Chapter 7

Useful Functions

7.1. Introduction

This chapter contains proofs omitted from Chapter 13 of the book, with the same title. As before, the theorems to be proved are restated here. The section and theorem numbers parallel Chapter 13 in the book, so that the proofs should be easy to find.

We consider four basic functions introduced in Section 3.5 of the book: composition, supremum, reflection and inverse. Another basic function is addition, but it has already been treated in Sections 12.6, 12.7 and 12.11 of the book. Our treatment of useful functions follows Whitt (1980), but the emphasis there was on the J_1 topology, even though the M_1 topology was used in places. In contrast, here the emphasis is on the M_1 and M_2 topologies.

Here is how this chapter is organized: We start in Section 7.2 by considering the composition map, which plays an important role in establishing FCLTs involving a random time change. We consider composition without centering in Section 7.2; then we consider composition with centering in Section 7.3.

In Section 7.4 we study the supremum function, both with and without centering. In Section 7.5 we apply the supremum results to treat the (one-sided one-dimensional) reflection map, which arises in queueing applications.

We start studying the inverse function in Section 7.6. We study the inverse map without centering in Section 7.6 and with centering in Section 7.7. In Section 7.8 we apply the results for inverse functions to obtain corresponding results for closely related counting functions.

In Section 7.9 we apply the previously established convergence-preservation results for the composition and inverse maps to establish stochastic-process limits for renewal-reward stochastic processes. When the times between the renewals in the renewal counting process have a heavy-tailed distribution, we need the M_1 topology.

In Chapter 3 of the Internet Supplement we discuss pointwise convergence and its preservation under mappings. The prservation of pointwise convergence focuses on relations for inidvidual sample paths, as in the queueing book by El-Taha and Stidham (1999). There we see that a function-space setting is not required for all convergence preservation.

7.2. Composition

This section is devoted to the composition function, mapping (x, y) into $x \circ y$, where

$$(x \circ y)(t) \equiv x(y(t)) \quad \text{for all} \quad t \ .$$

The composition map is useful to treat random sums and, more generally, processes modified by a random time change; e.g., see Section 13.9 of the book on renewal-reward processes.

Henceforth in this chapter, unless stipulated otherwise, when $D \equiv D^k$, so that the range of functions is $\mathbb{R}^k$, we let D be endowed with the strong version of the J_1, M_1 or M_2 topology, and simply write J_1, M_1 or M_2. It will be evident that most results also hold with the corresponding weaker product topology.

7.2.1. Preliminary Results

To ensure that $x \circ y \in D$, we will assume that y is also nondecreasing. We begin by defining subsets of $D \equiv D^k \equiv D([0, \infty), \mathbb{R}^k)$ that we will consider. Let D_0 be the subset of all $x \in D$ with $x^i(0) \geq 0$ for all i. Let $D_\uparrow$ and $D_{\uparrow\uparrow}$ be the subsets of functions in D_0 that are nondecreasing and strictly increasing in each coordinate. Let D_m be the subset of functions x in D_0 for which the coordinate functions x^i are monotone (either increasing or decreasing) for each i. Let C_0, $C_\uparrow$, $C_{\uparrow\uparrow}$ and C_m be the corresponding subsets of C; i.e., $C_0 \equiv C \cap D_0$, $C_\uparrow \equiv C \cap D_\uparrow$, $C_{\uparrow\uparrow} = C \cap D_{\uparrow\uparrow}$, and $C_m = C \cap D_m$.

It is important that all of these subsets are measurable subsets of D with the Borel σ-fields associated with the non-uniform Skorohod topologies, which all coincide with the Kolmogorov σ-field generated by the projection maps; see Theorems 11.5.2 and 11.5.3 in the book.

Returning to the composition map, we state the condition for $x \circ y \in D$ as a lemma.

Lemma 7.2.1. (criterion for $x \circ y$ to be in D) *For each* $x \in D([0,\infty),\mathbb{R}^k)$ *and* $y \in D_{\uparrow}([0,\infty),\mathbb{R}_+)$, $x \circ y \in D([0,\infty),\mathbb{R}^k)$.

A basic result, from pp. 145, 232 of Billingsley (1968), is the following. The continuity part involves the topology of uniform convergence on compact intervals.

Theorem 7.2.1. (continuity of composition at continuous limits) *The composition map from* $D^k \times D^1_{\uparrow}$ *to* D^k *is measurable and continuous at* $(x,y) \in C^k \times C^1_{\uparrow}$.

Our goal now is to obtain additional positive continuity results under extra conditions. We use the following elementary lemma.

Lemma 7.2.2. *If* $y(t) \in Disc(x)$ *and* y *is strictly increasing and continuous at* t, *then* $t \in Disc(x \circ y)$.

The following is the J_1 result.

Theorem 7.2.2. (J_1-continuity of composition) *The composition map from* $D^k \times D^1_{\uparrow}$ *to* D^k *taking* (x,y) *into* $(x \circ y)$ *is continuous at* $(x,y) \in (C^k \times D^1_{\uparrow}) \cup (D^k \times C^1_{\uparrow\uparrow})$ *using the* J_1 *topology throughout.*

Proof. First suppose that $(x_n, y_n) \to (x,y)$ in $D^k \times D^1_{\uparrow}$ with $(x,y) \in C^k \times D_{\uparrow}$. Choose $t_1 \in Disc(y)^c$. Then $y_n \to y$ for the restrictions to $[0,t_1]$; i.e., there exist $\lambda_n \in \Lambda([0,t_1])$ such that $\|y_n - y \circ \lambda_n\|_{t_1} \vee \|\lambda_n - e\|_{t_1} \to 0$. Choose t_2 such that $y(t_1) \le t_2$ and $y_n(t_1) \le t_2$ for all $n \ge 1$. Since $x \in C^k$, $\|x_n - x\|_{t_2} \to 0$. By the triangle inequality,

$$\|x_n \circ y_n - x \circ y \circ \lambda_n\|_{t_1} \le \|x_n \circ y_n - x \circ y_n\|_{t_1} + \|x \circ y_n - x \circ y \circ \lambda_n\|_{t_1} . \quad (2.1)$$

The first term on the right in (2.1) converges to 0 because $\|x_n - x\|_{t_2} \to 0$ and the range of y_n is contained in $[0,t_2]$. The second term on the right in (2.1) converges to 0 because x is uniformly continuous over $[0,t_2]$ and $\|y_n - y \circ \lambda_n\|_{t_1} \to 0$.

Next suppose that $(x_n, y_n) \to (x,y)$ in $D^k \times D^1_{\uparrow}$ with $(x,y) \in D \times C_{\uparrow\uparrow}$. By Lemma 7.2.2 below, $y(t) \in Disc(x)^c$ for each $t \in Disc(x \circ y)^c$. However, for each $t' \in Disc(x)^c$, we have local uniform convergence of x_n to x, i.e.,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n\to\infty}} \; v(x_n, x, t', \delta) = 0 \; ; \quad (2.2)$$

see Section 12.4 in the book. Since $y_n(t) \to y(t)$ as $n \to \infty$, as a consequence of (2.2), we have $(x_n \circ y_n)(t) \to (x \circ y)(t)$ for each $t \in Disc(x \circ y)^c$. Now we show that the closure of the sequence $\{x_n \circ y_n : n \geq 1\}$ is compact in the J_1 topology. Since $(x_n \circ y_n)(t) \to (x \circ y)(t)$ for t in a countable dense subset, all limits of convergent subsequences must coincide with $x \circ y$. Since all convergent subsequences have the same limit, compactness implies that the sequence itself must converge; i.e., $x_n \circ y_n \to x \circ y$ (J_1). Hence it suffices to show that the closure of $\{x_n \circ y_n\}$ is compact, for which we apply Theorem 14.4 of Billingsley (1968). For an arbitrary t_1, choose $t_2 > y(t_1)$ with $t_2 \in Disc(x)^c$. Then, for all sufficiently large n, $y_n(t_1) < t_2$ and $x_n \to x$ for the restrictions in $D([0,t_2], \mathbb{R}^k)$. It is easy to see that condition (14.49) and (14.50) in Billingsley (1968) hold. First, (14.49) holds because

$$\sup_{\substack{0 \leq s \leq t_1 \\ n \geq 1}} \|x_n \circ y_n(s)\| \leq \sup_{\substack{0 \leq s \leq t_2 \\ n \geq 1}} \|x_n\| < \infty \, , \tag{2.3}$$

since $x_n \to x$ in $D([0,t_2], \mathbb{R}^k, J_1)$. Next (14.50) holds because the oscillation functions for $x_n \circ y_n$ over $[0, t_1]$ be bounded above by the oscillation functions of x_n over $[0, t_2]$; e.g., since $y \in C_{\uparrow\uparrow}$ and $\|y_n - y\|_{t_1} \to 0$, for any δ_2 there exists n_0 and δ_1 such that $w''_{x_n \circ y_n}(\delta_1) \leq w''_{x_n}(\delta_2)$ for all $n \geq n_0$. ■

7.2.2. M-Topology Results

We have a different result for the M topologies.

Theorem 7.2.3. (M-continuity of composition) *If $(x_n, y_n) \to (x, y)$ in $D^k \times D^1_{\uparrow}$ and $(x, y) \in (D^k \times C^1_{\uparrow\uparrow}) \cup (C^k_m \times D^1_{\uparrow})$, then $x_n \circ y_n \to x \circ y$ in D^k, where the topology throughout is M_1 or M_2.*

In most applications we have $(x, y) \in D^k \times C^1_{\uparrow\uparrow}$, as is illustrated by the next section. That part of the M conditions is the same as for J_1. The mode of convergence in Theorem 7.2.3 for $y_n \to y$ does not matter, because on $D^1_{\uparrow}$, convergence in the M_1 and M_2 topologies coincides with pointwise convergence on a dense subset of $[0, \infty)$, including 0; see Corollary **??**.

It is easy to see that composition cannot in general yield convergence in a stronger topology, because $x \circ y = x$ and $x_n \circ y_n = x_n$, $n \geq 1$, when $y_n = y = e$, where $e(t) = t$, $t \geq 0$. Unlike for the J_1 topology, the composition map is in general *not* continuous at $(x, y) \in C \times D^1_{\uparrow}$ in the M topologies.

We actually prove a more general continuity result, which covers Theorem 7.2.3 as a special case.

Theorem 7.2.4. (more general M-continuity of composition) *Suppose that* $(x_n, y_n) \to (x, y)$ *in* $D^k \times D^1_{\uparrow}$. *If (i)* y *is continuous and strictly increasing at* t *whenever* $y(t) \in Disc(x)$ *and (ii)* x *is monotone on* $[y(t-), y(t)]$ *and* $y(t-), y(t) \notin Disc(x)$ *whenever* $t \in Disc(y)$, *then* $x_n \circ y_n \to x \circ y$ *in* D^k, *where the topology throughout is* M_1 *or* M_2.

Theorem 7.2.3 follows easily from Theorem 7.2.4: First, on $D^k \times C^1_{\uparrow}$, y is continuous, so only condition (i) need be considered; it is satisfied because y is continuous and strictly increasing everywhere. Second on $C^k_m \times D^1_{\uparrow}$, x is continuous so only condition (ii) need be considered; it is satisfied because x is monotone everywhere. Hence it suffices to prove Theorem 7.2.4, which is done in Section 1.8 of the Internet Supplement. The general idea in our proof of Theorem 7.2.4 is to work with the characterization of convergence using oscillation functions evaluated at single arguments, exploiting Theorems 6.5.1 (v), 6.5.2 (iv), 6.11.1 (v) and 6.11.2 (iv).

We obtain a stronger result (M_1 convergence of $x_n \circ y_n$ given only M_2 convergence of x_n) if we do not need to invoke condition (i) in Theorem 7.2.4. A sufficient condition is for x to be continuous.

Theorem 7.2.5. (obtaining SM_1 convergence from WM_2 convergence) *If the conditions of Theorem* 7.2.4 *hold with* $y(t) \notin Disc(x)$ *for all* t, *then* $x_n \circ y_n \to x \circ y$ *in* (D^k, SM_1) *even if* $x_n \to x$ *only in* (D^k, WM_2).

Proof. Apply Lemmas 7.2.4, 7.2.5 and 7.2.8 below. ∎

We prove Theorem 7.2.4 by identifying four different cases, with each either having $t \in Disc(x \circ y)$ or not.

Proof of Theorem 7.2.4. We will establish the appropriate characterization of convergence $x_n \circ y_n \to x \circ y$ at each t separately, using Theorems 12.5.1 (v), 12.5.2 (iv), 12.11.1 (v) and 12.11.2 (iv) in the book.

There are four cases to consider:

(i) $t \notin Disc(y)$ and $y(t) \notin Disc(x)$, so that $t \notin Disc(x \circ y)$;

(ii) $t \in Disc(y)$, $x(u) = x(y(t-)) = x(y(t))$ for all $u \in [y(t-), y(t)]$ and $y(t-)$, $y(t) \notin Disc(x)$, under which $t \notin Disc(x \circ y)$;

(iii) $t \in Disc(y)$, $x(y(t-)) \neq x(y(t))$, x is monotone on $[y(t-), y(t)]$ and $y(t-), y(t) \notin Disc(x)$, under which $t \in Disc(x \circ y)$;

(iv) $y(t) \in Disc(x)$ and y is continuous and strictly increasing at t so that $t \in Disc(x \circ y)$.

In case (ii) we have $t \notin Disc(x \circ y)$ even though $t \in Disc(y)$. The regularity conditions in case (ii) follow from condition (ii); since $x(y(t-)) = x(y(t))$, monotonicity reduces to a constant value over the subinterval. Case (iii) differs from case (ii) by having $x(y(t-)) \neq x(y(t))$, which makes $t \notin Disc(x \circ y)$. The regularity conditions in case (iii) again follow from condition (ii). The regularity conditions in case (iv) when $y(t) \in Disc(x)$ follow from condition (i). We use Lemma 7.2.2 in case (iv). In each case we know whether or not $t \in Disc(x \circ y)$. The four cases are covered by subsequent lemmas as follows: Case (i) by Lemmas 7.2.3–7.2.4; case (ii) by Lemma 7.2.5; case (iii) by Lemmas 7.2.6–7.2.8; and case (iv) by Lemma 7.2.10. ∎

We now establish several lemmas in order to complete the proof of Theorem 7.2.4. Throughout, we assume that (x_n, y_n), $n \geq 1$, and (x, y) are elements of $D^k \times D^1_\uparrow$. Refer to Section 12.4 of the book for the oscillation functions.

Lemma 7.2.3. *If* $v(y_n, y, t, \delta_1) \leq \delta_2$ *in* $D^1_\uparrow$, *then*

$$u(x_n \circ y_n, x \circ y, t, \delta_1) \leq v(x_n, x, y(t), \delta_2) + \bar{v}(x \circ y, t, \delta_1)$$

for v *in* (12.4.2), u *in* (12.4.1) *and* $\bar{v}$ *in* (12.4.3), *all in Section* 12.4 *of the book.*

Proof. By the condition, $|y_n(t_1) - y(t)| \leq \delta_2$ provided that $0 \vee (t - \delta_1) < t_1 < (t + \delta_1) \wedge T$. Hence, for t_1 in that range,

$$\begin{aligned} \|(x_n \circ y_n)(t_1) - (x \circ y)(t_1)\| &\leq \|x_n(y_n(t_1) - x(y(t))\| \\ &\quad + \|x(y(t)) - x(y(t_1))\| \\ &\leq v(x_n, x, y(t), \delta_2) + \bar{v}(x \circ y, t, \delta_1) \ . \end{aligned}$$ ∎

Lemma 7.2.4. *If* $t \notin Disc(y)$, $y(t) \notin Disc(x)$,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \ v(y_n, y, t, \delta) = 0 \tag{2.4}$$

and

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \ v(x_n, x, y(t), \delta) = 0 \ , \tag{2.5}$$

then

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \ v(x_n \circ y_n, x \circ y, t, \delta) = 0 \ . \tag{2.6}$$

Proof. Since $t \notin Disc(y)$ and $y(t) \notin Disc(x)$, $t \notin Disc(x \circ y)$ and $\bar{v}(x \circ y, t, \delta_1) \to 0$ as $\delta_1 \to 0$. We apply Lemma 7.2.3: For $\epsilon > 0$ given, choose δ_2 and n_1 so that

$$v(x_n, x, y(t), \delta_2) < \epsilon/2 \quad \text{for} \quad n \geq n_1 \ .$$

Then choose δ_1 and $n_2 \geq n_1$ so that $\bar{v}(x \circ y, t, \delta_1) < \epsilon/2$ and

$$v(y_n, y, t, \delta_1) \leq \delta_2 \quad \text{for} \quad n \geq n_2 \ .$$

By Lemma 7.2.3,

$$u(x_n \circ y_n, x \circ y, t, \delta_1) \leq \epsilon \quad \text{for} \quad n \geq n_2 \ .$$

Since ϵ was arbitrary, we have shown that

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \ u(x_n \circ y_n, x \circ y, t, \delta) = 0 \ ,$$

which is equivalent to (2.6) by Theorem 12.4.1 in the book. ■

Recall the m_p is the product metric inducing the WM_2 topology.

Lemma 7.2.5. *Suppose that $t \in Disc(y)$ but $y(t) \notin Disc(x)$, $y(t-) \notin Disc(x)$ and $x(y(t)) = x(y(t-))$ so that $t \notin Disc(x \circ y)$, i.e., case (ii) in Theorem 7.2.4. If $m_p(y_n, y) \to 0$ and $m_p(x_n, x) \to 0$ as $n \to \infty$, and $x(u) = x(y(t))$ for all $u \in [y(t-), y(t)]$, then* (2.6) *holds.*

Proof. Since $u \notin Disc(x)$ for all $u \in [y(t-), y(t)]$ and $m_p(x_n, x) \to 0$ as $n \to \infty$, for $\epsilon > 0$ given, we can choose δ_1 and n_0 so that

$$\sup_{0 \vee (y(t-) - \delta_1) \leq u \leq (y(t) + \delta_1) \wedge T} \{\|x_n(u) - x(u)\| \leq \epsilon \tag{2.7}$$

for all $n \geq n_0$ by Lemma 12.4.2 in the book. Since $x(u) = x(y(t))$ for $y(t-) \leq u \leq y(t)$ and x is continuous at $y(t-)$ and $y(t)$, from (2.7) we can obtain δ_2 such that

$$\sup_{0 \vee (y(t-) - \delta_2) \leq u \leq (y(t) + \delta_2) \wedge T} \{\|x_n(u) - x(y(t))\|\} \leq 2\epsilon \tag{2.8}$$

for $n \geq n_0$. By right continuity and the existence of left limits, we can choose $t_1 < t < t_2$ such that

$$y(t_1) < y(t-) < y(t_1) + \delta_2/2 \ , \tag{2.9}$$

$$y(t) < y(t_2) < y(t) + \delta_2/2 \ , \tag{2.10}$$

$$\|(x \circ y)(t_j) - (x \circ y)(t)\| < \epsilon \ , \tag{2.11}$$

and $t_j \notin Disc(y)$ for $j = 1, 2$. Applying (2.4), we can choose $\delta_3 > 0$ and $n_1 \geq n_0$ so that

$$v(y_n, y, t_j, \delta_3) < \delta_2/2 \tag{2.12}$$

for all $n \geq n_1$ and $j = 1, 2$. Combining (2.9)–(2.12), and using the monotonicity of y_n and y, we have for $0 \vee (t - \delta_3) \leq t', \ t'' \leq (t + \delta_3) \wedge T$, $\|y_n(t') - \{y(t-), y(t)\}\| < \delta_2$. Thus, by (2.8),

$$\|x_n \circ y_n(t') - x \circ y(t'')\| \leq \|x_n \circ y_n(t') - x \circ y(t)\| + \|x \circ y(t) - x \circ y(t'')\| \leq 3\epsilon \, .$$

Since ϵ was arbitrary, we have established (2.6). ■

We now turn to case (iii). We first show how we can exploit the monotonicity condition.

Lemma 7.2.6. (characterization of M_2 convergence at a monotone limit) *Suppose that x is monotone on $[a, b]$. Then $x_n \to x$ in $D([a, b], \mathbb{R}^k, WM_2)$ if and only if $x_n \to x$ pointwise on a dense subset of $[a, b]$ and*

$$\lim_{n\to\infty} w^*(x_n, [a, b]) = 0 \ , \tag{2.13}$$

where

$$w^*(x, [a, b]) \equiv \sup_{a \leq t_1 \leq t_2 \leq t_3 \leq b} \{\|x(t_2) - [x(t_1), x(t_3)]\|\} \ . \tag{2.14}$$

These imply that $x_n \to x$ as $n \to \infty$ in SM_1 as well.

Proof. Clearly $w_s(x, \delta) \leq w^*(x, [a, b])$ on $D([a, b], \mathbb{R}^k)$ for all $\delta > 0$, where

$$w_s(x, \delta) \equiv \sup_{a \leq t \leq b} w_s(x, t, \delta)$$

for $w_s(x, t, \delta)$ in equation (12.4.4) of the book, so that (2.13) plus the pointwise convergence implies that $x_n \to x$ as $n \to \infty$ in SM_1, by the basic characterization of SM_1 convergence, which in turn implies convergence in WM_2. To go the other way, suppose that $w^*(x_n, [a, b]) \not\to 0$ as $n \to \infty$. Then there exist $\epsilon > 0$ and subsequences $\{n_k\}$, $\{t_{n_k,j}\}$ for $j = 1, 2$ and 3 such that $n_k \to \infty$ and

$$\|x_{n_k}(t_{n_k,2}) - [x_{n_k}(t_{n_k,1}), x_{n_k}(t_{n_k,3})]\| > \epsilon \tag{2.15}$$

for all n_k. There are thus further subsequences $\{n'_k\}$, $\{t'_{n_k,2}\}$ for $j = 1, 2$, and 3 so that $t'_{n_k,j} \to t_j$ as $n'_k \to \infty$ for each j, where $t_1 \le t_2 \le t_3$. Assuming that $x_n \to x$ as $n \to \infty$ in WM_2, we have $x_{n'_k}(t'_{n_k,j}) \to [[x(t_j-), x(t_j)]]$ as $n'_k \to \infty$, by the characterization of WM_2 convergence. This, with (2.15) and the monotonicity of x, implies that

$$\max_{1 \le i \le k} \{\|x^i(t_2-) - [x^i(t_1-), x^i(t_3)]\|, \quad \|x^i(t_2) - [x^i(t_1-), x^i(t_3)]\| > 0\,,$$

which is impossible because x^i is monotone for each i. Hence, (2.13) must hold when $x_n \to x$ as $n \to \infty$ in WM_2. ∎

We will also apply the following elementary lemma, for which we omit the proof. We use the oscillation functions w_s in (12.4.4) and $\bar{v}$ in (12.4.3) of the book.

Lemma 7.2.7. *If*

$$y(t-) - \delta_2 \le y_n(t_1) \le y_n(t_2) \le y(t) + \delta_2$$

whenever $0 < t - \delta_1 \le t_1 \le t_2 \le t + \delta_1$, *then*

$$w_s(x_n \circ y_n, t, \delta_1) \le \bar{v}(x_n, y(t), \delta_2) + \bar{v}(x_n, y(t-), \delta_2) + w^*(x_n, [y(t-), y(t)])$$

for w^* *in (2.14).*

We apply Lemmas 7.2.6 and 7.2.7 to establish the following.

Lemma 7.2.8. *In case (iii), with* $t \in Disc(y)$, $y(t-)$, $y(t) \notin Disc(x)$ *and* x *monotone on* $[y(t-), y(t)]$, *if* $(x_n, y_n) \to (x, y)$ *in* $D^k(WM_2) \times D^1_{\uparrow}(WM_2)$, *then*

$$\lim_{\delta \downarrow 0} \; \overline{\lim_{n \to \infty}} \; w_s(x_n \circ y_n, t, \delta) = 0\ .$$

Proof. For any $\delta_2 > 0$ given, we can find δ_1 so that

$$y(t-) - \delta_2/2 \le y(t_1) \le y(t_2) \le y(t) + \delta_2/2$$

for $0 \vee (t - \delta_1) \le t_1 \le t_2 \le t + \delta_1$. By choosing continuity points of y, we can choose $n_2 \ge n_1$ so that

$$y(t-) - \delta_2 \le y_n(t_1) \le y_n(t_2) \le y(t) + \delta_2$$

for all $n \geq n_2$. Hence we can apply Lemmas 7.2.6 and 7.2.7. By Lemma 7.2.6, $w^*(x_n, [y(t-), y(t)]) \to 0$ as $n \to \infty$. Since $x_n \to x$ and $y(t-)$, $y(t) \notin Disc(x)$,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n\to\infty}} \ \bar{v}(x_n, y(t), \delta) = 0$$

and

$$\lim_{\delta \downarrow 0} \overline{\lim_{n\to\infty}} \ \bar{v}(x_n, y(t-), \delta) = 0 \ .$$

An application of Lemma 7.2.7 completes the proof. ∎

We now turn to case (iv). We first establish a preliminary result of independent interest, but which we do not directly need.

Lemma 7.2.9. *Suppose that $m_p(x_n, x) \to 0$ in D and $m(y_n, y) \to 0$ in $D^1_\uparrow$, but that $y(t) \in Disc(x)$. If y is strictly increasing and continuous in a neighborhood of t, then $(x_n \circ y_n)(t'_n) \to (x \circ y)(t')$ for all t' in a dense subset of neighborhood of t and all sequences $\{t'_n\}$ with $t'_n \to t'$.*

Proof. In the neighborhood of $y(t)$, there are at most countably many discontinuities of x. Since y is strictly increasing and continuous in a neighborhood of t, y is invertible there. Hence, for suitably small δ_2 and all but countably many t' in $(t - \delta_2, t + \delta_2)$, we simultaneously have y continuous at t' and x continuous at $y(t')$. At all such t', we have $y_n(t'_n) \to y(t')$ and $x_n(y_n(t')) \to x(y(t'))$ whenever $t'_n \to t'$, because m_p-convergence implies local uniform convergence at continuity points, by virtue of Theorem 12.4.1 in the book.

Corollary 7.2.1. *If y is strictly increasing and continuous whenever $y(t) \in Disc(x)$ and $(x_n, y_n) \to (x, y)$ in $D^1_\uparrow(M_1) \times D^1_\uparrow(M_1)$, then $x_n \circ y_n \to x \circ y$ in $D^1_\uparrow(M_1)$.*

Proof. By Lemma 7.2.6, M_1 convergence on $D^1_\uparrow$ coincides with pointwise convergence on a dense subset. Apply Lemma 7.2.9. ∎

Lemma 7.2.10. *If $m(y_n, y) \to 0$ in $D^1_\uparrow$, where y is continuous and strictly increasing at t, then for any $\delta > 0$, we can find $\delta_1 > 0$ such that, for all n sufficiently large,*

$$\begin{aligned} w_s(x_n \circ y_n, t, \delta_1) &\leq w_s(x_n, y(t), \delta) \ , \\ w_w(x_n \circ y_n, t, \delta_1) &\leq w_w(x_n, y(t), \delta) \ , \end{aligned}$$

$$\begin{aligned}\bar{w}_s(x_n \circ y_n, x \circ y, t, \delta_1) &\leq \bar{w}_s(x_n, x, y(t), \delta) ,\\ \bar{w}_s(x_n \circ y_n, x \circ y, t, \delta_1) &\leq \bar{w}_w(x_n, x, y(t), \delta) .\end{aligned}$$

Proof. Since y is continuous at t, we can find $t_1 < t < t_2$ such that y is continuous at t_1 and t_2 and $|y(t) - y(t_j)| < \delta/2$ for $j = 1, 2$. Since $y_n \to y$ we can find n_0 such that $|y_n(t_j) - y(t_j)| < \delta/2$ for $n \geq n_0$ and $j = 1, 2$. By the triangle inequality, $|y_n(t_j) - y(t)| < \delta$ for $n \geq n_0$ and $j = 1, 2$. Let $\delta_1 = \min\{|t-t_1|, |t-t_2|\}$. Since y_n and y are nondecreasing, $|y_n(t') - y(t)| < \delta$ whenever $|t' - t| < \delta_1$. Hence

$$w_s(x_n \circ y_n, t, \delta_1) \leq w_s(x_n, y(t), \delta)$$

and

$$w_w(x_n \circ y_n, t, \delta_1) \leq w_w(x_n, y(t), \delta) .$$

Moreover, since y is continuous and strictly increasing, $x(y(t)-) = x(y(t-))$. Hence

$$\bar{w}_s(x_n \circ y_n, x \circ y, t, \delta_1) \leq \bar{w}_s(x_n, x, y(t), \delta)$$

and

$$\bar{w}_w(x_n \circ y_n, x \circ y, t, \delta_1) \leq \bar{w}_w(x_n, x, y(t), \delta) . \quad \blacksquare$$

7.3. Composition with Centering

This section considers the composition map with centering. Nothing was omitted from the book here.

7.4. Supremum

In this section we consider the supremum function, mapping $D \equiv D([0, T], \mathbb{R})$ into itself according to

$$x^{\uparrow}(t) = \sup_{0 \leq s \leq t} x(s), \quad 0 \leq t \leq T . \tag{4.1}$$

7.4.1. The Supremum without Centering

The following elementary result is stated without proof.

Theorem 7.4.1. (Lipschitz property of the supremum function) *For any* $x_1, x_2 \in D([0,T],\mathbb{R})$,

$$\begin{aligned} d_{J_1}(x_1^{\uparrow}, x_2^{\uparrow}) &\leq d_{J_1}(x_1, x_2) \ , \\ d_{M_1}(x_1^{\uparrow}, x_2^{\uparrow}) &\leq d_{M_1}(x_1, x_2) \ , \\ d_{M_2}(x_1^{\uparrow}, x_2^{\uparrow}) &\leq d_{M_2}(x_1, x_2) \ . \end{aligned}$$

The conclusion in Theorem 7.4.1 can be recast in terms of pointwise convergence: Since $x^{\uparrow}$ is nondecreasing, convergence $x_n^{\uparrow} \to x^{\uparrow}$ in the M topologies is equivalent to pointwise convergence at continuity points of $x^{\uparrow}$, because on $D_{\uparrow}$ the M_1 and M_2 topologies coincide with pointwise convergence on a dense subset of $\mathbb{R}_+$ including 0; see Corollary 12.5.1 in the book. Thus the M topologies have not contributed much so far. We obtain more useful convergence-preservation results for the supremum map with the M topologies when we combine supremum with centering. As before, let e be the identity map, i.e., $e(t) = t$, $0 \leq t \leq T$.

7.4.2. The Supremum with Centering

The following is the main result stated as Theorem 13.4.2 in the book. Our object here is to prove it.

Theorem 7.4.2. (convergence preservation with the supremum function and centering) *Suppose that* $c_n(x_n - e) \to y$ *as* $n \to \infty$ *in* $D([0,T],\mathbb{R})$ *with one of the topologies* J_1, M_1 *or* M_2, *where* $c_n \to \infty$.

(a) If the topology is M_1 *or* M_2, *then* $c_n(x_n^{\uparrow} - e) \to y$ *in the same topology.*

(b) If the topology is J_1, *then* $c_n(x_n^{\uparrow} - e) \to y$ *if and only if* y *has no negative jumps.*

Before proving Theorem 7.4.2, we establish some preliminary lemmas. We first give an alternative expression for the result, in the form of a continuous mapping theorem. Let $y_n \equiv c_n(x_n - e)$. Then $s_n(y_n) = c_n(x_n^{\uparrow} - e)$, where

$$s_n(y) \equiv (y + c_n e)^{\uparrow} - c_n e \quad \text{for} \quad y \in D \ . \tag{4.2}$$

Thus the conclusion of Theorem 7.4.2 can be expressed as $s_n(y_n) \to s(y) \equiv y$ when $y_n \to y$, with the appropriate topology.

Note that, for $x \in D$ and s_n in (4.2), $s_n(x)$ cannot have any negative jumps. For any $x \in D$, we can characterize $s_n(x)$ as the majorant which decreases by at most slope c_n at any time; i.e.,

$$s_n(x) = \inf\{y \in D : y \geq x, \ y(t_2) - y(t_1) \geq -c_n(t_2 - t_1)\}, \tag{4.3}$$

where we allow $0 \le t_1 < t_2 \le T$.

Lemma 7.4.1. *For any $x \in D$, $s_n(x)$ defined by (4.2) satisfies (4.3).*

Proof. First note that $s_n(x) \ge x$. Next note that

$$\begin{aligned} s_n(x)(t_2) - s_n(x)(t_1) &= (x + c_n e)^{\uparrow}(t_2) - (x + c_n e)^{\uparrow}(t_1) - c_n(t_2 - t_1) \\ &\ge -c_n(t_2 - t_1)\,. \end{aligned}$$

Finally, suppose that $y \ge x$ and $y(t_2) - y(t_1) \ge -c_n(t_2 - t_1)$ for all $0 \le t_1 < t_2 < T$. Then $s_n(y) = y$. Since $y \ge x$, $s_n(y) \ge s_n(x)$. Hence $y \ge s_n(x)$. ∎

We can also bound $s_n(x)$ above for sufficiently large n by another majorant. Let the *left-local-majorant* of $x \in ([0,T], \mathbb{R})$ be

$$s_l^{\epsilon}(x)(t) = \sup_{0 \vee (t-\epsilon) \le s \le t} x(s), \quad 0 \le t \le T\,. \tag{4.4}$$

It is obvious that $x \le s_l^{\epsilon}(x)$ for all x and $\epsilon > 0$. Moreover $s_l^{\epsilon}(x)(t)$ is nonincreasing as $\epsilon \downarrow 0$. We now show that $s_l^{\epsilon}(x) \to x$ in (D, M_2) as $\epsilon \downarrow 0$.

Lemma 7.4.2. *For any $x \in D$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_{M_2}(x, s_l^{\delta}(x)) \le \epsilon\,. \tag{4.5}$$

Proof. First, for x and ϵ given, apply Theorem 12.2.2 in the book to choose $x_c \in D_c$ such that $\|x - x_c\| < \epsilon/3$. For x_c, it is evident that there exists δ with $0 < \delta < \epsilon/3$ such that

$$d_{M_2}(s_l^{\delta}(x_c), x_c) < \delta < \epsilon/3 \quad \text{and} \quad \|s_l^{\delta}(x_c) - s_l^{\delta}(x)\| < \epsilon/3\,.$$

Hence,

$$d_{M_2}(x, s_l^{\delta}(x)) \le \|x - x_c\| + d_{M_2}(x_c, s_l^{\delta}(x_c)) + \|s_l^{\delta}(x_c) - s_l^{\delta}(x)\| < \epsilon \quad \blacksquare \tag{4.6}$$

We now show that $s_n(x) \to x$ as $n \to \infty$ in the M_2 topology, uniformly over a large class of functions x.

Lemma 7.4.3. *Let s_n be as in (4.2), where $c_n \to \infty$. For any M and $\epsilon > 0$, there is an n_0 such that*

$$d_{M_2}(s_n(x), x) < \epsilon, \quad n \ge n_0\,, \tag{4.7}$$

for all x with $\|x\| \le M$.

Proof. Let ϵ, M and x be given with $\|x\| \leq M$. Apply Lemma 7.4.2 to find δ such that $m(s_l^\delta(x), x) < \delta < \epsilon$. Choose n_0 so that $c_n\delta > 2M$ for $n \geq n_0$. Then, for $n \geq n_0$,

$$x(s) + c_n s - c_n t \leq x(t) \tag{4.8}$$

for all s, $0 \leq s \leq t - \delta$, $0 \leq t \leq T$, because under those conditions

$$x(s) + c_n s - c_n t \leq M - c_n\delta \leq -M \leq x(t)\,. \tag{4.9}$$

Hence, for $n \geq n_0$,

$$x \leq s_n(x) \leq s_l^\delta(x)\,, \tag{4.10}$$

so that, by Lemma 7.4.2, $s_n(x)$ is contained in an M_2 ϵ-neighborhood of x; i.e., (4.7) holds. ■

Next, for the J_1 results we need the following.

Lemma 7.4.4. *If $x \in D([0,T],\mathbb{R})$ and x has no negative jumps, then for any $\epsilon > 0$ there is a $\delta > 0$ such that*

$$v^-(x,\delta) \equiv \sup_{\substack{0 \vee (t-\delta) \leq t' \leq t \\ 0 \leq t \leq T}} \{x(t') - x(t)\} < \epsilon\ . \tag{4.11}$$

Proof. Under the condition, for any $\epsilon > 0$ and all $t \in (0,T]$, there is a $\delta(t)$ such that $0 < t - \delta(t) < t$ and

$$x(t') \leq x(t) + \epsilon \quad \text{for all} \quad t' \in (t - \delta(t), t)\ . \tag{4.12}$$

By the right continuity of x at 0, there is a $\delta(0)$ such that $\|x(t') - x(0)\| < \epsilon$ for $0 \leq t' \leq \delta(0)$. The intervals $[0,\delta(0))$, $(t - \delta(t), t)$, $0 < t \leq T$, form an open cover of the compact set $[0,T]$. Hence there is a finite subcover. Let the subcover be chosen (modified) so that each t is in at most two subintervals. Let δ be the minimum length of the overlapping intervals, i.e.,

$$\delta = \min_i\{|t_i + \delta(t_{i+1}) - t_{i+1}|\} \wedge \delta(0)\,. \tag{4.13}$$

Then, if t is any point in $[0,T]$, it either belongs to the subinterval $[0,\delta(0))$ or it is at least δ away from the left endpoint of one of its subintervals. Hence property (4.11) holds for δ in (4.13). ■

Proof of Theorem 7.4.2. (a) We will show that $s_n(x_n) \to x$ whenever $x_n \to x$, for s_n in (4.2). First consider the M_2 topology. Let M be a constant so that $\|x\| \le M/2$. Since $d_{M_2}(x_n, x) \to 0$, there is an n_0 such that $\|x_n\| \le M$ for all $n \ge n_0$. By the condition and Lemma 7.4.3, for any $\epsilon > 0$ there is an $n_1 \ge n_0$ such that $d_{M_2}(x_n, x) < \epsilon/2$ and $d_{M_2}(s_n(x_n), x_n) < \epsilon/2$ for $n \ge n_1$. Hence, by the triangle inequality, for $n \ge n_1$,

$$d_{M_2}(s_n(x_n), x) \le d_{M_2}(s_n(x_n), x_n) + d_{M_2}(x_n, x) < \epsilon \ .$$

Next consider the M_1 topology. Since M_1 convergence implies M_2 convergence, we have $d_{M_2}(s_n(x_n), x) \to 0$ by the proof above. It thus suffices to strengthen convergence from M_2 to M_1. In particular, we can apply part (v) of Theorem 12.5.1 in the book. By Theorem 12.4.1 in the book, the M_2 convergence implies the local uniform convergence at continuity points in condition (12.5.4) in the book, so it only remains to establish the oscillation function limit at discontinuity points in condition (12.5.5) in the book; i.e.,

$$\lim_{\delta \downarrow 0} \ \overline{\lim_{n \to \infty}} \ w_s(s_n(x_n), t, \delta) = 0 \ . \tag{4.14}$$

We show that if (4.14) fails, then necessarily we cannot have

$$\lim_{\delta \downarrow 0} \ \overline{\lim_{n \to \infty}} \ w_s(x_n, t, \delta) = 0 \ , \tag{4.15}$$

so that $x_n \not\to x$ (M_1), which is a contradiction. If (4.14) fails, then there must exist $\epsilon > 0$, $\delta_k \downarrow 0$ and $n_k \uparrow \infty$ such that

$$w_s(s_{n_k}(x_{n_k}), t, \delta_k) > \epsilon \quad \text{for all} \quad k \ . \tag{4.16}$$

Let $y_{n_k} = s_{n_k}(x_{n_k})$. Given (4.16), there are two cases: In the first case, there exist $t_{n_k,1}$, $t_{n_k,2}$ and $t_{n_k,3}$ such that

$$0 \vee (t - \delta_k) \le t_{n_k,1} < t_{n_k,2} < t_{n_k,3} \le (t + \delta_k) \wedge T \ , \tag{4.17}$$

$y_{n_k}(t_{n_k,2}) > y_{n_k}(t_{n_k,1}) + \epsilon$ and $y_{n_k}(t_{n_k,2}) > y_{n_k}(t_{n_k,3}) + \epsilon$. However, $y_{n_k}(t_{n_k,2}) > y_{n_k}(t_{n_k,1}) + \epsilon$ implies that there must exist $t'_{n_k,2}$ with $t_{n_k,1} < t'_{n_k,2} \le t_{n_k,2}$ and $x_{n_k}(t'_{n_k,2}) \ge y_{n_k}(t_{n_k,2})$. Since $y_{n_k}(t_{n_k,1}) \ge x_{n_k}(t_{n_k,1})$ and $y_{n_k}(t_{n_k,3}) \ge x_{n_k}(t_{n_k,3})$, we then must have $w_s(x_{n_k}, t, \delta_k) > \epsilon$, which contradicts (4.15).

In the second case, there exist $t_{n_k,1}$, $t_{n_k,2}$ and $t_{n_k,3}$ such that (4.17) holds, $y_{n_k}(t_{n_k,2}) < y_{n_k}(t_{n_k,1}) - \epsilon$ and $y_{n_k}(t_{n_k,2}) < y_{n_k}(t_{n_k,3}) - \epsilon$. By the last inequality, there must exist $t'_{n_k,3}$ with $t_{n_k,2} < t'_{n_k,3} \le t_{n_k,3}$ such that

$x_{n_k}(t'_{n_k,3}) \geq y_{n_k}(t_{n_k,3}) - \epsilon$. Since $x_n \leq y_n$, $x_{n_k}(t_{n_k,2}) \leq y_{n_k}(t_{n_k,2})$. Finally, since $\{x_{n_k}\}$ is uniformly bounded, there is δ'_k where $\delta'_k \downarrow 0$ as $k \to \infty$, and $t'_{n_k,1}$ with $0 \vee (t - (\delta_k + \delta'_k)) \leq t'_{n_k,1} \leq t_{n_k,1}$ with $x_{n_k}(t'_{n_k,1}) \geq y_{n_k}(t_{n_k,1})$. Hence, we must have

$$w_s(x_{n_k}, t, \delta_k + \delta'_k) > \epsilon \quad \text{for all} \quad k \ . \tag{4.18}$$

Since $\delta_k + \delta'_k \downarrow 0$ as $k \to \infty$, (4.18) again contradicts (4.15) and thus $x_n \to x$ (M_1). Thus, $d_{M_1}(s_n(x_n), x) \to 0$ as claimed.

(b) We now turn to the J_1 result. Given $c_n(x_n - e) \to y$ (J_1), there exists $\lambda_n \in \Lambda$ such that $\|c_n(x_n - e) - y \circ \lambda_n\| \to 0$ as $n \to \infty$. We want to show that $\|c_n(x_n^{\uparrow} - e) - y \circ \lambda_n\| \to 0$. Since $x_n^{\uparrow} \geq x_n$, it suffices to show, for any $\epsilon > 0$, that there is n_1 such that

$$c_n x_n(s') - c_n s \leq y(\lambda_n(s)) + \epsilon \quad \text{for} \quad 0 \leq s' \leq s \leq T \tag{4.19}$$

for $n \geq n_1$. Choose n_0 such that $\|c_n(x_n - e) - y \circ \lambda_n\| < \epsilon/2$ for $n \geq n_0$. From (4.19), we see that it suffices to show that there is $n_1 \geq n_0$ such that

$$y(\lambda_n(s')) \leq y(\lambda_n(s)) + c_n(s - s') + \epsilon/2 \quad \text{for} \quad 0 \leq s' \leq s \leq T \, . \tag{4.20}$$

Since y has no negative jumps, we can apply Lemma 7.4.4 to conclude that there is a δ such that $v^-(y, \delta) < \epsilon/2$ for $v^-(y, \delta)$ in (4.11). Then choose $n_1 \geq n_0$ such that $\|\lambda_n - e\| < \delta$ and $c_n \delta \geq \|y\|$ for $n \geq n_1$, and we obtain (4.20). Finally, recall that the maximum negative jump function is continuous, e.g., see p. 301 of Jacod and Shiryaev (1987); i.e.,

$$J_-(x) \equiv \sup_{0 < t \leq 1} \{x(t-) - x(t)\} \ . \tag{4.21}$$

Clearly, $J_-(c_n(x_n^{\uparrow} - e)) = 0$, so that if $c_n(x_n^{\uparrow} - e) \to y$ (J_1), then y must have no negative jumps. ∎

We now obtain joint convergence in the stronger topologies on $D([0,T], \mathbb{R}^2)$ under the condition that the limit function have no negative jumps.

Theorem 7.4.3. (criterion for joint convergence) *Suppose that $c_n(x_n - e) \to y$ as $n \to \infty$ in $D([0,T], \mathbb{R})$ with one of the J_1, M_1 or M_2 topologies, where $c_n \to \infty$. If, in addition, y has no negative jumps, then*

$$c_n(x_n - e, x_n^{\uparrow} - e) \to (y, y) \quad as \quad n \to \infty \tag{4.22}$$

in $D([0,T], \mathbb{R}^2)$ with the strong version of the same topology, i.e., with SJ_1, SM_1 or SM_2.

Proof. For the SM_1 and SM_2 topologies, we will work with parametric representations, using the parametric representation $((u,u),r)$ for (y,y). Given that $(c_n(x_n - e) \to y$, there exist parametric representations $(u_n, r_n) \in \Pi_s(c_n(x_n - e))$ and $(u,r) \in \Pi(y)$ such that $\|u_n - u\| \vee \|r_n - r\| \to 0$ as $n \to \infty$. We construct the desired parametric representations from these. Note that $(c_n^{-1}u_n + r_n, r_n) \in \Pi(x_n)$ and $(u_n', r_n) \in \Pi(c_n(x_n^{\uparrow} - e))$ for

$$u_n' = c_n((c_n^{-1}u_n + r_n)^{\uparrow} - r_n) = (u_n + c_n r_n)^{\uparrow} - c_n r_n \,. \tag{4.23}$$

Note that $x_n^{\uparrow}$ has the jumps up of x_n, while $x_n^{\uparrow}$ is continuous when x_n has a jump down. Thus $((u_n, u_n'), r_n) \in \Pi_s(y_n, y_n')$ for $y_n \equiv c_n(x_n - e)$ and $y_n' \equiv c_n(x_n^{\uparrow} - e)$. Of course $((u,u),r) \in \Pi_s((y,y))$. Thus it remains to show that

$$\|(u_n, u_n') - (u,u)\| \vee \|r_n - r\| \to 0 \quad \text{as} \quad n \to \infty \,. \tag{4.24}$$

Given that $\|u_n - u\| \vee \|r_n - r\| \to 0$, it suffices to show that $\|u_n' - u\| \to 0$. Clearly, $u_n' \geq u_n$ for all n, so that it suffices to show that, for all $\epsilon > 0$, there exist n_1 such that $u_n'(s) < u(s) + \epsilon$ for all $n \geq n_1$ and $s \in [0,1]$. Equivalently, by (4.23), it suffices to show that

$$u_n(s') + c_n(r_n(s') - r_n(s)) < u(s) + \epsilon, \quad 0 \leq s' \leq s \leq 1 \;, \tag{4.25}$$

for all $n \geq n_1$. However, if we assume that the limit y has no negative jumps, then Lemma 7.4.4 implies that there is a $\delta > 0$ such that

$$u(s') \leq u(s) + \epsilon/2 \tag{4.26}$$

for all s, s' with $0 \leq s' \leq s \leq 1$ and $r(s) - r(s') < \delta$. Choose n_0 so that

$$\|u_n - u\| \vee \|r_n - r\| \leq (\delta \wedge \epsilon)/4 \quad \text{for} \quad n \geq n_0 \,.$$

Choose $n_1 \geq n_0$ so that

$$c_n \delta/4 \geq 2\|x\| \quad \text{for} \quad n \geq n_1 \;. \tag{4.27}$$

There are two cases: (i) $r_n(s) - r_n(s') \leq \delta/4$ and (ii) $r_n(s) - r_n(s') > \delta/4$. In case (i), $r(s) - r(s') < \delta$, so that by (4.26)

$$u_n(s') + c_n(r_n(s') - r_n(s)) \leq u_n(s') \leq u(s') + \epsilon/4 \leq u(s) + \epsilon \,, \tag{4.28}$$

so that (4.25) holds. In case (ii), by (4.27),

$$\begin{aligned} u_n(s') + c_n(r_n(s') - r_n(s)) &\leq u(s') + \epsilon/2 - c_n\delta/4 \\ &\leq u(s) + 2\|u\| - c_n\delta/4 + \epsilon/2 \\ &\leq u(s) + \epsilon \,, \end{aligned} \tag{4.29}$$

so that again (4.25) holds. Turning to J_1, we note that the result already follows from the proof of Theorem 7.4.2 (ii) because the same homeomorphims $\lambda_n \in \Lambda$ were used for both $c_n(x_n - e) \to y$ and $c_n(x_n^{\uparrow} - e) \to y$. ■

Corollary 7.4.1. *Under the conditions of Theorem 7.4.3,*

$$\|c_n(x_n^{\uparrow} - x_n)\| \to 0 \quad as \quad n \to \infty .$$

Proof. Apply subtraction to get

$$c_n(x_n - x_n^{\uparrow}) = c_n(x_n - e) - c_n(x_n^{\uparrow} - e) \to x - x(M_2) .$$

Since the limit is continuous, the convergence holds in the uniform topology. ■

We next give an elementary result about the supremum function when the centering is in the other direction, so that x_n must be rapidly decreasing. Convergence $x_n^{\uparrow}(t) \to x(0)$ as $n \to \infty$ is to be expected, but that conclusion can not be drawn if the M_2 convergence in the condition is replaced by pointwise convergence.

Theorem 7.4.4. (convergence preservation with the supremum function when the centering is in the other direction) *Suppose that* $c_n \to \infty$ *and* $x_n + c_n e \to y$ *in* $D([0,T], \mathbb{R}, M_2)$. *Then*

$$\|x_n^{\uparrow} - z(y)\| \to 0 \quad as \quad n \to \infty ,$$

where $z(y)(t) \equiv y(0)$, $0 \le t \le T$.

Proof. The assumed M_2 convergence implies local uniform convergence at the origin: For any $\epsilon > 0$, there is a δ and an n_0 such that

$$\sup_{0 \le t \le \delta} |x_n(t) + c_n t - y(0)| \le v(x_n, y, 0, \delta) < \epsilon$$

for $n \ge n_0$, where $v(x_1, x_2, t, \delta)$ is the modulus of continuity in (4.2) in Section 6.4. Hence, $x_n(t) \le y(0) + \epsilon$ for all t, $0 \le t \le \delta$, and $n \ge n_0$. Use the conditions to find $n_1 \ge n_0$ such that $\|x_n + c_n e\| \le \|y\| + \epsilon$ and $c_n\delta > 2\|y\|$ for $n \ge n_1$. Then, for $t > \delta$ and $n \ge n_1$,

$$x_n(t) = -c_n\delta + x_n(t) + c_n\delta \le -c_n\delta + \|x_n + c_n e\| \le -c_n\delta + \|y\| + \epsilon \le y(0) + \epsilon .$$

Hence, $x_n^{\uparrow}(t) \le y(0) + \epsilon$ for all t, $0 \le t \le T$, and $n \ge n_1$. On the other hand, for all t, $x_n^{\uparrow}(t) \ge x_n(0) \to y(0)$ as $n \to \infty$. ■

7.5. One-Dimensional Reflection

Closely related to the supremum function is the one-dimensional (one-sided) reflection mapping, which we have used to construct queueing processes. Indeed, the reflection mapping can be defined in terms of the supremum mapping as

$$\phi(x) \equiv x + (-x \vee 0)^{\uparrow} \ ;$$

i.e.,

$$\phi(x)(t) = x(t) - (\inf\{x(s) : 0 \le s \le t\} \wedge 0) \ , \quad 0 \le t \le T \ , \qquad (5.1)$$

as in equation (2.5) in Section 5.2 of the book.

The Lipschitz property for the supremum function with the uniform topology in Lemma **??** immediately implies a corresponding result for the reflection map ϕ in (5.1).

Unfortunately, however, the Lipschitz property for the reflection map ϕ with the uniform topology does not even imply continuity in all the Skorohod topologies. In particular, ϕ is not continuous in the M_2 topology.

We do obtain positive results with the J_1 and M_1 topologies. As before, let d_{J_1} and d_{M_1} be the metrics in equations 3.2 and 3.4 in Section 3.3 of the book. For the J_1 result, we use the following elementary lemma.

Lemma 7.5.1. *For any $x \in D$ and $\lambda \in \Lambda$,*

$$\phi(x) \circ \lambda = \phi(x \circ \lambda) \ .$$

For the M_1 result, we use the following lemma. A fundamental difficulty for treating the more general multidimensional reflection map is that Lemma 7.5.2 below does not extend to the multidimensional reflection map; see Chapter 8.

Lemma 7.5.2. (preservation of parametric representations under reflections) *For any $x \in D$, if $(u, r) \in \Pi(x)$, then $(\phi(u), r) \in \Pi(\phi(x))$.*

Proof. In book. ∎

Theorem 7.5.1. (Lipschitz property with the J_1 and M_1 metrics) *For any $x_1, x_2 \in D([0,T], \mathbb{R})$,*

$$d_{J_1}(\phi(x_1), \phi(x_2)) \le 2 d_{J_1}(x_1, x_2)$$

and

$$d_{M_1}(\phi(x_1), \phi(x_2)) \le 2 d_{M_1}(x_1, x_2)) \ ,$$

where ϕ is the reflection map in (5.1).

Proof. In book. ■

Theorem 7.5.1 covers the standard heavy-traffic regime for one single-server queue when $\rho = 1$, where ρ is the traffic intensity. The next result covers the other cases: $\rho < 1$ and $\rho > 1$. We use the following elementary lemma in the easy case of the uniform metric.

Lemma 7.5.3. *Let d be the metric for the U, J_1, M_1 or M_2 topology. Let $x \vee a : D \to D$ be defined by*

$$(x \vee a)(t) \equiv x(t) \vee a, \quad 0 \leq t \leq T. \tag{5.2}$$

Then, for any $x_1, x_2 \in D$,

$$d(x \vee a(x_1), x \vee a(x_2)) \leq d(x_1, x_2) .$$

Theorem 7.5.2. (convergence preservation with centering) *Suppose that $x_n - c_n e \to y$ in $D([0,T], \mathbb{R})$ with the U, J_1, M_1 or M_2 topology.*

(a) If $c_n \to +\infty$, then

$$\phi(x_n) - c_n e \to y + \gamma(y) \quad as \quad n \to \infty \quad in \quad D$$

with the same topology, where

$$\gamma(y)(t) \equiv (-y(0)) \vee 0 = -(y(0) \wedge 0), \quad 0 \leq t \leq T .$$

(b) If $c_n \to -\infty$, $y(0) \leq 0$ and y has no positive jumps, then

$$\|\phi(x_n) - 0e\| \to 0 \quad as \quad n \to \infty \quad in \quad D ,$$

where $e(t) = t$, $0 \leq t \leq T$.

Proof. (a) Note that

$$\phi(x_n) - c_n e = x_n - c_n e + (-x_n \vee 0)^{\uparrow} ,$$

where $(-x_n \vee 0)^{\uparrow} = (-x_n)^{\uparrow} \vee 0$. By assumption, $x_n - c_n e \to y$. By Theorem 7.4.4,

$$\|(-x_n)^{\uparrow} - z(-y)\| \to 0 \quad \text{as} \quad n \to \infty$$

where $z(-y)(t) = -y(0)$, $0 \leq t \leq T$. By Lemma 7.5.3,

$$\|(-x_n)^{\uparrow} \vee 0 - z(-y) \vee 0\| \to 0 \quad \text{as} \quad n \to \infty .$$

We obtain the desired convergence by adding, using the fact that the second term has a continuous limit.

(b) Apply the argument of Theorem 7.4.4 to show that, for all $\epsilon > 0$, there exists n_1 such that $-x_n(t) > -y(0) - \epsilon$ for all t, $0 \leq t \leq T$, and all $n \geq n_1$. Since $y(0) \leq 0$, $-x_n(t) > -\epsilon$ for all t, $0 \leq t \leq T$, and all $n \geq n_1$. Thus,

$$(-x_n + c_n e, (-x_n) \vee 0 + c_n e) \to (-y, -y)$$

in $D([0,T], \mathbb{R}^2)$ with the appropriate strong topology. Then, by Theorem 7.4.3,

$$(-x_n + c_n e, (-x_n) \vee 0 + c_n e, (-x_n \vee 0)^{\uparrow} + c_n e) \to (-y, -y, -y) \quad (5.3)$$

in $D([0,T], \mathbb{R}^3)$ with the appropriate strong topology. Then, by applying subtraction to the first and third terms in (5.3), we get

$$\begin{aligned} \phi(x_n) &\equiv x_n + (-x_n \vee 0)^{\uparrow} \\ &= [(-x_n \vee 0)^{\uparrow} + c_n e] - [-x_n + c_n e] \\ &\to -y + y = 0e \end{aligned} \quad (5.4)$$

as $n \to \infty$. ■

7.6. Inverse

We now consider the inverse map.. It is convenient to consider the inverse map on the subset D_u of x in $D \equiv D([0,\infty), \mathbb{R})$ that are unbounded above and satisfy $x(0) \geq 0$. For $x \in D_u$, let the inverse of x be

$$x^{-1}(t) = \inf\{s \geq 0 : x(s) > t\}, \quad t \geq 0 \ . \quad (6.1)$$

As before, let D_0 be the subset of x in D with $x(0) \geq 0$, and let $D_{\uparrow}$ and $D_{\uparrow\uparrow}$ be the subsets of nondecreasing and strictly increasing functions in D_0. Let $D_{u\uparrow} \equiv D_u \cap D_{\uparrow}$ and $D_{u\uparrow\uparrow} \equiv D_u \cap D_{\uparrow\uparrow}$. Clearly,

$$D_{\uparrow\uparrow} \subseteq D_{\uparrow} \subseteq D_u \subseteq D_0 \ .$$

7.6.1. The M_1 Topology

Even for the M_1 topology, there are complications at the left endpoint of the domain $[0,\infty)$.

Example 7.6.1. *Complications at the left endpoint of the domain.* To see that the inverse map from $(D_\uparrow, U)$ to $(D_\uparrow, M_1)$ is in general not continuous, let $x(t) = 0$, $0 \le t < 1$, and $x(t) = t$, $t \ge 1$; Let $x_n = t/n$, $0 \le t < 1$ and $x_n(t) = t$, $t \ge 1$. Then $\|x_n - x\|_\infty = n^{-1} \to 0$, but $x_n^{-1}(0) = 0 \not\to 1 = x^{-1}(0)$, so that $x_n^{-1} \not\to x^{-1}$ (M_1). ■

To avoid the problem in Example 7.6.1, we can require that $x^{-1}(0) = 0$. To develop an equivalent condition, let $D_\epsilon^\uparrow$ be the subset of functions x in D_u such that $x(t) = 0$ for $0 \le t \le \epsilon$.

Then let

$$D_u^* \equiv \cap_{n=1}^\infty (D_{u,n^{-1}})^c \ . \tag{6.2}$$

Lemma 7.6.1. (measurability of D_u^*) *With the J_1, M_1 or M_2 topology, D_u^* in* (6.2) *is a G_δ subset of D_u and*

$$D_u^* = \{x \in D_u : x^{-1}(0) = 0\} \ . \tag{6.3}$$

Let $D_{u\uparrow}^* \equiv D_\uparrow \cap D_u^*$. A key property of $D_{u,\uparrow}^*$, not shared by $D_{u,\uparrow}$ because of the complication at the origin, is that parametric representation (u, r) for x directly serve as parametric representations for x^{-1} when we switch the roles of the components u and r.

Lemma 7.6.2. (switching the roles of u and r) *For $x \in D_{u,\uparrow}^*$, the graph Γ_x serves as the graph of $\Gamma_{x^{-1}}$ with the axes switched. Thus, $(u, r) \in \Pi(x)$ if and only if $(r, u) \in \Pi(x^{-1})$, where $\Pi(x)$ is the set of M_1 parametric representations.*

Corollary 7.6.1. (continuity on (D_u^*, M_1)) *The inverse map from (D_u^*, M_1) to $(D_{u,\uparrow}, M_1)$ is continuous.*

Proof. First apply Theorem 7.4.1 for the supremum. Then apply Lemma 7.6.2. ■

We now generalize Corollary 7.6.1 by only requiring that the limit be in D_u^*.

Theorem 7.6.1. (measurability and continuity at limits in D_u^*) *The inverse map in* (6.1) *from (D_u, M_2) to $(D_{u,\uparrow}, M_1)$ is measurable and continuous at $x \in D_u^*$, i.e., for which $x^{-1}(0) = 0$.*

Proof. First, recalling that the Borel σ-field on D coincides with the Kolmogorov σ-field generated by the projections, measurability follows from Lemma **??**; it suffices to show that $\{x : x^{-1}(t) \leq a\}$ is measurable. However,

$$\begin{aligned} \{x : x^{-1}(t) \leq a\} &= \cap_{j=1}^{\infty} \cap_{k=1}^{\infty} \{x : x^{-1}((t + j^{-1})-) \leq a + k^{-1}\} \\ &= \cap_{j=1}^{\infty} \cap_{k=1}^{\infty} \{x : x^{\leftarrow}((t + j^{-1})) \leq a + k^{-1}\} \\ &= \cap_{j=1}^{\infty} \cap_{k=1}^{\infty} \{x : x(a + k^{-1}) \geq t + j^{-1}\} , \end{aligned} \qquad (6.4)$$

which is measurable. Next we turn to continuity. For any $x \in D_u$, $x^{-1} = (x^{\uparrow})^{-1}$, so it suffices to start from $x_n^{\uparrow} \to x^{\uparrow}$. By Theorem 7.4.1, the assumed convergence $x_n \to x$ in (D_u, M_2) implies that $x_n^{\uparrow} \to x^{\uparrow}$ in $(D_{\uparrow}, M_2)$. However, the M_1 and M_2 topologies coincide in $D_{\uparrow}$. So $x_n^{\uparrow} \to x^{\uparrow}$ in $(D_{\uparrow}, M_1)$. Since $x \in D_u^*$, $x^{\uparrow} \in D_{u,\uparrow}^*$. However, we need not have $x_n^{\uparrow} \in D_{u,\uparrow}^*$. We could directly apply Lemma 7.6.2 if $x_n^{\uparrow} \in D_{\uparrow}^*$ for all sufficiently large n. Hence suppose that is not the case. Then there exists a subsequence $\{x_{n_k}^{\uparrow}\}$ with $x_{n_k}^{\uparrow} \notin D_{u,\uparrow}^*$ for all n_k. Necessarily, then, $x_{n_k}^{\uparrow}(0) = 0$ for all n_k. Since $x_n^{\uparrow} \to x^{\uparrow}$, we can conclude that $x^{\uparrow}(0) = 0$. Since $x^{\uparrow}$ is right continuous and $x^{\uparrow} \in D_{u,\uparrow}^*$, for any $\epsilon > 0$, there exists δ, $0 < \delta < \epsilon/2$, such that $\delta \in Disc(x^{\uparrow})^c$ and $0 < x^{\uparrow}(\delta) < \epsilon/2$. Let n_0 then be such that $|x_n^{\uparrow}(0) - x^{\uparrow}(0)| < \epsilon/2$ and $|x_n^{\uparrow}(\delta) - x^{\uparrow}(\delta)| < \epsilon/2$ for all $n \geq n_0$. Hence, for $n \geq n_0$, we can define an approximation to $x_{n_k}^{\uparrow}$ which belongs to $D_{u,\uparrow}^*$. In particular, let $x_{n_k}^*(0) = x_{n_k}^{\uparrow}(0) = 0$ and let $x_{n_k}^*(t) = x_{n_k}^{\uparrow}(t)$ for all $t \geq \delta$ and let $x_{n_k}^*$ be defined by linear interpolation in $[0, \delta]$. Then $x_{n_k}^* \in D_{u,\uparrow}^*$, $\|x_{n_k}^* - x_{n_k}^{\uparrow}\| < \epsilon$ and $\|(x_{n_k}^*)^{-1} - x_{n_k}^{-1}\| < \epsilon$ for all $n_k \geq n_0$. For $n \geq n_0$ such that $x_n^{\uparrow} \in D_{u,\uparrow}^*$, let $x_n^* = x_n^{\uparrow}$. Since ϵ was arbitrary, we can choose x_n^* such that $x_n^* \to x^{\uparrow}$ (M_1), $\|x_n^* - x_n^{\uparrow}\| \to 0$ and $\|(x_n^*)^{-1} - x_n^{-1}\| \to 0$ as $n \to \infty$. By Lemma 7.6.2, $(x_n^*)^{-1} \to x^{-1}$ (M_1). Since $\|(x_n^*)^{-1} - x_n^{-1}\| \to 0$, $x_n^{-1} \to x^{-1}$ (M_1) as well. ∎

Corollary 7.6.2. . (continuity at strictly increasing functions) *The inverse map from (D_u, M_2) to $(D_{u,\uparrow}, U)$ is continuous at $x \in D_{u,\uparrow\uparrow}$.*

Proof. First, $D_{u,\uparrow\uparrow} \subseteq D_{u,\uparrow}^*$, so that we can apply Theorem 7.6.1 to get $x_n^{-1} \to x^{-1}$ in $(D_{u,\uparrow}, M_1)$. However, by Lemma **??**, $x^{-1} \in C$ when $x \in D_{u,\uparrow\uparrow}$. Hence the M_1 convergence $x_n^{-1} \to x^{-1}$ actually holds in the stronger topology of uniform convergence over compact subsets. ∎

7.6.2. The M_1' Topology

For cases in which the condition $x^{-1}(0) = 0$ in Theorem 7.6.1 is not satisfied, we can modify the M_1 and M_2 topologies to obtain convergence, following Puhalskii and Whitt (1997). With these new weaker topologies, which we call M_1' and M_2', we do not require that $x_n(0) \to x(0)$ when $x_n \to x$. We construct the new topologies by extending the graph of each function x by appending the segment $[0, x(0)] \equiv \{\alpha 0 + (1-\alpha)x(0) : 0 \le \alpha \le 1\}$. Let the new graph of $x \in D$ be

$$\begin{aligned} \Gamma_x' &= \{(z,t) \in \mathbb{R}^k \times [0,\infty) : z = \alpha x(t) + (1-\alpha)x(t-) \\ &\qquad \text{for } 0 \le \alpha \le 1 \text{ and } t \ge 0\}, \end{aligned} \tag{6.5}$$

where $x(0-) \equiv 0$. Let $\Pi'(x)$ and $\Pi_2'(x)$ be the sets of all M_1 and M_2 parametric representations of Γ_x', defined just as before. We say that $x_n \to x$ in (D, M_i') if there exist parametric representations $(u_n, r_n) \in \Pi'(x_n)$ and $(u, r) \in \Pi'(x)$, where Π' is the set of M_1' and M_2' parametric representations, such that

$$\|u_n - u\|_t \vee \|r_n - r\|_t \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad t > 0\,. \tag{6.6}$$

With the M_1' topologies, we obtain a cleaner statement than Lemma 7.6.2.

Lemma 7.6.3. (graphs of the inverse with the M_1' topology) *For $x \in D_{u,\uparrow}$, the graph Γ_x' serves as the graph $\Gamma_{x^{-1}}'$ with the axes switched, so that $(u,r) \in \Pi'(x)$ $(\Pi_2'(x))$ if and only if $(r,u) \in \Pi'(x^{-1})$ $(\Pi_2'(x^{-1}))$.*

Thus we get an alternative to Theorem 7.6.1.

Theorem 7.6.2. (continuity in the M_1' topology) *The inverse map in (6.1) from (D_u, M_2') to $(D_{u,\uparrow}, M_1')$ is continuous.*

Proof. By the M_2' analog of Theorem 7.4.1, if $x_n \to x$ in (D_u, M_2'), then $x_n^\uparrow \to x^\uparrow$ in $(D_{u,\uparrow}, M_2')$. Since the M_2' topology coincides with the M_i' topology on $D_\uparrow$, we get $x_n^\uparrow \to x^\uparrow$ in $(D_{u,\uparrow}, M_1')$. By Lemma 7.6.3, we get $(x_n^\uparrow)^{-1} \to (x^\uparrow)^{-1}$ in $(D_{u,\uparrow}, M_1')$. That gives the desired result because $(x^\uparrow)^{-1} = x^{-1}$ for all $x \in D_u$. ∎

An alternative approach to the difficulty at the origin besides M_i' topology on $D_u([0,\infty),\mathbb{R})$ is the ordinary M_i topology on $D_u((0,\infty),\mathbb{R})$. The difficulty at the origin goes away if we ignore it entirely, which we can do by making the function domain $(0,\infty)$ for the image of the inverse functions.

In particular, Theorem 7.6.2 implies the following corollary.

Corollary 7.6.3. (continuity when the origin is removed from the domain) *The inverse map in* (6.1) *from* $D_u([0,\infty), M_2)$ *to* $D_{u,\uparrow}((0,\infty), M_1)$ *is continuous.*

Proof. Since the M_2' topology is weaker than M_2, if $x_n \to x$ in $D_u([0,\infty), M_2)$, then $x_n \to x$ in $D_u([0,\infty), M_2')$. Apply Theorem 7.6.2 to get $x_n^{-1} \to x^{-1}$ in $D_{u,\uparrow}([0,\infty), M_1')$. That implies $x_n^{-1} \to x^{-1}$ for the restrictions in $D_\uparrow([t_1, t_2], M_1)$ for all $t_1, t_2 \in Disc(x^{-1})^c$, which in turn implies that $x_n^{-1} \to x^{-1}$ in $D_{u,\uparrow}((0,\infty), M_1)$. ∎

However, in general we cannot work with the inverse on $D_u((0,\infty), \mathbb{R})$. We can obtain positive results if all the functions are required to be monotone. The following result is elementary.

Theorem 7.6.3. (equivalent characterizations of convergence for monotone functions) *For* x_n, $n \geq 1$, $x \in D_{u,\uparrow}([0,\infty), \mathbb{R})$, *the following are equivalent:*

$$x_n \to x \quad in \quad D_{u,\uparrow}((0,\infty), \mathbb{R}, M_1)\ ; \tag{6.7}$$

$$x_n \to x \quad in \quad D_{u,\uparrow}([0,\infty), \mathbb{R}, M_1')\ ; \tag{6.8}$$

$$x_n(t) \to x(t) \quad for\ all\ t\ in\ a\ dense\ subset\ of\ (0,\infty)\ ; \tag{6.9}$$

$$x_n^{-1} \to x^{-1} \quad in \quad D((0,\infty), \mathbb{R}, M_1)\ ; \tag{6.10}$$

$$x_n^{-1} \to x^{-1} \quad in \quad D([0,\infty), \mathbb{R}, M_1')\ ; \tag{6.11}$$

$$x_n^{-1}(t) \to x^{-1}(t) \quad for\ all\ t\ in\ a\ dense\ subset\ of\ (0,\infty). \tag{6.12}$$

Proof. Theorem 7.6.2 implies the equivalence of (6.8) and (6.11). Clearly, (6.8)→(6.7)→(6.9), so that (6.11)→(6.10)→(6.12). It thus suffices to show that (6.9)→(6.8). For any $\epsilon > 0$, we can find t and n_0 such that $0 < t < \epsilon$, $t \in Disc(x)$ and $|x_n(t) - x(t)| < \epsilon$ for $n \geq n_0$. Let $n_1 \geq n_0$ be such that $d_{M_2'}(x_n, x) < \epsilon$ for the restrictions to $[t, t']$ for any $t' > t$ with $t' \in Disc(x)^c$. Since x_n and x are nondecreasing and nonnegative, the bounds $d_{M_2'}(x_n, x) < \epsilon$ over $[t, t']$ and $|x_n(t) - x(t)| < \epsilon$ imply that $d_{M_2'}(x_n, x) < \epsilon$ for the restrictions over$[0, t']$. Since ϵ and t' were arbitrary, $x_n \to x$ in $D_\uparrow([0,\infty), \mathbb{R}, M_2')$, but the M_2' and M_1' topologies are equivalent on $D_\uparrow$. ∎

In general, convergence in $D([0,\infty), \mathbb{R}, M_1')$ provides stronger control of the behavior at the origin than convergence in $D((0,\infty), \mathbb{R}, M_1)$. Nothing more is omitted from Section 13.6 of the book.

7.7. Inverse with Centering

We continue considering the inverse map, but now with centering. We start by considering linear centering. In particular, we consider when a limit for $c_n(x_n - e)$ implies a limit for $c_n(x_n^{-1} - e)$ when $x_n \in D_u \equiv D_u([0,\infty), \mathbb{R})$ and $c_n \to \infty$. By considering the behavior at one t, it is natural to anticipate that we should have $c_n(x_n^{-1} - e) \to -y$ when $c_n(x_n - e) \to y$. A first step for the M topologies is to apply Theorem 7.4.2, which yields limits for $c_n(x_n^{\uparrow} - e)$. Thus for the M topologies, it suffices to assume that $x_n \in D_{\uparrow}$.

Now we state the main limit theorem for inverse functions with centering.

Theorem 7.7.1. *Suppose that $c_n(x_n - e) \to y$ as $n \to \infty$ in $D([0,\infty), \mathbb{R})$ with one of the topologies M_2, M_1 or J_1, where $x_n \in D_u$, $c_n \to \infty$ and $y(0) = 0$.*

(a) If the topology is M_2 or M_1, then $c_n(x_n^{-1} - e) \to -y$ as $n \to \infty$ with the same topology.

(b) If the topology is J_1 and if y has no positive jumps, then $c_n(x_n^{-1} - e) \to -y$ as $n \to \infty$.

Proof. (a) The proof is easy for the M_i topologies when $x_n \in D_u^*$ for all sufficiently large n. First, given $c_n(x_n - e) \to y$ (M_i), we can apply Theorem 7.4.2 (a) to conclude that $c_n(x_n^{\uparrow} - e) \to y$ (M_i). Hence we can assume that $x_n \in D_{\uparrow}^*$. Thus there exist parametric representations $(u_n, r_n) \in \Pi(c_n(x_n - e))$ and $(u, r) \in \Pi(x)$ of the appropriate type such that $\|u_n - u\|_t \vee \|r_n - r\|_t \to 0$ as $n \to \infty$ for all $t > 0$. Then $(u_n', r_n) \in \Pi(x_n)$ for $u_n' = c_n^{-1} u_n + r_n$. Since $x_n \in D_{\uparrow}^*$ for n sufficiently large, $(r_n, u_n') \in \Pi(x_n^{-1})$ and $(c_n(r_n - u_n'), u_n') \in \Pi(c_n(x_n^{-1} - e))$ for sufficiently large n. However,

$$c_n(r_n - u_n') = -u_n \tag{7.1}$$

and

$$\|u_n' - r\|_t \to 0 \quad \text{as } n \to \infty \text{ for all } t > 0\,, \tag{7.2}$$

so that $c_n(x_n^{-1} - e) \to -y$ (M_i) as $n \to \infty$. However, in general we need not have $x_n \in D_u^*$ for all sufficiently large n. So, suppose that we do not. We then only have $x_n \in D_u$ for all n. As before, we can apply Theorem 7.4.2 to show that it suffices to assume that $x_n \in D_{\uparrow}$ for all n. We now show that we can approximate $x_n \in D_{\uparrow}$ by $x_n^* \in D_{\uparrow}^*$ for all n sufficiently large, so that

$$c_n\|x_n - x_n^*\| \to 0 \quad \text{and} \quad c_n\|x_n^{-1} - (x_n^*)^{-1}\| \to 0 \quad \text{as} \quad n \to \infty\,. \tag{7.3}$$

The limits in (7.3) plus the triangle inequality imply that

$$d(c_n(x_n^* - e), y) \le d(c_n(x_n - e), y) + c_n\|x_n - x_n^*\| \to 0 \quad \text{as} \quad n \to \infty \quad (7.4)$$

and

$$\begin{aligned} &d(c_n(x_n^{-1} - e), -y) \\ &\quad \le \|c_n(x_n^{-1} - (x_n^*)^{-1}\| + d(c_n((x_n^*)^{-1} - e), -y) \to 0 \end{aligned} \quad (7.5)$$

as $n \to \infty$, where d is the M_i metric. Thus, the remaining problem is to construct $x_n^* \in D_\uparrow^*$ satisfying (7.3). Since $y(0) = 0$ and $y \in D$, for all $\epsilon > 0$, there exists δ_1 such that $v(y, 0, \delta_1) < \epsilon/2$. Since $c_n(x_n - e) \to y$ (M_2), there exists n_0 and δ_2 such that $v(c_n(x_n - e), y, 0, \delta/2) < \epsilon/2$ for all $n \ge n_0$. Thus

$$t - c_n^{-1}\epsilon < x_n(t) \le t + c_n^{-1}\epsilon \quad (7.6)$$

for all $n \ge n_0$ and t with $0 \le t \le \delta \equiv \delta_1 \wedge \delta_2$. By Lemma **??**,

$$t + c_n^{-1}\epsilon > x_n^{-1}(t-) \ge t - c_n^{-1}\epsilon \quad (7.7)$$

for all $n \ge n_0$ and t with $0 \le t \le \delta - c_n^{-1}\epsilon$. Now choose $n_1 \ge n_0$ so that $c_n^{-1}\epsilon < \delta/4$ for all $n \ge n_1$. Then, by (7.6), for $n \ge n_1$,

$$0 < x_n(\delta/4) < \delta/2 \quad (7.8)$$

and (7.7) holds for $0 \le t \le 3\delta/4$. Hence, if $n \ge n_1$ and $x_n \notin D_\uparrow^*$, we can construct $x_n^* \in D_\uparrow^*$ by letting $x_n^*(0) = x_n(0) = 0$, $x_n^*(t) = x_n(t)$, $t \ge \delta/4$, and letting x_n^* be defined by linear interpolation for t in $[0, \delta/4]$. By (7.8), $x_n^* \in D_\uparrow^*$. Since x_n^* is defined by linear interpolation over $[0, \delta/4]$, for $n \ge n_1$,

$$\|c_n(x_n^* - e)\|_{\delta/4} = \max\{c_n(x_n - e)(0), c_n(x_n - e)(\delta/4)\} \le \epsilon\ , \quad (7.9)$$

so that

$$\|c_n(x_n^* - x_n)\| \le \|c_n(x_n - e)\|_{\delta/4} + \|c_n(x_n^* - e)\|_{\delta/4} \le 2\epsilon\,. \quad (7.10)$$

Similarly, $(x_n^*)^{-1}(t) = x_n^{-1}(t)$ for $t \le x_n(\delta/4) < \delta/2$ and $n \ge n_1$, so that by (7.7)

$$\|c_n((x_n^*)^{-1} - x_n^{-1})\| \le \|c_n(x_n^{-1} - e)\|_{\delta/2} + \|c_n((x_n^*)^{-1} - e)\|_{\delta/2} \le 2\epsilon\,. \quad (7.11)$$

Since ϵ was arbitrary, (7.10) and (7.11) imply (7.3), as required.

(b) Since $c_n(x_n - e) \to y$ (J_1) and $c_n \to \infty$, $\|x_n - e\|_t \to 0$ as $n \to \infty$ for all $t > 0$. By Corollary 7.6.2, $\|x_n^{-1} - e\|_t \to 0$ as $n \to \infty$ for each

$t > 0$. By Theorem 7.2.2, we can apply the composition map to obtain $c_n(x_n \circ x_n^{-1} - x_n^{-1}) \to y$ (J_1). Hence it suffices to show that $c_n \|x_n \circ x_n^{-1} - e\|_t \to 0$ as $n \to \infty$ for all $t > 0$. However, by Corollary **??**,

$$\begin{aligned} c_n \|x_n \circ x_n^{-1} - e\|_t &\leq c_n J_{x_n^{-1}(t)}(x_n) \\ &= J_{x_n^{-1}(t)}(c_n(x_n - e)) , \end{aligned} \tag{7.12}$$

where $J_t(x)$ is the maximum jump of x over $[0, t]$, treating $x(0-)$ as 0. Since $c_n(x_n - e) \to y$, $y(0) = 0$ and y has no positive jumps, $J_t(c_n(x_n - e)) \to 0$ as $n \to \infty$ for all $t > 0$, which implies the desired conclusion. ■

Nothing else is omitted from Section 13.7 of the book.

7.8. Counting Functions

Inverse functions or first-passage-time functions are closely related to counting functions. A counting function is defined in terms of a sequence $\{s_n : n \geq 0\}$ of nondecreasing nonnegative real numbers with $s_0 = 0$. We can think of s_n as the partial sum

$$s_n \equiv x_1 + \cdots + x_n, \quad n \geq 1 , \tag{8.1}$$

by simply writing $x_i \equiv s_i - s_{i-1}$, $i \geq 1$. The associated *counting function* $\{c(t) : t \geq 0\}$ is defined by

$$c(t) \equiv \max\{k \geq 0 : s_k \leq t\}, \quad t \geq 0 . \tag{8.2}$$

To have $c(t)$ finite for all $t > 0$, we assume that $s_n \to \infty$ as $n \to \infty$. We can reconstruct the sequence $\{s_n\}$ from $\{c(t) : t \geq 0\}$ by

$$s_n = \inf\{t \geq 0 : c(t) \geq n\}, \quad n \geq 0 . \tag{8.3}$$

The sequence $\{s_n\}$ and the associated function $\{c(t) : t \geq 0\}$ can serve as sample paths for a stochastic point process on the nonnegative real line. Then there are (countably) infinitely many points with the n^{th} point being located at s_n. The summands x_n are then the intervals between successive points. The most familiar case is when the sequence $\{x_n : n \geq 1\}$ constitutes the possible values from a sequence $\{X_n : n \geq 1\}$ of i.i.d. random variables with values in $\mathbb{R}_+$. Then the counting function $\{c(t) : t \geq 0\}$ constitutes a possible sample path of an associated renewal counting process $\{C(t) : t \geq 0\}$; see Section 7.3 of the book.

Paralleling Lemma 13.6.3 in the book, we have the following basic inverse relation for counting functions.

Lemma 7.8.1. *For any nonnegative integer n and nonnegative real number t,*

$$s_n \leq t \quad \textit{if and only if} \quad c(t) \geq n\,. \tag{8.4}$$

We can put counting functions in the setting of inverse functions on $D_\uparrow$ by letting

$$y(t) \equiv s_{\lfloor t \rfloor}, t \geq 0\,. \tag{8.5}$$

To have $y \in D_\uparrow$, we use the assumption that $s_n \to \infty$ as $n \to \infty$. if all the summands are strictly positive then

$$y^{-1}(t) = c(t) + 1, \quad t \geq 0\,, \tag{8.6}$$

where y^{-1} is the image of the inverse map in (6.1) applied to y in (8.5). With (8.6), limits for counting functions can be obtained by applying results in the previous two sections.

The connection to the inverse map can also be made when the summands x_i are only nonnegative. To do so, we observe that the counting function c is a time-transformation of y^{-1}. both are right-continuous, but $c(t) < y^{-1}(t)$. In particular, c and y can be expressed in terms of each other.

Lemma 7.8.2. (relation between counting functions and inverse functions) *For y in* (8.5) *and c in* (8.2),

$$\begin{aligned} c(t) &= y^{-1}(y(y^{-1}(t)-)-), \quad t \geq 0\,, & (8.7)\\ c(t) &= y^{-1}(t-) \quad \textit{for all} \quad t \in Disc(c) = Disc(y^{-1})\,, & (8.8)\\ y^{-1}(t) &= c(c^{-1}(c(t)), \quad t \geq 0\,. & (8.9) \end{aligned}$$

The three functions y, y^{-1} and c are depicted for a typical initial segment of a sequence $\{s_n : n \geq 0\}$ in Figure 13.1 of the book. We can apply (8.7)–(8.9) in Lemma 7.8.1 to show that limits for scaled counting functions with centering, are equivalent to limits for scaled inverse functions. We use the fact that the M topologies are not altered by changing to the left limits, because the graph is unchanged. We first consider the case of no centering; afterwards we consider the case of centering. When there is no centering, the M_1 and M_2 topologies coincide and reduce to pointwise convergence on a dense subset of $\mathbb{R}_+$ including 0.

Consider a sequence of counting functions $\{\{c_n(t) : t \geq 0\} : n \geq 1\}$ with associated processes

$$y_n^{-1}(t) \equiv c_n(c_n^{-1}(c_n(t))), \quad t \geq 0\,, \tag{8.10}$$

$y_n = (y_n^{-1})^{-1}$. Form scaled functions by setting

$$\mathbf{c}_n(t) = n^{-1}c_n(a_n t) \quad \text{and} \quad \mathbf{y}_n(t) = a_n^{-1}y_n(nt), \quad t \geq 0\,, \tag{8.11}$$

where a_n are positive real numbers with $a_n \to \infty$. Note that

$$\mathbf{c}_n^{-1}(t) = a_n^{-1}c_n^{-1}(nt) \quad \text{and} \quad \mathbf{y}_n^{-1}(t) = n^{-1}y_n(a_n t), \quad t \geq 0\,. \tag{8.12}$$

Theorem 7.8.1. (asymptotic equivalence of limits for scaled processes) *Suppose that $\mathbf{y}_n \in D_{u,\uparrow}$, $n \geq 1$, for $\mathbf{y}_n$ in (8.11). Then any one of the limits $\mathbf{y}_n \to y$, $\mathbf{y}_n^{-1} \to y^{-1}$, $\mathbf{c}_n \to y^{-1}$ or $\mathbf{c}_n^{-1} \to y^{-1}$ in $D_\uparrow([0,\infty),\mathbb{R})$ with the M_2 $(= M_1)$ topology, for y_n^{-1}, $\mathbf{c}_n$ and $\mathbf{c}_n^{-1}$ in (8.11) and (8.12), implies the others.*

Proof. The equivalence between $\mathbf{y}_n \to y$ and $\mathbf{y}_n^{-1} \to y^{-1}$, and between $\mathbf{c}_n \to y^{-1}$ and $\mathbf{c}_n^{-1} \to y$ follow from Theorem 7.6.1. We can relate the limits $\mathbf{c}_n \to y^{-1}$ and $\mathbf{y}_n \to y$ by applying (8.6), after modifying the summands $x_{n,i}$ in the sequences $\{s_{n,k} : k \geq 0\}$ to make them strictly positive. We can show that the limits are unaltered by adding suitably small positive values to the summands. Given $\epsilon > 0$ and $\{x_n : n \geq 1\}$, let

$$x_n' = x_n + \epsilon 2^{-n}, n \geq 1\,, \tag{8.13}$$

and let $x_n' = x_1' + \cdots + x_n'$, $n \geq 1$, and $c'(t) = \max\{k \geq 0 : s_n' \leq t\}$, $t \geq 0$. Then

$$s_n \leq s_n' \leq s_n + \epsilon, \quad n \geq 0\,, \tag{8.14}$$

and

$$c((t-\epsilon) \vee 0) \leq c'(t) \leq c(t), \quad t \geq 0\,. \tag{8.15}$$

The actual limits we want to consider involve a sequence of sequences $\{\{s_{n,k} : k \geq 0\}, n \geq 1\}$ with $s_{n,0} = 0$ for each n. Let $\{\{c_n(t) : t \geq 0\}\}$ be the associated sequence of counting functions. Let $x_{n,k}'$ $s_{n,k}'$, $n_n'(t)$, $\mathbf{s}_n'$ and $\mathbf{n}_n'$ be associated quantities defined by the modification in (8.7), i.e., by letting

$$x_{n,k}' \equiv x_{n,k} + \epsilon_n 2^{-k}, \quad k \geq 1\,. \tag{8.16}$$

Given that scaled processes are formed as in (8.11) and (8.12). It is elementary that

$$\|\mathbf{y}_n - \mathbf{y}_n'\|_\infty \leq \epsilon_n / a_n \to 0 \tag{8.17}$$

so that, for appropriate choice of ϵ_n, e.g., $\epsilon_n = \epsilon$, $\epsilon_n / a_n \to 0$. The bound in (8.15) enables us to conclude that $\mathbf{c}_n \to c$ (M_2) if and only if $\mathbf{c}'_n \to c$ (M_2) by applying Corollary 12.11.6 in the book. Hence it suffices to assume that the sequences $\{s_{n,k} : k \geq 0\}$ are strictly increasing, which implies that (8.6) holds. Then, after scaling as in (8.11) and (8.12),

$$\|\mathbf{y}_n^{-1} - \mathbf{c}_n\|_\infty \leq 1/n \to 0\,,$$

which completes the proof. ■

We now apply the results for inverse maps with centering in Section 7.7 to obtain limits for counting functions with centering. Consider a sequence of counting functions $\{\{c_n(t) : t \geq 0\} : n \geq 1\}$ associated with a sequence of nondecreasing sequences of nonnegative numbers $\{\{s_{n,k} : k \geq 0\} : n \geq 1\}$ defined as in (8.2). Let the scaled functions $\mathbf{c}_n$, $\mathbf{y}_n$, $\mathbf{c}_n^{-1}$ and $\mathbf{y}_n^{-1}$ be defined as in (8.10)–(8.12).

Theorem 7.8.2. (asymptotic equivalence of counting and inverse functions with centering) *Suppose that* $\mathbf{y}_n \in D_\uparrow$, $n \geq 1$, $b_n \to \infty$ *and* $y(0) = 0$. *Then any one of the limits* $b_n(\mathbf{y}_n - e) \to y$, $b_n(\mathbf{c}_n - e) \to -y$, $b_n(\mathbf{y}_n^{-1} - e) \to -y$ *or* $b_n(\mathbf{c}_n^{-1} - e) \to y$ *in* $D([0,\infty), \mathbb{R})$ *with the* M_1 *or* M_2 *topology, for* $\mathbf{y}_n$, $\mathbf{c}_n$, *and* $\mathbf{y}_n^{-1}$ *and* $\mathbf{c}_n^{-1}$ *in* (8.11) *and* (8.12), *implies the others with the same topology.*

Proof. The equivalence between $b_n(\mathbf{y}_n - e) \to y$ and $b_n(\mathbf{y}_n^{-1} - e) \to -y$ is contained in Theorem 7.7.1. Similarly, the equivalence between $b_n(\mathbf{c}_n - e) \to -y$ and $b_n(\mathbf{c}_n^{-1} - e) \to y$ is contained in Theorem 7.7.1. Let the topology be fixed at either M_1 or M_2. Given $b_n(\mathbf{y}_n^{-1} - e) \to -y$, we have $\|\mathbf{y}_n^{-1} - e\|_t \to 0$ and $\|\mathbf{y}_n - e\|_t \to 0$ as $n \to \infty$ for each $t > 0$. For any $x \in D$, let $\hat{x}$ denote the associated left-limit function; i.e., $\hat{x}(t) = x(t-)$. Then $\mathbf{c}_n = \hat{\mathbf{y}}_n^{-1} \circ \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1}$. Given $b_n(\mathbf{y}_n^{-1} - e) \to -y$, we have $\hat{\mathbf{y}}_n^{-1} \to e$, $\hat{\mathbf{y}}_n \to e$, $b_n(\hat{\mathbf{y}}_n^{-1} - e) \to -y$ and $b_n(\hat{\mathbf{y}}_n - e) \to y$, because the graphs are unchanged. Now we can apply the composition map to get $b_n(\hat{\mathbf{y}}_n^{-1} \circ \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1} - \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1}) \to -y$ and $b_n(\hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1} - \mathbf{y}_n^{-1}) \to y$. Hence, by Proposition **??**, for each $t \in Disc(y)^c$, we have

$$\begin{aligned} b_n(\mathbf{c}_n - e)(t) &= b_n(\hat{\mathbf{y}}_n^{-1} \circ \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1} - e)(t) \\ &= b_n(\hat{\mathbf{y}}_n^{-1} \circ \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1} - \hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1})(t) \\ &\quad + b_n(\hat{\mathbf{y}}_n \circ \mathbf{y}_n^{-1} - \mathbf{y}_n^{-1})(t) + b_n(\mathbf{y}_n^{-1} - e)(t) \\ &\quad \to -y(t) + y(t) - y(t) = -y(t)\,. \end{aligned} \tag{8.18}$$

Now we apply Theorems 6.5.1 (iv) and 6.11.1 (iv). Let $w(x,\delta)$ be the M_i oscillation function over the interval $[0,t]$. By (8.8), the oscillations of $b_n(\mathbf{c}_n - e)$ coincide with the oscillations of $b_n(\mathbf{y}_n^{-1} - e)$ at discontinuity points of $\mathbf{c}_n$ and $\mathbf{y}_n^{-1}$. Moreover, in between such discontinuity points, they have identical maximum oscillations. Hence, for any interval $[0,t]$ with $t \in Disc(y)^c$,

$$w(b_n(\mathbf{c}_n - e), \delta) < w(b_n(\mathbf{y}_n^{-1} - e), \delta) \ . \tag{8.19}$$

Since $b_n(\mathbf{y}_n^{-1} - e) \to -y$ by assumption,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n\to\infty}} \ w(b_n(\mathbf{y}_n^{-1} - e), \delta) = 0 \tag{8.20}$$

Consequently,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n\to\infty}} \ w(b_n(\mathbf{c}_n - e), \delta) = 0 \ . \tag{8.21}$$

Hence, we can conclude that $b_n(\mathbf{c}_n - e) \to -y$.

To go the other way, suppose that $b_n(\mathbf{c}_n - e) \to -y$. Applying Theorem 7.7.1, we have $b_n(\mathbf{c}_n^{-1} - e) \to y$, $\mathbf{c}_n \to e$ and $\mathbf{c}_n^{-1} \to e$. Then, paralleling (8.18), we can apply (8.9) to obtain

$$\begin{aligned} b_n(\mathbf{y}_n^{-1} - e)(t) &= b_n(\mathbf{c}_n \circ \mathbf{c}_n^{-1} \circ \mathbf{c}_n - e)(t) \\ &= b_n(\mathbf{c}_n \circ \mathbf{c}_n^{-1} \circ \mathbf{c}_n - \mathbf{c}_n^{-1} \circ \mathbf{c}_n)(t) \\ &\quad + b_n(\mathbf{c}_n^{-1} \circ \mathbf{c}_n - \mathbf{c}_n)(t) + b_n(\mathbf{c}_n - e)(t) \\ &\quad \to -y(t) + y(t) - y(t) = -y(t) \end{aligned} \tag{8.22}$$

for each $t \in Disc(y)^c$. Now let $w(x,\delta,t)$ denote the M_i oscillation function over the interval $[0,t]$ as a function of the right endpoint t. Then, paralleling (8.19), by (8.8), for all $t_1 \in Disc(y)^c$, there exists $t_2 > t_1$ with $t_2 \in Disc(y)^c$ such that

$$w(b_n(\mathbf{y}_n^{-1} - e), \delta, t_1) < w(b_n(\mathbf{c}_n - e), \delta, t_2) \tag{8.23}$$

for all n sufficiently large. Hence we can use the previous oscillation argument to conclude that $b_n(\mathbf{y}_n^{-1} - e) \to -y$. ∎

7.9. Renewal-Reward Processes

Nothing was omitted from Section 13.9 in the book.