leq K\|x\| \quad \text{for all} \quad x \in D \ ,$$

but that follows from Theorem 13.2.5. We apply the bounded convergence theorem with (5.2) and (5.3) to establish (5.4). We now turn to (5.5). Since $\psi(x_n)$ and $\psi(x)$ are nondecreasing in each coordinate the pointwise convergence established in (5.2) actually implies WM_1 convergence in (5.5); see Corollary 12.5.1 in the book.

(b) First, we use the assumed convergence $x_n \to x$ in (D, SM_1) to pick $(u, r) \in \Pi_s(x)$ and $(u_n, r_n) \in \Pi_s(x_n)$, $n \geq 1$, with

$$\|u_n - u\| \vee \|r_n - r\| \to 0 \ .$$

Since R is continuous on (D, U), we also have $\|R(u_n) - R(u)\| \to 0$. By part (a), we know that there is local uniform convergence of $R(x_n)$ to $R(x)$ at each continuity point of $R(x)$. Thus, by Theorem 12.5.1 (v) in the book, to establish $R(x_n) \to R(x)$ in (D, WM_1), it suffices to show that

$$\lim_{\delta \downarrow 0} \ \overline{\lim_{n \to \infty}} \ w_s(R^i(x_n), t, \delta) = 0 \tag{5.8}$$

for each i, $1 \leq i \leq 2k$, and $t \in Disc(R(x))$, where

$$w_s(x, t, \delta) \equiv \sup\{\|x(t_2) - [x(t_1), x(t_3)]\| : (t_1, t_2, t_3) \in A(t, \delta)\} \tag{5.9}$$

for

$$A(t, \delta) \equiv \{(t_1, t_2, t_3) : (t - \delta) \vee 0 \leq t_1 < t_2 < t_3 \leq (t + \delta) \wedge T\} \ .$$

(Since we are considering the i^{th} coordinate function $R^i(x_n)$, the function x in (5.9) is real-valued here.) Suppose that (5.8) fails for some i and t. Then there exist $\epsilon > 0$ and subsequences $\{\delta_k\}$ and $\{n_k\}$ such that $\delta_k \downarrow 0$, $n_k \to \infty$ and

$$w_s(R^i(x_{n_k}), t, \delta_k) > \epsilon \quad \text{for all} \quad \delta_k \quad \text{and} \quad n_k \ .$$

That is, there exist time points t_{1,n_k}, t_{2,n_k} and t_{3,n_k} with

$$(t - \delta_k) \vee 0 \leq t_{1,n_k} < t_{2,n_k} < t_{3,n_k} \leq (t + \delta_k) \wedge T \tag{5.10}$$

and

$$\|R^i(x_{n_k})(t_{2,n_k}) - [R^i(x_{n_k}(t_{1,n_k}), R^i(x_{n_k}(t_{3,n_k})]\| > \epsilon \ . \tag{5.11}$$

Since the values $R^i(x_{n_k})(t)$ are contained in the values $R^i(u_{n_k})(s)$ where $(u_{n_k}, r_{n_k}) \in \Pi_s(x_{n_k})$, we can deduce that there are points s_{j,n_k} for $j = 1, 2, 3$ such that $0 \leq s_{1,n_k} < s_{2,n_k} < s_{3,n_k} \leq 1$, $r_{n_k}(s_{j,n_k}) = t_{j,n_k}$ for $j = 1, 2, 3$ and all n_k, and

$$\|R^i(u_{n_k})(s_{2,n_k}) - [R^i(u_{n_k})(s_{1,n_k}), R^i(u_{n_k})(s_{3,n_k})]\| > \epsilon \ . \tag{5.12}$$

By (5.10) and (5.12), there then exists a further subsequence $\{n'_k\}$ such that $t_{j,n'_k} \to t$ and $s_{j,n'_k} \to s_j$ as $n'_k \to \infty$ for $j = 1, 2, 3$, where $0 \leq s_1 \leq s_2 \leq s_3 \leq 1$, $r_{n'_k}(s_{j,n'_k}) \to r(s_j) = t$ and

$$\|R^i(u)(s_2) - [R^i(u)(s_1), R^i(u)(s_3)]\| \geq \epsilon > 0 \ . \tag{5.13}$$

However, by Theorem 14.4.2 in the book, $(R(u), r) \in \Pi_w(R(x))$ since $x \in D_s$, so that $(R^i(u), r) \in \Pi_s(R^i(x))$. Hence $(R^i(u), r) \in \Pi_s(R^i(x))$. Since $R^i(u)$ is monotone on $[s_-(t), s_+(t)]$, (5.13) cannot occur. Hence (5.8) must in fact hold and $R^i(x_n) \to R^i(x)$ in (D, M_1). Since that is true for all i, we must have $R(x_n) \to R(x)$ in (D, WM_1). ■

Under the extra condition in part (b), the mode of convergence on the domain actually can be weakened. However, little positive can be said if only $x_n \to x$ in (D, WM_1) without $x \in D_s$; see Example 14.5.3 in the book.

Theorem 8.5.2. (continuity with the WM_1 topology on the domain) *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then* (5.7) *holds.*

The proof of Theorem 8.5.2 is more difficult. We now work towards its proof. By Theorem 8.4.3, R is Lipschitz on (D_s, WM_1), but x_n need not be in D_s. We show that we can approximate x_n by elements of D_s.

We first restate Corollary 12.11.2 in the book as a lemma. It states that Convergence in WM_2, which of course is implied by convergence in WM_1, has the advantage that jumps in the converging functions must be inherited by the limit function.

Lemma 8.5.1. (inheritance of jumps) *If $x_n \to x$ in (D, WM_2), $t_n \to t$ in $[0, T]$ and $x^i_n(t_n) - x^i_n(t_n-) \geq c > 0$ for all n, then $x^i(t) - x^i(t-) \geq c$.*

For $x \in D$ and $t \in Disc(x)$, let $\gamma(x, t)$ be the largest magnitude (absolute value) of the jumps in x at time t of opposite sign to the sign of the largest jump in x at time t. Let $\gamma(x)$ be the maximum of $\gamma(x, t)$ over all $t \in Disc(x)$. We apply Lemma 8.5.1 to establish the next result.

Lemma 8.5.2. *If $x_n \to x$ in (D, WM_1), then*

$$\overline{\lim_{n\to\infty}} \ \gamma(x_n) \leq \gamma(x) \ .$$

We only use the following consequence of Lemma 8.5.2.

Lemma 8.5.3. *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then $\gamma(x_n) \to 0$.*

We also use a generalization of Lemma 8.4.4 above, which is established in the same way.

Lemma 8.5.4. *For any $x \in D$, there exist $x_n \in D_{s,l}$ such that $\|x_n - x\| \to \gamma(x)$ as $n \to \infty$.*

We combine Lemmas 8.5.2 and 8.5.4 to obtain the tool we need.

Lemma 8.5.5. *If $x_n \to x$ in (D, WM_1) and $x \in D_s$, then there exists $x'_n \in D_{s,l}$ for $n \geq 1$ such that $\|x'_n - x_n\| \to 0$.*

Proof of Theorem 8.5.2. Given $x_n \to x$ in (D, WM_1), apply Lemma 8.5.5 to find $x'_n \in D_{s,l}$ for $n \geq 1$ such that $\|x'_n - x_n\| \to 0$ as $n \to \infty$. Then, by the triangle inequality, Theorem 14.2.5 in the book and Lemma 8.5.3 above,

$$\begin{aligned} d_p(R(x_n), R(x)) &\leq d_p(R(x_n), R(x'_n)) + d_p(R(x'_n), R(x)) \\ &\leq \|R(x_n) - R(x'_n)\| + d_w(R(x'_n), R(x)) \\ &\leq K\|x_n - x'_n\| + K d_w(x'_n, x). \end{aligned}$$

Since

$$\begin{aligned} d_p(x'_n, x) &\leq d_p(x'_n, x_n) + d_p(x_n, x) \\ &\leq \|x'_n - x_n\| + d_p(x_n, x) \\ &\to 0\ , \end{aligned}$$

$d_w(x'_n, x) \to 0$. Hence, $d_p(R(x_n), R(x)) \to 0$ as claimed. ∎

Example 12.3.1 in the book shows that convergence $x_n \to x$ can hold in (D, WM_1) but not in (D, SM_1) even when $x \in D_s$. Thus Theorems 8.5.1 (a) and 8.5.2 cover distinct cases. An important special case of both occurs when $x \in D_1$, where D_1 is the subset of x in D with discontinuities in only one coordinate at a time; i.e., $x \in D_1$ if $t \in Disc(x^i)$ for at most one i when $t \in Disc(x)$, with the coordinate i allowed to depend upon t. In Section 6.7 it is shown that WM_1 convergence $x_n \to x$ is equivalent to SM_1 convergence when $x \in D_1$.

Just as with D_s above, D_1 is a closed subset of (D, J_1) and thus a Borel measurable subset of (D, SM_1). Since $D_1 \subseteq D_s$, the following corollary to Theorem 8.5.2 is immediate.

Corollary 8.5.1. (common case for applications) *If $x_n \to x$ in (D, WM_1) and $x \in D_1$, then $R(x_n) \to R(x)$ in (D, WM_1).*

We can obtain stronger Lipschitz properties on special subsets. Let D_+ be the subset of x in D with only nonnegative jumps, i.e., for which $x^i(t) - x^i(t-) \geq 0$ for all i and t. As with D_s and D_1 above, D_+ is a closed subset of (D, J_1) and thus a measurable subset of (D, SM_1).

Theorem 8.5.3. (*Lipschitz properties*) *There is a constant K (the same as associated with the uniform norm from Theorem* 8.2.5) *such that*

$$d_s(R(x_1), R(x_2)) \leq K d_s(x_1, x_2) \tag{5.14}$$

for all $x_1, x_2 \in D_+$, and

$$d_p(R(x_1), R(x_2)) \leq d_w(R(x_1), R(x_2)) \leq K d_w(x_1, x_2) \leq K d_s(x_1, x_2) \tag{5.15}$$

for all $x_1, x_2 \in D_s$.

Proof. Given that $x \in D_+$, apply Theorem 14.4.2 (a) in the book to get $(R(u), r) \in \Pi_s(R(x))$ when $(u, r) \in \Pi_s(x)$. Then

$$\begin{aligned} d_s(R(x_1), R(x_2)) &\equiv \inf_{\substack{(u_i', r_i) \in \Pi_s(R(x_i)) \\ i=1,2}} \{\|u_1' - u_2'\| \vee \|r_1 - r_2\|\} \\ &\leq \inf_{\substack{(u_i, r_i) \in \Pi_s(x_i) \\ i=1,2}} \{\|\phi(u_1) - \phi(u_2)\| \vee \|r_1 - r_2\|\} \\ &\leq \inf_{\substack{(u_i, r_i) \in \Pi_s(x_i) \\ i=1,2}} \{K\|u_1 - u_2\| \vee \|r_1 - r_2\|\} \\ &\leq K d_s(x_1, x_2) \end{aligned}$$

because $K \geq 1$. The other results are obtained in essentially the same way. Apply Theorem 14.4.3 in the book to get $(R(u), r) \in \Pi_w(R(x))$ when $(u, r) \in \Pi_w(x)$ and $x \in D_+$. When $x \in D_s$, apply Theorem 13.4.2 (b) to get $(R(u), r) \in \Pi_w(R(x))$ when $(u, r) \in \Pi_s(x)$. ∎

We can actually do somewhat better than in Theorem 8.5.1 when the limit is in D_+.

Theorem 8.5.4. (strong continuity when the limits is in D_+) *If*

$$x_n \to x \quad \text{in} \quad (D, SM_1) , \tag{5.16}$$

where $x \in D_+$, then

$$R(x_n) \to R(x) \quad \text{in} \quad (D, SM_1) . \tag{5.17}$$

Proof. Suppose that $x_n \to x$ in (D, SM_1). By Theorem 8.5.1(a), we have $\psi(x_n) \to \psi(x)$ in (D, WM_1). Since $x \in D_+$, $\psi(x) \in C$, by Corollary 14.3.5 in the book. Hence the WM_1 convergence is equivalent to uniform convergence; i.e.,

$$\psi(x_n) \to \psi(x) \quad \text{in} \quad D([0,T], \mathbb{R}^k, U) \ .$$

We can then apply addition with equation (14.2.6) in the book to get

$$R(x_n) \to R(x) \quad \text{in} \quad D([0,T], \mathbb{R}^{2k}, SM_1) \ . \quad \blacksquare$$

Our final result shows how the reflection map behaves as a function of the reflection matrix Q, as well as x, with the M_1 topologies.

Theorem 8.5.5. (continuity as a function of (x, Q)) *Suppose that $Q_n \to Q$ in $\mathcal{Q}$.*

(a) If $x_n \to x$ in (D^k, WM_1) and $x \in D_s$, then

$$R_{Q_n}(x_n) \to R_Q(x) \quad \text{in} \quad (D^{2k}, WM_1) \ . \tag{5.18}$$

(b) If $x_n \to x$ in (D^k, SM_1) and $x \in D_+$, then

$$R_{Q_n}(x_n) \to R_Q(x) \quad \text{in} \quad (D^{2k}, SM_1) \ . \tag{5.19}$$

Proof. We only prove the first of the two results, since the two proofs are essentially the same. If $x_n \to x$ in (D, WM_1) with $x \in D_s$, then we can find $x'_n \in D_{s,l}$ for $n \geq 1$ such that $\|x_n - x'_n\| \to 0$ by Lemma 8.5.5. By Theorem 14.2.5 in the book,

$$\|R_{Q_n}(x_n) - R_{Q_n}(x'_n)\| \leq K_n \|x_n - x'_n\| \to 0 \tag{5.20}$$

because $K_n \to K < \infty$. By Theorem 14.4.3 in the book, $(R_Q(u), r) \in \Pi_w(R(x))$ when $x \in D_s$. So, for any $\epsilon > 0$ given, let $(u, r) \in \Pi_w(x)$ and $(u_n, r_n) \in \Pi_w(x'_n)$ such that $\|u_n - u\| \vee \|r_n - r\| \leq \epsilon$. Then $(R_Q(u), r) \in \Pi_w(R_Q(x))$, $(R_{Q_n}(u_n), r_n) \in \Pi_w(R_{Q_n}(x'_n))$ for $n \geq 1$ and

$$\|R_{Q_n}(u_n) - R_Q(u)\| < K(\epsilon + \|Q_n - Q\|) \tag{5.21}$$

by Theorem 14.2.9 and equation (14.2.35) in the book, so that

$$R_{Q_n}(x'_n) \to R_Q(x) \quad \text{in} \quad (D^{2k}, WM_1) \ . \tag{5.22}$$

Combining (5.20), (5.22) and the triangle inequality with the metric d_p, we obtain (5.18). $\blacksquare$

We can apply Section 6.9 to extend the continuity and Lipschitz results to the space $D([0, \infty), \mathbb{R}^k)$.

Theorem 8.5.6. (extension of continuity results to $D([0,\infty),\mathbb{R}^k)$) *The convergence-preservation results in Theorems 8.5.1, 8.5.2 and 8.5.4 and Corollary 8.5.1 extend to* $D([0,\infty),\mathbb{R}^k)$.

Proof. Suppose that $x_n \to x$ in $D([0,\infty),\mathbb{R}^k)$ with the appropriate topology and that $\{t_j : j \geq 1\}$ is a sequence of positive numbers with $t_j \in Disc(x)^c$ and $t_j \to \infty$ as $j \to \infty$. Then, $r_{t_j}(x_n) \to r_{t_j}(x)$ in $D([0,\infty),\mathbb{R}^k)$ with the same topology as $n \to \infty$ for each j, where r_t is the restriction map to $D([0,t],\mathbb{R}^k)$. Under the specified assumptions,

$$r_{t_j}(R(x_n)) = R_{t_j}(r_{t_j}(x_n)) \to R_{t_j}(r_{t_j}(x)) = r_{t_j}(R(x)) \tag{5.23}$$

in $D([0,t_j],\mathbb{R}^{2k})$ with the specified topology as $n \to \infty$ for each j, which implies that

$$R(x_n) \to R(x) \quad \text{in} \quad D([0,\infty),\mathbb{R}^{2k}) \tag{5.24}$$

with the same topology as in (5.23). ∎

Theorem 8.5.7. (extension of Lipschitz properties to $D([0,\infty),\mathbb{R}^k)$) *Let* $R : D([0,\infty),\mathbb{R}^k) \to D([0,\infty),\mathbb{R}^{2k})$ *be the reflection map with function domain* $[0,\infty)$ *defined by Definition* 8.2.1. *Let metrics associated with domain* $[0,\infty)$ *be defined in terms of restrictions by* (**??**) *in Section* 6.9. *Then the conclusions of Theorems* 8.2.5, 8.2.7 *and* 8.5.3 *also hold for domain* $[0,\infty)$.

Proof. Apply Theorem 12.9.4 in the book. ∎

8.6. Limits for Stochastic Fluid Networks

Nothing has been omitted from Section 14.6 of the book.

8.7. Queueing Networks with Service Interruptions

Nothing has been omitted from Section 14.7 of the book.

8.8. The Two-Sided Regulator

Nothing has been omitted from Section 14.8 of the book.

8.9. Existence of a Limiting Stationary Version

In this section, drawing on and extending Kella and Whitt (1996), we show that there exists a proper limiting stationary version of a reflected stochastic process under natural conditions. We establish existence and uniqueness of the limiting stochastic process, but we do not otherwise characterize the limiting marginal distribution on $\mathbb{R}^k$ or determine how to calculate it.

Our existence and uniqueness results with general initial conditions cover the case of the reflected Lévy process obtained as the heavy-traffic limit of the vector-valued buffer-content stochastic processes in a stochastic fluid network, as in Section 14.6 of the book, when the exogenous input processes at the different nodes are independent Lévy processes (i.e., processes with stationary independent increments) under a natural condition on the net input rates. We also obtain useful results about more general reflected processes without the independence conditions.

8.9.1. The Main Results

We are given a net-input stochastic process $\{X(t) : t \geq 0\}$ and the associated reflected content stochastic process

$$Z(t) \equiv \phi(X)(t) \equiv X(t) + (I - Q)Y(t), \quad t \geq 0 , \tag{9.1}$$

where $Y \equiv \psi(X)$ is the minimal nondecreasing nonnegative stochastic process such that $Z \geq 0$, as in Definition 8.2.1. We want to consider the limiting behavior as $t \to \infty$. We want to determine conditions under which

$$(Z_s(t_1), \ldots, Z_s(t_m)) \Rightarrow (Z_*(t_1), \ldots, Z_*(t_m)) \quad \text{in} \quad \mathbb{R}^{km} \quad \text{as} \quad s \to \infty \tag{9.2}$$

for all positive integers m and any m time points t_i with $0 \leq t_1 < \cdots < t_m$, where

$$Z_s(t) \equiv Z(s + t), \quad t \geq 0, \quad s \geq 0\,, \tag{9.3}$$

and the limiting stochastic process $Z_* \equiv \{Z_*(t) : t \geq 0\}$ is a *stationary stochastic process*, i.e., where

$$(Z_*(t_1 + h), \ldots, Z_*(t_m + h)) \stackrel{\mathrm{d}}{=} (Z_*(t_1), \ldots, Z_*(t_m))$$

for all positive integers m, any m time points t_i with $0 \leq t_1 < \cdots < t_m$ and all $h > 0$. We also want the limit process to be proper, i.e., we want to have

$$P(Z_*(t) < \infty) = 1 \quad \text{for all} \quad t\ .$$

We then call the stochastic process Z_* the *limiting stationary version* of Z.

We first observe that convergence of the finite-dimensional distributions in (9.2) for processes Z_s defined as in (9.3) directly implies that the limit process Z_* is stationary.

Lemma 8.9.1. (stationarity from convergence) *If*

$$(Z(s+t_1),\ldots,Z(s+t_m)) \Rightarrow (Z_*(t_1),\ldots,Z_*(t_m)) \quad \textit{in} \quad \mathbb{R}^{km} \tag{9.4}$$

as $s \to \infty$ for all positive integers m and all m time points t_i with $0 \le t_1 < \cdots < t_m$, then Z_* is a stationary process.

Proof. If (9.4) holds, then

$$(Z(s+t_1+h),\ldots,Z(s+t_m+h)) \Rightarrow (Z_*(t_1+u),\ldots,Z_*(t_m+u)) \tag{9.5}$$

as $s \to \infty$ for any u, $0 \le u \le h$, because we can let $s' = s+h-u$, $t_i' = t_i + u$, $1 \le i \le m$, and let $s' \to \infty$ with (9.4). Hence the distribution of the random vector on the right in (9.5) must be independent of u. ■

In order to obtain a unique limiting stationary version, we will assume that the net-input process X has *stationary increments*, i.e., the joint distribution of the random vector

$$(X(t_1+s)-X(u_1+s),\ldots,X(t_m+s)-X(u_m+s))$$

in $\mathbb{R}^{km}$ is independent of s for all positive integers m and all m-tuples of real numbers $(t_1,\ldots,t_m)$ and $(u_1,\ldots,u_m)$. We assume that X is defined on the whole real line $(-\infty,\infty)$. As a consequence,

$$X_s \equiv \{X(t+s)-X(s) : t \ge 0\} \tag{9.6}$$

has a distribution as a random element of D^k independent of s. We will also assume that X has *ergodic increment*, i.e., the increment $X(t+s)-X(s)$ have finite mean and

$$t^{-1}X(t) \to E[X(1)-X(0)] \quad \text{w.p.1} \quad \text{as} \quad t \to \infty\,.$$

Here is our main result: In addition to the assumptions above, it depends on the special initial condition $X(0) = 0$, which forces $Z(0) = Y(0) = 0$. The proof of the following result and several others are given at the end of the section.

Theorem 8.9.1. (existence of a limiting stationary version) *If X has stationary ergodic increments with $X(0) = 0$ and*

$$((I - Q)^{-1} E[X(1) - X(0)])^i < 0, \quad 1 \le i \le k \ , \tag{9.7}$$

then (9.2) *holds, i.e., the finite-dimensional distributions of Z_s in* (9.3) *converge as $s \to \infty$ to the finite-dimensional distributions of a proper stationary stochastic process Z_*.*

We now show the necessity of condition (9.7), leaving untouched the boundary case of equality. In particular, we show that a proper limit cannot exist if the strict inequality in (9.7) is reversed in any coordinate i. Indeed, then the i^{th} coordinate of the reflected process grows without bound.

Theorem 8.9.2. (necessity of the drift condition) *Suppose that*

$$t^{-1} X(t) \to x \quad \text{in} \quad \mathbb{R}^k \quad \text{w.p.1} \quad \text{as} \quad t \to \infty \, . \tag{9.8}$$

If

$$(I - Q)^{-1} x \le 0 \ , \tag{9.9}$$

then

$$t^{-1} Z(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{w.p.1} \tag{9.10}$$

for Z in (9.1). *On the other hand, if $((I - Q^{-1})x)^i > 0$ for some i, then*

$$\liminf_{t \to \infty} t^{-1} Z^i(t) > 0 \quad \text{for that} \quad i \ . \tag{9.11}$$

Proof. By Corollary 3.2.1 in the Internet Supplement, the SLLN in condition (9.8) implies the stronger FSLLN

$$\mathbf{X}_n \to x\mathbf{e} \quad \text{in} \quad D \quad \text{w.p.1}$$

for

$$\mathbf{X}_n(t) \equiv n^{-1} X(nt), \quad t \ge 0 \, .$$

By Theorem 8.2.5,

$$\phi(\mathbf{X}_n) \to \phi(x\mathbf{e}) \quad \text{in} \quad D \quad \text{w.p.1} \quad \text{as} \quad n \to \infty \, .$$

However, condition (9.9) implies that $\phi(x\mathbf{e}) = \mathbf{0}$. Hence, (9.10) is obtained by applying the projection map $\pi_1(x) = x(1)$. Finally, we obtain (9.11) from (9.8) after noting from (9.1) that $(I - Q)^{-1} Z \ge (I - Q)^{-1} X$. ∎

Theorem 8.9.1 does not cover all cases, because it requires the special initial condition $X(0) = 0$. However, we also obtain additional results with other initial conditions below. A difficulty occurs because in general the initial condition $X(0)$ and the remaining net-input process $\{X(t) - X(0) : t \geq 0\}$ are dependent. Hence, in general we cannot talk about the increments process as if it did not depend upon the initial condition. Nevertheless, we are able to obtain some positive results. We first establish a tightness result; see Section 11.6 of the book.

Theorem 8.9.3. (tightness under general initial conditions) *If X has stationary ergodic increments, and if condition* (9.7) *holds, then the family of random variables $\{Z(t) : t \geq 0\}$ is tight in $\mathbb{R}^k$.*

Since tightness in product spaces is equivalent to tightness of the components in each coordinate by Theorem 11.6.7 in the book, Theorem 8.9.3 implies the following.

Corollary 8.9.1. (tightness of the finite-dimensional distributions) *Under the conditions of Theorem* 8.9.3, *the family $\{Z_s(t_1), \ldots, Z_s(t_m) : s \geq 0\}$ is tight in $\mathbb{R}^{km}$ for every positive integer m and m time points $0 \leq t_1 < \cdots < t_m$.*

We can combine Prohorov's theorem (Theorem 11.6.1 in the book) with monotonicity to obtain the following result.

Corollary 8.9.2. (convergence of subsequences) *Under the conditions of Theorem* 8.9.3, *every subsequence $\{Z(t_k) : k \geq 1\}$ based on a sequence $\{t_k : k \geq 1\}$ of nonnegative numbers has a convergent subsequence $\{Z(t'_k) : k \geq 1\}$. If $Z(t_k) \Rightarrow L$ in $\mathbb{R}^k$ as $t_k \to \infty$, then*

$$Z_*(0) \leq_{st} L \; , \tag{9.12}$$

where Z_* is the stationary process obtained in Theorem 8.9.1 *and*

$$P(L^i < \infty) = 1, \quad 1 \leq i \leq k \, .$$

If we can conclude that the process Z gets arbitrarily close to the origin, then we can replace tightness in Theorem 8.9.3 with convergence.

Theorem 8.9.4. (convergence if the origin is approached) *If, in addition to the assumptions of Theorem* 8.9.3, *for any $\epsilon > 0$ there exists random time T_ϵ with*

$$P(T_\epsilon < \infty) = 1 \tag{9.13}$$

such that

$$\|Z(T_\epsilon)\| < \epsilon \ , \tag{9.14}$$

then the finite-dimensional distributions of Z_s in (9.3) *converge as $s \to \infty$ to the finite-dimensional distributions of the limit process Z_* in Theorem* 8.9.1.

We can obtain a stronger conclusion if the origin is actually hit for all initial positions.

Theorem 8.9.5. (coupling if the origin is always hit) *If, in addition to the assumptions of Theorem* 8.9.3, *for each initial value $X(0)$, there exists a random time T with $P(T < \infty) = 1$ such that $Z(T) = 0$, then the process $\{Z(t) : t \geq 0\}$ couples with the stationary version in finite time, so that*

$$\lim_{s\to\infty} Ef(Z_s) = Ef(Z_*)$$

for all measurable real-valued functions f on D^k.

However in general $\{Z(t) : t \geq 0\}$ need never visit a neighborhood of the origin.

Example 8.9.1. *The process Z need not vist a neighborhood of the origin.* To see that it is possible to have $Z(t) \neq (0, \ldots, 0)$, and even $\|Z(t)\| > c > 0$ for some constant c, for all $t \geq 0$ under the conditions of Theorem 8.9.3, consider a two-dimensional case in which *either* $X^1(t+\epsilon) - X^1(t) > \delta\epsilon$ or $X^2(t+\epsilon) - X^2(t) > \delta\epsilon$ for all t, where ϵ and δ are small positive constants. For example, let

$$V^1(t) = \begin{cases} \delta, & 3k \leq t < 3k+2 \\ -1, & 3k+2 \leq t < 3k+3 \end{cases}$$

and

$$V^2(t) = \begin{cases} \delta, & 3k+1 \leq t < 3k+3 \\ -1, & 3k \leq t < 3k+1 \end{cases}$$

for all nonnegative integers k. Let U be uniformly distributed on $[0, 3]$. Then $\{V(t) : t \geq 0\} \equiv \{(V^1(t+U), V^2(t+U)) : t \geq 0\}$ is a stationary process on the positive half line, so that $X(t) \equiv \int_0^t V(u)du$ is a net input process with stationary increments. It is easy to see that the content process associated with $Q = 0$ never hits the origin after time 0, and yet for $\delta < 1/2$

it has a proper steady-state distribution. Indeed, eventually $Z(t)$ follows the deterministic trajectory with $Z(3k-U) = (2\delta, 0)$, $Z(3k+1-U) = (0, \delta)$ and $Z(3k+2-U) = (\delta, 2\delta)$. This steady-state trajectory is reached for

$$t \geq 3\left(1 + \frac{\max\{Z^1(0), Z^2(0)\}}{1-2\delta}\right) .$$

By an appropriate choice of units, the limiting trajectory falls outside any neighborhood of the origin.

We can also modify Example 8.9.1 to construct two stable content processes which differ only in their initial conditions but do *not* couple in finite time.

Example 8.9.2. *Failure to couple in finite time.* We modify Example 8.9.1 by letting $Q^t_{1,2} = P^t_{2,1} = \epsilon$ for $0 < \epsilon < \delta$. The content process now approaches the deterministic trajectory with $Z(3k-U) = (2\delta-\epsilon+\epsilon', 0)$, $Z(3k+1-U) = (0, \delta-\epsilon+\epsilon')$ and $Z(3k+2-U) = (\delta, 2\delta-\epsilon+\epsilon')$, where $\epsilon' = (2\delta^2-\epsilon\delta)/(1+\delta)$. However, unlike Example 8.9.1, the content process typically does not reach this cycle in finite time. Suppose one of the two content processes starts above another, where they have the same net input process X. They move together until they hit a boundary. However, when the lower process is on a boundary and the other is not, the other coordinate of the two processes moves away from each other at rate ϵ. Hence the processes cannot couple on any boundary, although they do get closer in an appropriate metric as they hit the boundaries.

Since many of the limiting net-input processes X will be Lévy processes (i.e., will have stationary independent increments), we now add the independent increments property.

Theorem 8.9.6. (existence and uniqueness for Lévy net-input processes with independent coordinate processes) *Suppose that* $X \equiv (X^1, \ldots, X^k)$ *has mutually independent marginal processes* X^i, $1 \leq i \leq k$, *each with stationary and independent increments,* $X(0)$ *is proper and condition* (9.7) *holds. Then the limit* (9.2) *holds and the limit has the same distribution as the limit* $Z_*(0)$ *associated with* $X(0) = 0$.

As mentioned in the beginning of this section, Theorem 8.9.6 applies to the limit process in Section 8.6 when the scaled versions of the exogenous arrival process C converge to a Lévy process with mutually independent coordinate processes, because the only stochastic component in the net-input process X^i is C^i. However, in general, Theorem 8.9.6 does not apply to the heavy-traffic limits for the queueing network in Section 8.7. It does in the special case in which the coordinate limit process $\mathbf{X}^i$ depends only on the limit of the scaled process associated with the i^{th} coordinate arrival process.

It remains to establish more general conditions under which the assumptions of Theorems 8.9.4 and 8.9.5 are satisfied. It also remains to find useful expressions for the limiting distributions. Explicit expressions for the Laplace transforms of non-product-form two-dimensional stationary buffer-content distributions of stochastic fluid networks with Lévy exogenous input processes have been determined by Kella and Whitt (1992a) and Kella (1993).

8.9.2. Proofs

We now provide the missing proofs for the results above. We first establish some bounds and inequalities to be used in the proofs. Let $D^k_\downarrow$ be the subset of nonnegative nonincreasing functions in D^k. As before, let $D^k_\uparrow$ be the subset of nonnegative nondecreasing functions in D^k.

Theorem 8.9.7. (bounds and inequalities for the reflection map) *Assume that* $x_1, x_2 \in D$ *with* $x_2 - x_1 \in D^k_\uparrow$, $x_3 = x_1 + (I-Q)\psi(x_2)$ *and* $w \geq 0$ *in* $\mathbb{R}^k$. *Then*

(i) $\phi(x_2) \geq \phi(x_1)$,

(ii) $\psi(x_1) - \psi(x_2) \in D^k_\uparrow$,

(iii) $\psi(x_1) - \psi(x_2) \leq (I-Q)^{-1}(x_2 - x_1)$,

(iv) $\psi(x_3) = \psi(x_1) - \psi(x_2)$,

(v) $0 \leq (I-Q)^{-1}(\phi(x_2) - \phi(x_1)) \leq (I-Q)^{-1}(x_2 - x_1)$,

(vi) $0 \leq 1(\phi(x_2) - \phi(x_1)) \leq 1(x_2 - x_1)$,

(vii) $(I-Q)^{-1}(\phi(x_1 + w) - \phi(x_1)) \in D^k_\downarrow$,

(viii) $1(\phi(x_1 + w) - \phi(x_1)) \in D^k_\downarrow$.

Proof. Parts (i) and (ii) follow for $x_1, x_2 \in D_c$ by induction from Corollary 14.3.2 and Lemma 14.3.3 in the book. They then follow for $x_1, x_2 \in D$ by taking limits: Given $x_1, x_2 \in D$ with $x_2 - x_1 \in D_\uparrow$, it is possible to find $x_{1,n}$ and $x_{2,n} \in D_c$ with $x_{2,n} - x_{1,n} \in D_\uparrow$ for all n and $\|x_{j,n} - x_j\| \to 0$ as $n \to \infty$ for $j = 1, 2$. Part (iii) follows from Theorem 14.2.4 in the book because

$$\eta_1(x_2 - x_1) = x_2 - x_1 \quad \text{for} \quad x_2 - x_1 \in D_\uparrow \; .$$

Turning to (iv), note that

$$0 \le \phi(x_3) = x_1 + (I - Q)(\psi(x_2) + \psi(x_3)) \tag{9.15}$$

and

$$0 \le \phi(x_1) = x_3 + (I - Q)(\psi(x_1) - \psi(x_2)) \; . \tag{9.16}$$

From (9.15) and minimality of $\psi(x_1)$, it follows that $\psi(x_1) \le \psi(x_2) + \psi(x_3)$ for any choice of x_1 and x_2. From (9.16) and minimality of $\psi(x_3)$, it follows that $\psi(x_3) \le \psi(x_1) - \psi(x_2)$. Hence we must have $\psi(x_3) = \psi(x_1) + \psi(x_2)$ as claimed. Parts (v) –(viii) follow from the relations $(I - Q)^{-1}\phi(x) = (I - Q)^{-1}x + \psi(x)$ and $1(I - Q) \ge 0$, and Theorem 14.2.4 in the book. ■

We now apply Theorem 8.9.7 to determine the shape of several mean values as a function of time.

Corollary 8.9.3. (concavity of mean values) *If X has stationary increments with $X(0) = 0$, then the functions $((I - Q)^{-1}E\phi(X)(t))^i$, $E\psi^i(X)(t)$ and $1E\phi(X)(t)$ are concave functions of t for each i.*

Proof. Apply parts (vii), (ii) and (viii) of Theorem 8.9.7, respectively. We will only prove the first result because the three proofs are essentially the same. It suffices to show that

$$((I - Q)^{-1}E[\phi(X)(t + s) - \phi(X)(s)])^i$$

is nonincreasing in s for all t, but that follows from Theorem 8.9.7(vii), because $\phi(X)(t + s)$ is distributed as the reflection of $X_s(t) \equiv X(s + t) - X(s)$ starting at $\phi(X)(s)$ evaluated at t, while $\phi(X)(s)$ is distributed as the reflection of X_s starting at 0 evaluated at t, since the law of X_s is independent of s. ■

We say that a real-valued function f on $\mathbb{R}_+$ is *subadditive* if

$$f(t_1 + t_2) \le f(t_1) + f(t_2)$$

for all $t_1, t_2 \in \mathbb{R}_+$. We say that an $\mathbb{R}^k$-valued stochastic process $\{X(t) : t \geq 0\}$ is *stochastically increasing and subadditive* (SIS) if

$$Ef(X(t_1 + t_2)) \leq Ef(X(t_1)) + Ef(X(t_2))$$

for all nondecreasing subadditive real-valued functions f on $\mathbb{R}^k$.

Corollary 8.9.4. (SIS property) *If X has stationary increments with $X(0) = 0$, then $(I - Q)^{-1}Z$ and $1Z$ are stochastically increasing and subadditive stochastic processes.*

Proof. Since the two results are proved similarly, we only prove the first. Let

$$\tilde{Z}_{s_1,s_2}(t) \equiv (I - Q)^{-1}Z(t)$$

with Z having initial value $Z(s_1)$ and net input $X_{s_2}(t) \equiv X(s_2 + t) - X(s_2)$, $t \geq 0$, where $0 \leq s_1 \leq s_2$. By Theorem 8.9.7(vii),

$$\tilde{Z}_{s,s}(t) - \tilde{Z}_{0,s}(t) \leq \tilde{Z}_{s,s}(0) - \tilde{Z}_{0,s}(0) = \tilde{Z}_{s,s}(0)$$

for all $s, t \geq 0$, or

$$\tilde{Z}_{0,0}(s + t) = \tilde{Z}_{s,s}(t) \leq \tilde{Z}_{0,s}(t) + \tilde{Z}_{s,s}(0) \ ,$$

so that, for any subadditive function f,

$$\begin{aligned} E[f(\tilde{Z}_{0,0}(s + t)) &\leq E[f(\tilde{Z}_{0,s}(t) + \tilde{Z}_{s,s}(0))] \\ &\leq E[f(\tilde{Z}_{0,s}(t))] + E[f(\tilde{Z}_{s,s}(0))] \\ &\leq E[f(Z_{0,0}(t))] + E[f(Z_{0,0}(s))] \ , \end{aligned}$$

with the last line holding because there is equality in distribution for the respective terms. ■

A key to establishing the important Theorems 14.8.1 and 14.8.6 in the book is the following stochastic increasing property, which we deduce from Theorem 8.9.7.

Theorem 8.9.8. (stochastic increasing starting empty) *If X has stationary increments and $X(0) = 0$, then the family of processes $\{Z_s : s \geq 0\}$ in (9.3) is stochastically increasing in s, i.e.,*

$$Ef(Z_{s_1}) \leq Ef(Z_{s_2})$$

for $0 \leq s_1 < s_2$ and all bounded measurable nondecreasing real-valued functions f on $D \equiv D([0, \infty), \mathbb{R}^k)$, using the componentwise order on D.

Proof. Let $\hat{Z}_s(t)$ $(Z_s(t))$ be the content with $Z(0) = 0$ $(Z(0) = Z(s))$ and input increments from X_s in equation (14.8.6) in the book. Then, for $0 \le s_1 < s_2$,

$$\hat{Z}_{s_2-s_1}(s_1+t) \le Z_{s_2-s_1}(s_1+t) \quad \text{for all} \quad t \ge 0 \quad \text{w.p.1}\,,$$

by Theorem 8.9.7 because $\hat{Z}_{s_2-s_1}(0) \equiv 0 \le Z_{s_2-s_1}(0) \equiv Z(s_2-s_1)$ and both processes have the common input increments from X_s. Hence,

$$Ef(\hat{Z}_{s_2-s_1}) \le Ef(Z_{s_2-s_1})$$

for all nondecreasing bounded measurable real-valued functions f on D, using the usual componentwise order. However, since $X_s \stackrel{\mathrm{d}}{=} X$,

$$\{\hat{Z}_{s_1-s_1}(s_1+t) : t \ge 0\} \stackrel{\mathrm{d}}{=} \{Z(s_1+t) : t \ge 0\} \equiv Z_{s_1}$$

and

$$\{Z_{s_2-s_1}(s_1+t) : t \ge 0\} = \{Z(s_2+t) : t \ge 0\} \equiv Z_{s_2}\,.$$

These last three relations combine to establish the desired conclusion. ∎

We use the following result to establish Theorem 14.8.3 in the book.

Theorem 8.9.9. (tightness solidarity) *Suppose that X has stationary increments. Then $\{Z(t) : t \ge 0\}$ is tight for all proper distributions of $X(0)$ if and only if it is tight for any one.*

Proof. Note that $\{Z(t) : t \ge 0\}$ is tight if and only if $\{(I-Q)^{-1}Z(t) : t \ge 0\}$ is tight. By Theorem 8.9.7, the processes $(I-Q)^{-1}Z(t)$ starting at $X(0)$ and 0, with common increments from X, differ by at most $(I-Q)^{-1}\|X(0)\|$. Hence they are tight or non-tight together. Hence, the tightness of the process with one proper initial condition implies the tightness of the process starting at 0. Then the tightness of the process starting at 0 implies the tightness of any other process with another initial condition. ∎

The key to our tightness results, and thus also our convergence results, is our ability to bound the marginal processes Z^i associated with a k-dimensional reflected process $Z \equiv (Z^1, \ldots, Z^k)$ by related well-studied and well-understood one-dimensional reflections. For that purpose, we have the following bounds.

Theorem 8.9.10. (one-dimensional reflection bounds) *For any $x \in D^k$ and $Q \in \mathcal{Q}$,*

$$\psi_1((I-Q)^{-1}x) \le \psi(x) \le (I-Q)^{-1}\psi_1(x) \qquad (9.17)$$

and

$$\phi_1((I-Q)^{-1}x) \le (I-Q)^{-1}\phi(x) \le (I-Q)^{-1}\phi_1(x) \ , \tag{9.18}$$

where $(\psi_1, \phi_1) : D^k \to D^{2k}$ *with*

$$(\psi_1(x)^i, \phi_1(x)^i) \equiv \hat{\psi}_1(x^i), \hat{\phi}_1(x^i)), \quad 1 \le i \le k \ ,$$

and $(\hat{\psi}_1, \hat{\phi}_1) : D \to D^2$ *being the one-dimensional reflection map, i.e.,*

$$\hat{\phi}_1(x^i) \equiv x^i + \hat{\psi}_1(x^i)) \tag{9.19}$$

and

$$\hat{\psi}_1(x^i) \equiv -\inf_{0\le s\le t}\{x_i(s)^-\}, \quad t \ge 0\,. \tag{9.20}$$

Proof. For the upper bounds, note that

$$\phi_1(x) = x + \psi_1(x) = x + (I-Q)(I-Q)^{-1}\psi_1(x) \ .$$

By the minimality of $\psi(x)$ in the definition of (ψ, ϕ),

$$\psi(x) \le (I-Q)^{-1}\psi_1(x) \ .$$

Therefore,

$$(I-Q)^{-1}\phi(x) = (I-Q)^{-1}x+\psi(x) \le (I-Q)^{-1}x+(I-Q)^{-1}\psi_1(x) = (I-Q)^{-1}\phi_1(x)\,.$$

Similarly, for the lower bound,

$$\phi_1((I-Q)^{-1}x) = (I-Q)^{-1}x + \psi_1((I-Q)^{-1}x) \tag{9.21}$$

and

$$(I-Q)^{-1}\phi(x) = (I-Q)^{-1}x + \psi(x)\,.$$

Since $(I-Q)^{-1}\phi(x) \ge 0$, we can apply the minimality of ψ_1 in (9.21) to deduce that

$$\psi_1((I-Q)^{-1}x) \le \psi(x)$$

and

$$\phi_1((I-Q)^{-1}x) \le (I-Q)^{-1}\phi(x) \ . \quad \blacksquare$$

In order to apply the one-dimensional reflection bounds in Theorem 8.9.10, we need to have a net input process X with negative drift in each coordinate. However, from (9.7), we only have X such that $(I-Q)^{-1}X$ has negative drift in each coordinate. We now show that, given X such that $(I-Q)^{-1}X$ has negative drift, we can bound $(I-Q)^{-1}\phi(X)$ above by $(I-Q)^{-1}\phi(X_y)$, where $X_y(t) \equiv X(t) - yt$, $t \ge 0$ and X_y has negative drift in each coordinate.

Theorem 8.9.11. (upper bound with negative drift) *Let X be a random element of D^k with stationary increments such that*

$$E[X(1) - X(0)] = x \quad \textit{and} \quad ((I-Q)^{-1}x)^i < 0, \quad 1 \le i \le k\,.$$

For any $y \in \mathbb{R}^k$ with $y^i > x^i$ and $((I-Q)^{-1}y)^i < 0$, $1 \le i \le k$ (there necessarily is one), let

$$X_y(t) \equiv X(t) - yt, \quad t \ge 0\,.$$

Then X_y has stationary increments (and ergodic increments if X does) with

$$E[X_y(1) - X_y(0)]^i = x^i - y^i < 0, \quad 1 \le i \le k\,,$$

and

$$(I-Q)^{-1}\phi(X) \le (I-Q)^{-1}\phi(X_y)\ . \tag{9.22}$$

Proof. Only the final conclusion (9.22) requires discussion. Let e be the identity map, i.e., $e(t) = t$, $t \ge 0$. Recall that

$$\begin{aligned}
\phi(X)(t) &\equiv X(t) + (I-Q)\psi(X)(t)\\
\phi(X_y)(t) &\equiv X(t) - yt + (I-Q)\psi(X_y)(t)\\
\phi(ye)(t) &\equiv yt + (I-Q)\psi(ye)(t), \quad t \ge 0\,.
\end{aligned}$$

First, since $(I-Q)^{-1}y \le 0$, it is easy to see that

$$\phi(ye)(t) = 0 \quad \text{and} \quad \psi(ye)(t) = -(I-Q)^{-1}yt\ .$$

Then

$$\phi(X_y)(t) \equiv \phi(X_y)(t) + \phi(ye)(t) = X(t) + (I-Q)(\psi(X_y)(t) + \psi(ye)(t))\,.$$

By the minimality of $\psi(X)$,

$$\psi(X) \le \psi(X_y) + \psi(ye)$$

and

$$\begin{aligned}
(I-Q)^{-1}\phi(X) &= (I-Q)^{-1}X + \psi(X)\\
&\le (I-Q)^{-1}X + \psi(X_y) + \psi(ye) = (I-Q)^{-1}\phi(X_y)\,. \quad \blacksquare
\end{aligned}$$

We now state the classical one-dimensional result, which depends on the fact that the reflected content $\phi(X)(t)$ has the same distribution as the supremum of the time-reversed net-input process for each t (but not for multiple t).

Theorem 8.9.12. (classical one-dimensional result) *If X is a real-valued stochastic process with stationary increments such that*

$$X_r(t) \equiv -X(-t) \to -\infty$$

as $t \to \infty$ and $X(0)$ is proper, then there exists a proper random variable L such that

$$\phi(X)(t) \Rightarrow L \quad in \quad \mathbb{R} \quad as \quad t \to \infty\,.$$

Proof. First assume that $X(0) = 0$. Given the time reversed process $X_r(t) \equiv -X(-t)$, $t \geq 0$, note that

$$\phi(X)(t) \stackrel{\mathrm{d}}{=} X_r^{\uparrow}(t) \quad \text{for each} \quad t \geq 0\,.$$

Since $X_r(t) \to -\infty$ as $t \to \infty$ and $X_r \in D$,

$$X_r^{\uparrow}(t) \to X_r^{\uparrow}(\infty) < \infty \quad \text{as } t \to \infty \qquad \text{w.p.1}\,.$$

Hence the desired conclusion holds with the proper limit $L \stackrel{\mathrm{d}}{=} X_r^{\uparrow}(\infty)$. Now suppose that $X(0) \neq 0$. Since $X(t) \to -\infty$ w.p.1, the processes $Z(t)$ starting at 0 and $X(0)$, with common net input process X, couple w.p.1. Hence we can invoke Theorem 8.9.5. ■

We now provide the missing proofs of theorems earlier in this section.

Proof of Theorem 8.9.1. By Theorem 8.9.8, the family of processes Z_s in (9.3) are stochastically increasing in s. Consequently, the finite-dimensional distributions of Z_s are stochastically increasing in s. The cumulative distribution functions (cdf's) of $(Z_s(t_1), \ldots, Z_s(t))$ in $\mathbb{R}^{km}$ thus converge as $s \to \infty$ to a possibly improper cdf; e.g., see Chapter VIII of Feller (1971). It thus suffices to show that $\{Z^i(t) : t \geq 0\}$ is tight for each i, for which it suffices to show that $\{((I - Q)^{-1}Z(t))^i : t \geq 0\}$ is tight for each i. (The tightness implies that the limiting cdf is proper.) By Theorem 8.9.11, we can bound $(I - Q)^{-1}\phi(X)$ above by $(I - Q)^{-1}\phi(X_y)$, where $X_y(t) \equiv X(t) - yt$ for appropriate $y \in \mathbb{R}^k$ and

$$-\infty < E[X_y^i(1) - X_y^i(0)] < 0 \quad \text{for all} \quad i\,. \tag{9.23}$$

By (9.18) in Theorem 8.9.10, we can bound $(I - Q)^{-1}\phi(X_y)$ above by $(I - Q)^{-1}\phi_1(X_y)$, where ϕ_1 is the vector of one-dimensional reflection maps. Hence it suffices to show that $\{\hat{\phi}_1(X_y^i(t) : t \geq 0\}$ is tight for each i, where

$\hat{\phi}_1$ is the one-dimensional reflection map in (9.19). However, $\hat{\phi}_1(X_y^i)(t)$ converges to a proper limit by Theorem 8.9.12. The condition $-X(-t) \to -\infty$ in Theorem 8.9.12 holds for X_y by virtue of (9.23) and that fact that $\{-X(t)\}$ is a process with stationary ergodic increments (Stationarity and metric transitivity are invariant under time reversal, and ergodicity is equivalent to metric transitivity.) The assumptions imply that

$$-t^{-1}X_y^i(-t) \to E[X_y^i(1) - X_y^i(0)] \quad \text{as} \quad t \to \infty \quad \text{w.p.1}$$

for each i, which implies that $-X_y^i(-t) \to -\infty$ w.p.1 as $t \to \infty$ for each i. ■

Proof of Theorem 8.9.3. By Theorem 8.9.1, we have convergence to a proper limit L for the process $\{Z_0(t) : t \geq 0\}$ starting from the origin. By the continuous mapping theorem,

$$(I - Q)^{-1} Z_0(t) \Rightarrow (I - Q)^{-1} L \quad \text{as} \quad t \to \infty .$$

If $X(0)$ is proper, then so is $X(0)^+ \equiv (X^1(0)^+, \ldots, X^k(0)^+)$. Then, from Theorem 8.9.7(i) and (v),

$$0 \leq (I-Q)^{-1} Z_{X(0)}(t) \leq (I-Q)^{-1} Z_{X(0)^+}(t) \leq (I-Q)^{-1} Z_0(t) + (I-Q)^{-1} X(0)^+ ,$$

where here $Z_w(t)$ denotes the process governed by X with initial position w. Hence

$$P(|((I - Q)^{-1} Z_{X(0)}(t))^i| > 2K) \leq P(|(I - Q)^{-1} Z_0(t))^i| > K) + P(|(I - Q)^{-1} X(0)^+)^i| > K) ,$$

so that the tightness holds by the results above. ■

Proof of Corollary 8.9.2. We can combine Prohorov's theorem (Theorem 11.6.1 in the book) with monotonicity. By Theorem 8.9.7,

$$Z_0(t) \leq Z_{X(0)}(t) \quad \text{for all} \quad t . \tag{9.24}$$

Since $Z_0(t) \Rightarrow Z_*(0)$ by Theorem 8.9.1, (9.12) must hold. (Stochastic order on $\mathbb{R}^k$ is preserved under weak convergence.) ■

Proof of Theorem 8.9.4. Since (9.24) holds and

$$(I - Q)^{-1}(Z_{X_0} - Z_0) \in D_{\downarrow}^k ,$$

by Theorem 8.9.7(vii),

$$0 \leq (I - Q)^{-1}(Z_{X(0)}(t) - Z_0(t)) \leq (I - Q)^{-1} 1\epsilon \quad \text{for all} \quad t \geq T_\epsilon .$$

Since $Z_0(t) \Rightarrow L$ as $t \to \infty$ by Theorem 8.9.1 and ϵ is arbitrary, we must have $Z_{X(0)}(t) \Rightarrow L$ too. ■

Proof of Theorem 8.9.5. The processes starting at 0, $X(0)$ or $Z_*(0)$ can all be given a common net input process $X(t) - X(0)$, $t \geq 0$. Hence, they all must couple when the process starting at $Z(0) \vee Z^*(0)$ first hits the origin. ∎

In preparation for the proof of Theorem 8.9.6, we now establish a property of the limiting distribution in the one-dimensional case when X is a Lévy process.

Theorem 8.9.13. (mass near the origin) *If, in addition to the assumptions of Theorem* 8.9.12*, the one-dimensional net-input process X has independent increments, then*

$$P(L < \epsilon) > 0 \quad \textit{for all} \quad \epsilon > 0\,,$$

where L is the limiting random variable.

Proof. Consider the time reversed process X_r defined in Theorem 8.9.12. It suffices to show that $P(X_r^{\uparrow}(\infty) < \epsilon) > 0$. Suppose not. Then $P(X_r^{\uparrow}(\infty) \geq \epsilon) = 1$, which implies that $P(T_\epsilon < \infty) = 1$, where

$$T_\epsilon = \inf\{t > 0 : X_r(t) \geq \epsilon\}\ .$$

Using the regeneration property associated with the stationary independent increments, that in turn implies that

$$\limsup_{t\to\infty} X_r(t) = +\infty \quad \text{w.p.1}\ ,$$

which contradicts the limit $X_r(t) \to -\infty$ w.p.1. Hence we must have $P(X_r^{\uparrow}(\infty) < \epsilon) > 0$ for all $\epsilon > 0$ as claimed. ∎

Proof of Theorem 8.9.6. The conditions allow us to apply Theorem 8.9.4. Theorems 8.9.10 and 8.9.11 allow us to bound the process $(I - Q)^{-1}\phi(X)(t)$ above by $(I-Q)^{-1}\phi_1(X_y)(t)$, as in the proof of Theorem 8.9.1. However, $\phi_1(X_y)$ has mutually independent coordinate processes. Let L^i be the limit random variable for the one-dimensional process associated with $\phi_1(X_y)$ and coordinate i. Since, for any $\epsilon > 0$,

$$P(L^1 \leq \epsilon, \ldots, L^k \leq \epsilon) = \prod_{i=1}^{k} P(L^i \leq \epsilon) > 0$$

by the independence and Theorem 8.9.13 we must have $P(T_\epsilon < \infty) = 1$ for the random time T_ϵ in Theorem 8.9.4. ∎

As mentioned earlier, Theorem 8.9.6 applies to the limit process in Section 14.6 in the book when the scaled versions of the exogenous arrival process C converge to a Lévy process with mutually independent coordinate processes, because the only stochastic component in the net-input process X^i is C^i. However, in general, Theorem 8.9.6 does not apply to the heavy-traffic limits for the queueing network in Section 14.7 of the book. It does in the special case in which the coordinate limit process $\mathbf{X}^i$ depends only on the limit of the scaled process associated with the i^{th} coordinate arrival process.