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WEAK CONVERGENCE THEOREMS FOR PRIORITY QUEUES: PREEMPTIVE-RESUME DISCIPLINE

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1. Introduction and summary

We shall consider a single-server queue with r priority classes of customers and a preemptive-resume discipline. In this system customers are served in order of their priority while customers of the same priority are served in order of their arrival. Higher priority customers, immediately upon arrival, replace lower priority customers at the server, while customers displaced in this way return to the server before any other customers of the same priority receive service. When a displaced customer returns to the server, his remaining service time is the uncompleted portion of his original service time (cf. Jaiswal (1968)).

As basic data, we assume that we are given a set of initial conditions and a sequence of ordered $2r$-tuples of non-negative random variables, $\{(u_n^1,\cdots,u_n^r, v_n^1,\cdots,v_n^r), n = 1,2,\cdots\}$, all defined on a common probability space, where the variable u_n^i represents the interarrival time between the $(n-1)$th and nth customers in the ith priority class and the variable v_n^i represents the service time of the nth customer of the ith priority class. The most important special case occurs when the sequence of $2r$-tuples can be represented as $2r$ independent sequences of i.i.d. random variables with finite, positive variances, but we permit other possibilities. The sequence $\{u_n^i\}$ might result from the superposition of several renewal processes (corresponding to multiple arrival channels), several of the successive u_n^i might be 0 corresponding to batch processing, or u_n^i and u_n^j might be dependent (corresponding to customers of different priority classes arriving in the same channel for example); our basic assumption, just as in [9], is that certain counting or partial sum processes associated with $\{(u_n^1,\cdots,u_n^r,v_n^1,\cdots,v_n^r)\}$ satisfy a weak convergence theorem.

Our object is to obtain weak convergence theorems for sequences of random functions induced in the function space $D[0,1]$ by the basic stochastic processes characterizing this system. Thus this paper is a continuation of the research reported in [12], [8], [9], [14], [5], and [13]. The critical tool is a functional central limit theorem for the sum of a random number of random variables

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with non-zero mean due to Iglehart and Kennedy (1970). Since such a theorem exists for double sequences as well as single sequences, our analysis applies to sequences of queueing systems as well as single queueing systems (cf. [9] and [13]) but for simplicity we shall only consider a single queueing system here. With a single queueing system, the weak convergence theory leads to limit theorems in heavy traffic; that is, when the queue is unstable and a steady state is never achieved. Using sequences of queueing systems admits more possibilities. Of course, heavy traffic limit theorems for sequences of queueing systems are also possible, some of which are potentially useful as approximations for stable queues, but weak convergence theorems for sequences of queueing systems are also possible in situations other than heavy traffic. Here as well as in [12], [9], and [13] we obtain weak convergence theorems for queues corresponding to each weak convergence theorem for random functions induced by the partial sums associated with the interarrival times and service times. Neither the normalization, nor the form of the limit is critical. Although we focus primarily on heavy traffic, it is important to note that with such generality other situations could be treated. Forthcoming work by Iglehart and Kennedy will demonstrate that the weak convergence approach is not limited to heavy traffic. Again, we shall consider only a single queue in heavy traffic, but the same arguments apply to sequences of queueing systems using [13], Theorem 3.19.

We should also remark that functional laws of the iterated logarithm, weak laws of large numbers, and strong laws of large numbers all hold for the random functions induced in $D[0,1]$ by the queueing processes if corresponding behavior is exhibited by the random functions induced by the basic counting or partial sum processes. The statement and proof of the functional law of the iterated logarithm for priority queues is essentially the same as for standard multiple channel queues (cf. [5]). The strong laws of large numbers are easy byproducts of the proof of the functional laws of the iterated logarithm and the weak laws of large numbers are easy consequences of the continuous mapping theorem for convergence in probability. We shall only discuss weak convergence here, however.

Heavy traffic limit theorems (not weak convergence) for the single-server queue with two priority classes and a preemptive-resume discipline were first proved by Hooke and Prabhu (1969) and Hooke (1969). Hooke and Prabhu (1969) mainly studied the virtual waiting time processes for the high and low priority customers when arrivals occur in independent Poisson processes. The work in [3] appears in Chapter 3 of [4] with a slightly different argument and is extended to include arrivals occuring in independent renewal processes in Chapter 5 of [4].

We shall prove limit theorems for other processes in the same system with less restrictive assumptions. Our results tend to answer questions from the point of view of the server rather than the customer. We shall prove heavy traffic limit theorems for the number of customers of each priority in the system at time t, the number of customers of each priority that have departed in the interval $[0,t]$, the work load in service time of each priority class facing the server at time t, the accumulated time in $[0,t]$ during which the server has been available and there have been no customers of a given priority in the system, and corresponding characteristics of the entire system. We remark that the system studied here is a standard single-server queue in which customers are served in order of their arrival without defections from the point of view of the highest priority customers. The total service load is also the same for many priority disciplines. Thus, the weak convergence theorems in these cases are contained in our earlier work. A significant aspect of this paper in comparison with our earlier work is that we obtain weak convergence jointly for all the processes characterizing the systems. We extend the work of Hooke and Prabhu by treating new processes. We do not discuss the virtual waiting time of the lower priority customers as they do and they do not discuss the processes we treat here. Our work intersects theirs only for those processes, such as the total service load, for which heavy traffic limits were already known because the priority discipline is not relevant. The setting of our theorems is much more general, however, with weak convergence on the function space instead of R^1 and the relaxing of the i.i.d. assumptions for interarrival times and service times. We prove weak convergence theorems for sequences of random functions induced in $D[0,1]$ and the product space $D[0,1]^s$ by the processes above. For a discussion of the relevant weak convergence theory, see [2], or [12], Chapter 3.

We now chart the way ahead. In Section 2 we prove the functional central limit theorem for random sums which we will need. We use this theorem to prove weak convergence theorems for the processes above in Sections 3 and 4. We treat embedded sequences in Section 5. Finally, in Section 6 we discuss several specific situations in which the hypotheses of the theorems in Sections 2, 3, 4, and 5 are satisfied and exhibit a few corollaries.

2. Weak convergence for random sums

The fundamental tool in this paper is a functional central limit theorem for random sums of random variables with possibly non-zero mean due to Iglehart and Kennedy (1970). The theorems of this kind in Billingsley ((1968), Section 17) apply to random variables with mean 0. Random sum theorems have previously been used for proving weak convergence theorems for queues in [12] Section 7.4, [9] Sections 3.4 and 3.5, [6], and [13] Section 5; in [6], [9], and [13] non-zero mean versions have been employed. A random sum theorem (not weak convergence) was also central to the work of Hooke (1969). Hooke ((1969), Theorem A2) obtains his result using characteristic functions, but it is also a special case of Theorem 2.1 below.

For $r \geqq 1$, let $\{(u_n^1, \cdots, u_n^r, v_n^1, \cdots, v_n^r),\ n \geqq 1\}$ be a sequence of $2r$-tuples of random variables all defined on a common probability space. We consider single sequences of random variables here, but a corresponding theorem exists for double sequences (cf. [13]). Assume $u_n^i \geqq 0\ (1 \leqq i \leqq r, n \geqq 1)$ and define counting processes $\{A^i(t), t \geqq 0\}$ associated with each sequence $\{u_n^i\}$:

$$A^i(t) = \begin{cases} \max\{k \geqq 1: u_1^i + \cdots + u_k^i \leqq t\}, & u_1^i \leqq t \\ 0, & u_1^i > t. \end{cases} \tag{2.1}$$

Now define various random functions in $D[0,c]$, $0 < c < \infty$, induced by $A^i(t)$, the partial sums of $\{u_n^i\}$ and $\{v_n^i\}$, the random sums $v_1^i + \cdots + v_{A^i(t)}^i$, and the constants λ_i and μ_i:

$$\begin{aligned} U_n^i &\equiv n^{-1/2} \sum_{j=1}^{[nt]} (u_j^i - \lambda_i^{-1}), \qquad \lambda_i^{-1} > 0, \\ A_n^i &\equiv [A^i(nt) - \lambda_i nt]/n^{1/2}, \qquad \lambda_i > 0, \\ \Phi_n^i &\equiv (A^i(nt)/n) \wedge c, \\ S_n^i &\equiv n^{-1/2} \sum_{j=1}^{[nt]} (v_j^i - \mu_i^{-1}),\ \mu_i^{-1} \geqq 0, \end{aligned} \tag{2.2}$$

and

$$X_n^i \equiv n^{-1/2} \left[\sum_{j=1}^{N^i(nt)} v_j^i - \lambda_i \mu_i^{-1} nt \right],$$

for $1 \leqq i \leqq r$ and $0 \leqq t \leqq c$. It is also convenient to define associated random functions on the product space $D[0,c]^r$:

$$\begin{aligned} \boldsymbol{A}_n &\equiv (A_n^1, \cdots, A_n^r), \\ \boldsymbol{\Phi}_n &\equiv (\Phi_n^1, \cdots, \Phi_n^r), \\ \boldsymbol{S}_n &\equiv (S_n^1, \cdots, S_n^r), \end{aligned} \tag{2.3}$$

and

$$\boldsymbol{X}_n \equiv (X_n^1, \cdots, X_n^r).$$

The metric d_r: $D[0,c]^r \times D[0,c]^r \to R$, defined for any $\boldsymbol{x} = (x_1, \cdots, x_r)$ and $\boldsymbol{y} = (y_1, \cdots, y_r) \in D[0,c]^r$ by $d_r(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \leqq i \leqq r} d(x_i, y_i)$, where d is the metric on $D[0,c]$, gives $D[0,c]^r$ the product topology and makes it a complete separable metric space (cf. [2], Chapter 3). By Theorem 1 of [8], $A_n^i \Rightarrow A^i$ if and only if $U_n^i \Rightarrow -\mu_i^{-3/2} A^i$ if $P\{A^i \in C[0,c]\} = 1$. Hence, we can start with weak convergence of either U_n^i or A_n^i. Actually, by [8] convergence of one sequence in

$D[0,c]$ implies convergence of the other in $D[0,c-\varepsilon]$ for any ε, $0<\varepsilon<c$, but since c is more or less arbitrary, this does not affect our argument and we shall not make the distinction.

Theorem 2.1. If $(A_n,S_n)\Rightarrow(A,S)$ in $D[0,c]^{2r}$ where $c=1+\max\{1,\lambda_1,\cdots,\lambda_r\}$ and $P\{(A,S)\in C[0,c]^{2r}\}=1$, then

$$(A_n,S_n,\Phi_n,X_n)\Rightarrow(A,S,\Phi,X) \text{ in } D[0,1]^{4r},$$

and

$$X_n\Rightarrow X \qquad \text{in } D[0,1]^r,$$

$$X_n^i\Rightarrow X^i \qquad \text{in } D[0,1],$$

where $X=(X^1,\cdots,X^r)$, $\Phi=(\Phi^1,\cdots,\Phi^r)$, $X^i=S^i\circ\Phi^i+\mu_i^{-1}A^i$, and $\Phi(t)=\lambda_i t$, $0\leqq t\leqq 1$, $1\leqq i\leqq r$. If A^i and S^i are independent, then

$$X^i\sim\lambda_i^{1/2}S^i+\mu_i^{-1}A^i.$$

Proof. Our argument will be essentially that of Iglehart and Kennedy ((1970), Section 2), employing the random time change of [2], Section 17.

We first show that

$$(2.4) \qquad (A_n,S_n,\Phi_n)\Rightarrow(A,S,\Phi).$$

Let $g\colon D[0,c]\to R$ be defined for any $x\in D[0,c]$ by $g(x)=\sup_{0\leqq t\leqq c}|x(t)|$. Since $(A_n,S_n)\Rightarrow(A,S)$, we obtain $A_n^i\Rightarrow A^i$ and $g(A_n^i)\Rightarrow g(A^i)$ using Theorem 5.1 of [2]. Now let $h_n\colon R\to R$ and $h\colon R\to R$ be defined for any $x\in R$ by $h_n(x)=xn^{-1/2}$ and $h(x)=0$. By Theorem 5.5 of [2], $h_n[g(A_n^i)]\Rightarrow h[g(A^i)]=0$. Since $d(\Phi_n^i,\Phi^i)\leqq\rho(\Phi_n^i,\Phi^i)\equiv\sup_{0\leqq t\leqq c}|\Phi_n^i(t)-\Phi^i(t)|\leqq h_n[g(A_n^i)]$, $d(\Phi_n^i,\Phi^i)\Rightarrow 0$. By Theorem 4.1 of [2], (2.4) is established.

Now we are ready to apply the composition function of [2], page 144, but note that $\Phi_n(t)$ and $\Phi(t)$ may assume values greater than 1. Therefore, let $D_c[0,1]$ consist of all those functions $y\in D[0,1]$ which are non-decreasing and satisfy $0\leqq y(t)\leqq c$. Note that Φ_n and Φ when restricted to [0,1] are elements of $D_c[0,1]$ so that (2.4) holds in $D[0,c]^{2r}\times D_c[0,1]^r$. Then we use the composition which maps $D[0,c]\times D_c[0,1]$ into $D[0,1]$, defined for any $x\in D[0,c]$ and $y\in D_c[0,1]$ by

$$(x\circ y)(t)=x(y(t)), \qquad 0\leqq t\leqq 1.$$

Hence, we can apply Theorem 5.1 of [2] to obtain

$$(A_n,S_n,\Phi_n,S_n^1\circ\Phi_n^1+\mu_1^{-1}A_n^1,\cdots,S_n^r\circ\Phi_n^r+\mu_r^{-1}A_n^r)$$
$$\Rightarrow(A,S,\Phi,S^1\circ\Phi^1+\mu_1^{-1}A^1,\cdots,S^r\circ\Phi^r+\mu_r^{-1}A^r)$$

in $D[0,1]^{4r}$. Since $c>1$, we have applied a projection to get from $D[0,c]$ to $D[0,1]$ for some components. Next observe that

$d_{4r}[(A_n, S_n, \Phi_n, S_n^1 \circ \Phi_n^1 + \mu_1^{-1}A_n^1, \cdots, S_n^r \circ \Phi_n^r + \mu_r^{-1}A_n^r), (A_n, S_n, \Phi_n, X_n^1, \cdots, X_n^r)] \Rightarrow 0$

and then apply Theorem 4.1 of [2]. Note that $X_n^i = S_n^i \circ \Phi_n^i + \mu_i^{-1}A_n^i$ if $\Phi_n^i(1) < c$, but $P\{\Phi_n^i(1) < c, 1 \leqq i \leqq r\} \to 1$ because $\mathbf{\Phi}_n \Rightarrow \mathbf{\Phi}$. Finally, note that $S^i \circ \Phi^i \sim \lambda_i^{1/2} S^i$ because $\lambda_i^{1/2} S_{\lambda n}^i = S_n^i \circ \Phi^i$, $\lambda_i^{1/2} S_{\lambda n}^i \Rightarrow \lambda_i^{1/2} S^i$, and $S_n^i \circ \Phi^i \Rightarrow S^i \circ \Phi^i$. Consequently, with the independence assumption,

$$\lambda_i^{1/2} S^i + \mu_i^{-1} A^i \sim S^i \circ \Phi^i + \mu_i^{-1} A^i.$$

As a special case, we have the ordinary central limit theorem for random sums:

Corollary 2.1. Let $\{u_n, n \geqq 1\}$ and $\{v_n, n \geqq 1\}$ be two independent sequences of i.i.d. random variables with $Eu_n = \lambda^{-1} > 0$, $Ev_n = \mu^{-1}$, $\mathrm{Var}(u_n) = \sigma_u^2$, $0 < \sigma_u^2 < \infty$, $\mathrm{Var}(v_n) = \sigma_v^2$, and $0 < \sigma_v^2 < \infty$. Then

$$\lim_{t \to \infty} P \left\{ \frac{X(t) - \rho t}{\alpha t^{1/2}} \leqq x \right\} = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy,$$

where

$$X(t) = v_1 + \cdots + v_{A(t)},$$

$$A(t) = \begin{cases} \max\{k \geqq 1, u_1 + \cdots + u_k \leqq t\}, & u_1 \leqq t \\ 0, & u_1 > t, \end{cases}$$

$$\rho = Ev/Eu = \lambda/\mu,$$

and

$$\alpha = (\lambda\sigma_v^2 + \mu^{-2}\lambda^3\sigma_u^2)^{1/2}.$$

Proof. Theorems 10.1 and 17.3 of [2] imply that $A_n^1 \Rightarrow (\lambda^3\sigma_u^2)^{1/2}\xi^1$ and $S_n^1 \Rightarrow \sigma_v \xi^2$, where ξ^1 and ξ^2 are Wiener processes. The independence implies that $(A_n^1, S_n^1) \Rightarrow [(\lambda^3\sigma_u^2)^{1/2}\xi^1, \sigma_v\xi^2]$ where ξ^1 and ξ^2 are independent Wiener processes (cf. [2], page 27). Theorem 2.1 implies that $X_n^1 \Rightarrow \lambda_1^{1/2}S^1 + \mu_1^{-1}A^1 = (\lambda_1\sigma_v^2)^{1/2}\xi^2 + (\mu_1^{-2}\lambda^3\sigma_u^2)^{1/2}\xi^1$, but $a\xi^1 + b\xi^2 \sim (a^2 + b^2)^{1/2}\xi$, where a and b are arbitrary constants, ξ^1 and ξ^2 are independent Wiener processes, and ξ is a Wiener process ($\sim$ means equality in distribution). Finally, it only remains to apply Theorem 5.1 of [2] with the projection for $t = 1$. Also let n go to infinity in a continuous manner (cf. [2], page 16).

3. The basic processes, workload and idle time processes

Let $\{A^i(t), t \geqq 0\}$ be the process which counts the number of arrivals in priority class i $(1 \leqq i \leqq r)$ in $[0, t]$, that is, let

$$A^i(t) = \begin{cases} \max\{k \geqq 1: u_1^i + \cdots + u_k^i \leqq t\}, & u_1^i \leqq t \\ 0, & u_1^i > t. \end{cases} \tag{3.1}$$

We shall assume that the system is initially empty, but [2], Theorem 4.1 can be used to verify that our weak convergence theorems are invariant under different initial conditions. Assuming the system is initially empty, the processes $\{X^i(t), t \geqq 0\}$ depict the total input of work of priority class i $(1 \leqq i \leqq r)$ in $[0,t]$ if

$$X^i(t) = \begin{cases} v_1^i + \cdots + v_{A^i(t)}^i, & u_1^i \leqq t \\ 0, & u_1^i > t. \end{cases} \tag{3.2}$$

Now assume that the priority increases from 1 to r. Then for the highest priority customers the process $\{Y^r(t), t \geqq 0\}$, where

$$Y^r(t) = X^r(t) - t, \tag{3.3}$$

is the net input process. The process depicting the workload of priority class r facing the server at time t can then be defined by

$$L^r(t) = Y^r(t) - \inf_{0 \leqq s \leqq t} Y^r(s). \tag{3.4}$$

For priority class r, $L^r(t) = W^r(t)$, where $W^r(t)$ is the virtual waiting time process for customers in priority class r. The accumulated idle time for priority class r in $[0,t]$ can then be depicted by the process $\{I^r(t), t \geqq 0\}$, where

$$I^r(t) = L^r(t) - Y^r(t) = - \inf_{0 \leqq s \leqq t} Y^r(s). \tag{3.5}$$

The process $\{I^r(t), t \geqq 0\}$ records the accumulated time in $[0,t]$ during which there is no customer of priority class r waiting or being served. Notice that our system is just like the standard single-server queue from the point of view of the highest priority customer. Thus, weak convergence results for the highest priority class agree with earlier ones for the standard single-server queue (cf. [12], [9], or [13]). The relationships depicted in (3.1)–(3.5) are basic notions in queueing (cf. Beneš (1963)); we mention them only to be complete. We should remark however that we regard (3.1)–(3.10) as definitions and not theorems. We do not wish to follow the practice of some authors who try to prove that basic defining formulas satisfy verbal interpretations. In other words, we *define* $L^r(t)$ by (3.4) and *interpret* $L^r(t)$ as the total workload of priority class r facing the server at time t. Obviously some authors might choose different initial definitions, but as long as they are equivalent, it will not affect our discussion here.

It is now easy to define the processes $\{Y^i(t), t \geqq 0\}$, $\{I^i(t), t \geqq 0\}$ and $\{L^i(t), t \geqq 0\}$ for $1 \leqq i \leqq r-1$. We do this recursively:

$$\begin{aligned} Y^i(t) &= X^i(t) - I^{i+1}(t), \\ I^i(t) &= - \inf_{0 \leqq s \leqq t} Y^i(s), \end{aligned} \tag{3.6}$$

and
$$L^i(t) = Y^i(t) - \inf_{0 \leqq s \leqq t} Y^i(s)$$
$$= Y^i(t) + I^i(t).$$

The process $Y^i(t)$ is the net input process for priority class i; the process $I^i(t)$ depicts the accumulated time in $[0,t]$ during which the server is available for priority class i but there is no customer of priority class i in the system; and the process $L^i(t)$ depicts the workload of priority class i facing the server at time t. Note that $I^i(t)$ does not represent the accumulated time in $[0,t]$ during which there is no customer of priority class i because $I^i(t)$ depends on the availability of the server. For $i < r$, $L^i(t)$ does not coincide with the virtual waiting time process for priority class i. We refer the reader to Hooke (1969) for a treatment of the virtual waiting time processes when $i < r$.

It is easy to combine these processes to obtain similar processes characterizing the entire system. The process depicting the total input of work into the entire system is $\{X(t), t \geqq 0\}$, where

(3.7)
$$X(t) = X^1(t) + \cdots + X^r(t).$$

The net input process into the system is $\{Y(t), t \geqq 0\}$, where

(3.8)
$$Y(t) = X(t) - t.$$

Then the idle time for the entire system is $\{I(t), t \geqq 0\}$, where

(3.9)
$$I(t) = -\inf_{0 \leqq s \leqq t} Y(s)$$

and the total service load facing the server at time t is $\{L(t), t \geqq 0\}$, where

(3.10)
$$L(t) = Y(t) - \inf_{0 \leqq s \leqq t} Y(s) \text{ or}$$
$$= L^1(t) + \cdots + L^r(t).$$

Note that these total system processes do not depend on the priority discipline as long as the server is always working whenever there is a customer somewhere in the system. Hence, weak convergence theorems for the processes in (3.7)–(3.10) are contained in our earlier work. All the important new processes here are contained in (3.6). However, our weak convergence theorems here are for all the processes jointly.

We shall now define random functions induced in the function spaces $D[0,1]$ or $D[0,c]$, $0 < c < \infty$, by the processes in (3.1)–(3.10). The normalization by $n^{1/2}$ is to be expected, but is not used explicitly.

$$
\begin{aligned}
A_n^i &\equiv [A^i(nt) - \lambda_i nt]/n^{1/2}, \qquad \lambda_i > 0, \qquad 1 \leqq i \leqq r,\\
S_n^i &\equiv n^{-1/2} \sum_{j=1}^{[nt]} (v_j^i - \mu_i^{-1}), \qquad 1 \leqq i \leqq r,\\
X_n^i &\equiv [X^i(nt) - \rho_i nt]/n^{1/2}, \qquad \rho_i = \lambda_i/\mu_i,\ 1 \leqq i \leqq r,\\
Y_n^r &\equiv [Y^r(nt) - (\rho_r - 1)nt]/n^{1/2},\\
X_n &\equiv [X(nt) - \rho nt]/n^{1/2}, \qquad \rho = \rho_1 + \cdots + \rho_r,\\
Y_n &\equiv [Y(nt) - (\rho - 1)nt]/n^{1/2},\\
Y_n' &\equiv Y(nt)/n^{1/2},\\
I_n &\equiv [I(nt) - (1 - \rho)nt]/n^{1/2},\\
I_n' &\equiv I(nt)/n^{1/2},\\
L_n &\equiv [L(nt) - (\rho - 1)nt]/n^{1/2},
\end{aligned}
\tag{3.11}
$$

and

$$L_n' \equiv L(nt)/n^{1/2}.$$

We shall also define various random functions in the product spaces $D[0,1]^r$ or $D[0,c]^r$, $0 < c < \infty$. Let

$$
\begin{aligned}
A_n &\equiv (A_n^1, \cdots, A_n^r),\\
S_n &\equiv (S_n^1, \cdots, S_n^r),
\end{aligned}
\tag{3.12}
$$

and

$$X_n \equiv (X_n^1, \cdots, X_n^r).$$

Theorem 3.1. If $(A_n, S_n) \Rightarrow (A, S)$ in $D[0,c]^{2r}$ where $c = 1 + \max\{1, \lambda_1, \cdots, \lambda_r\}$ and $P\{(A,S) \in C[0,c]^{2r}\} = 1$, then

(a) $(A_n, S_n, X_n, X_n, Y_n^r, Y_n) \Rightarrow (A, S, X, X, X^r, X)$ in $D[0,1]^{3r+3}$,

(b) if $\rho > 1$, $(A_n, S_n, X_n, X_n, Y_n^r, Y_n, L_n, I_n') \Rightarrow (A, S, X, X^r, X, X, \theta)$,

(c) if $\rho = 1$, $(A_n, S_n, X_n, X_n, Y_n^r, Y_n, L_n', I_n') \Rightarrow (A, S, X, X, X^r, X, f(X), g(X))$, and

(d) if $\rho < 1$, $(A_n, S_n, X_n, X_n, Y_n^r, Y_n, L_n', I_n) \Rightarrow (A, S, X, X, X^r, X, \theta, -X)$ all in $D[0,1]^{3r+5}$, where $\rho = \rho_1 + \cdots + \rho_r$, $\rho_i = \lambda_i/\mu_i$, and

(i) $X^i = S^i \circ \Phi^i + \mu_i^{-1} A^i, \qquad 1 \leqq i \leqq r,$

(ii) $X = X^1 + \cdots + X^r,$

(iii) $\Phi^i(t) = \lambda_i t, \qquad 0 \leqq t \leqq 1,$

(iv) $\theta(t) = 0, \qquad 0 \leqq t \leqq 1,$

(v) $f\colon D[0,1] \to D[0,1]$ is defined for any $x \in D[0,1]$ by

$$f(x)(t) = x(t) - \inf_{0 \leqq s \leqq t} x(s), \qquad 0 \leqq t \leqq 1,$$

and

(vi) $g\colon D[0,1] \to D[0,1]$ is defined for any $x \in D[0,1]$ by

$$g(x)(t) = -\inf_{0 \leqq s \leqq t} x(s), \qquad 0 \leqq t \leqq 1.$$

Proof. Theorem 2.1 implies that $(A_n, S_n, X_n) \Rightarrow (A, S, X)$ in $D[0,1]^{3r}$. The continuous mapping theorem ([2], Theorem 5.1) with $h\colon D[0,1]^{3r} \to D[0,1]^{3r+3}$ defined for any $(x, y, z) \in D[0,1]^{3r}$ by $h(x, y, z) = (x, y, z, z_1 + \cdots + z_r, z_r, z_1 + \cdots + z_r)$ gives us part (a), after observing that $X_n^r = Y_n^r$ and $X_n = Y_n$. Parts (b)–(d) follow by the usual arguments since $L_n = f(Y_n)$ and $I_n = g(Y_n)$ (cf. [7] Theorem 1, [14] Theorem 1, or [13] Theorems 4.2, 4.3, and 4.4).

In order to obtain weak convergence theorems for the processes $Y^i(t)$, $I^i(t)$, and $L^i(t)$ $(1 \leqq i \leqq r)$, we shall consider only two priority classes, but the method applies to an arbitrary number. Even with only two classes there are five different cases. Let $U_n = (A_n, S_n, X_n, X_n, Y_n^r, Y_n)$ and $U = (A, S, X, X, X^r, X)$.

Theorem 3.2. Let $(A_n^1, A_n^2, S_n^1, S_n^2) \Rightarrow (A^1, A^2, S^1, S^2)$ in $D[0,c]^4$ where $c = 1 + \max\{1, \lambda_1, \lambda_2\}$ and $P\{(A^1, A^2, S^1, S^2) \in C[0,c]^4\} = 1$. Then

(a) if $\rho_2 > 1$, if

$$Z_n(t) = \left(U_n(t), \frac{L(nt) - (\rho_1 + \rho_2 - 1)nt}{n^{1/2}}, \frac{I(nt)}{n^{1/2}}, \frac{I^2(nt)}{n^{1/2}}, \frac{L^2(nt) - (\rho_2 - 1)nt}{n^{1/2}}, \right.$$
$$\left. \frac{Y^1(nt) - \rho_1 nt}{n^{1/2}}, \frac{I^1(nt)}{n^{1/2}}, \frac{L^1(nt) - \rho_1 nt}{n^{1/2}}\right), 0 \leqq t \leqq 1,$$

and if

$$Z = (U, X^1 + X^2, \theta, \theta, X^2, X^1, \theta, X^1),$$

then $Z_n \Rightarrow Z$ in $D[0,1]^{3r+10}$;

(b) if $\rho_2 = 1$, if

$$Z_n(t) = \left(U_n(t), \frac{L(nt) - \rho_1 nt}{n^{1/2}}, \frac{I(nt)}{n^{1/2}}, \frac{I^2(nt)}{n^{1/2}}, \frac{L^2(nt)}{n^{1/2}}, \frac{Y^1(nt) - \rho_1 nt}{n^{1/2}}, \right.$$
$$\left. \frac{I^1(nt)}{n^{1/2}}, \frac{L^1(nt) - \rho_1 nt}{n^{1/2}}\right), \qquad 0 \leqq t \leqq 1,$$

and if $Z = (U, X^1 + X^2, \theta, g(X^2), f(X^2), X^1 - g(X^2), \theta, X^1 - g(X^2))$, then $Z_n \Rightarrow Z$ in $D[0,1]^{3r+10}$;

(c) if $\rho_2 < 1$ and $\rho_1 + \rho_2 > 1$, if

$$Z_n(t) = \left(U_n(t), \frac{L(nt) - (\rho_1 + \rho_2 - 1)nt}{n^{1/2}}, \frac{I(nt)}{n^{1/2}}, \frac{I^2(nt) - (1 - \rho_2)nt}{n^{1/2}}, \frac{L^2(nt)}{n^{1/2}}, \frac{Y^1(nt) - (\rho_1 + \rho_2 - 1)nt}{n^{1/2}}, \frac{I^1(nt)}{n^{1/2}}, \frac{L^1(nt) - (\rho_1 + \rho_2 - 1)nt}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $Z = (U, X^1 + X^2, \theta, -X^2, \theta, X^1 + X^2, \theta, X^1 + X^2)$, then $Z_n \Rightarrow Z$ in $D[0,1]^{3r+10}$;

(d) if $\rho_1 + \rho_2 = 1$, if

$$Z_n(t) = \left(U_n(t), \frac{L(nt)}{n^{1/2}}, \frac{I(nt)}{n^{1/2}}, \frac{I^2(nt) - (1 - \rho_2)nt}{n^{1/2}}, \frac{L^2(nt)}{n^{1/2}}, \frac{Y^1(nt)}{n^{1/2}}, \frac{I^1(nt)}{n^{1/2}}, \frac{L^1(nt)}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $Z = (U, f(X^1 + X^2), g(X^1 + X^2), -X^2, \theta, X^1 + X^2, g(X^1 + X^2), f(X^1 + X^2))$, then $Z_n \Rightarrow Z$ in $D[0,1]^{3r+10}$;

(e) if $\rho_1 + \rho_2 < 1$, if

$$Z_n(t) = \left(U_n(t), \frac{L(nt)}{n^{1/2}}, \frac{I(nt) - (1 - \rho_1 - \rho_2)rt}{n^{1/2}}, \frac{I^2(nt) - (1 - \rho_2)nt}{n^{1/2}}, \frac{L^2(nt)}{n^{1/2}} \frac{Y^1(nt) - (\rho_1 + \rho_2 - 1)nt}{n^{1/2}}, \frac{I^1(nt) - (1 - \rho_1 - \rho_2)nt}{n^{1/2}}, \frac{L^1(nt)}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $Z = (U, \theta, -X^1 - X^2, -X^2, \theta, X^1 + X^2, -X^1 - X^2, \theta)$, then $Z_n \Rightarrow Z$ in $D[0,1]^{3r+10}$.

(Definitions of X^i, θ, f, and g are given in Theorem 3.1.)

Proof. Limits for the first three terms of Z_n follow from Theorem 3.1 in each case. The rest can be obtained by successively applying Theorems 4.1 and 5.1 of [2] using the relationships outlined in (3.1)–(3.10). Each component of Z_n is a continuous function of earlier components, differs from such components by an amount converging to 0, or converges to θ itself.

4. The number in the system and the number of departures

Let $\{Q^i(t), t \geqq 0\}$ be the processes counting the number of priority class i customers in the system at time t and let $\{D^i(t), t \geqq 0\}$ be the processes counting the number of priority class i customers that have been served and have departed in $[0,t]$, $1 \leqq i \leqq r$. Let $\{Q(t), t \geqq 0\}$ and $\{D(t), t \geqq 0\}$ be the processes recording the number of customers of all priorities in the system at time t and having departed in $[0,t]$. Obviously $Q(t) = Q^1(t) + \cdots + Q^r(t)$, $D(t) = D^1(t) + \cdots + D^r(t)$ and $Q^i(t) = A^i(t) - D^i(t)$. In this section we shall prove weak convergence theorems for these processes in heavy traffic.

It is now convenient to introduce counting processes associated with the sequences of service times $\{v_n^i, n \geqq 1\}$, $1 \leqq i \leqq r$. Let $\{N^i(t), t \geqq 0\}$ be defined by

$$N^i(t) = \begin{cases} \max\{k \geqq 1: v_1^i + \cdots + v_k^i \leqq t\}, & v_1^i \leqq t \\ 0, & v_1^i > t, \end{cases} \tag{4.1}$$

for $1 \leqq i \leqq r$. Also define random functions in $D[0,1]$ or $D[0,c]$, $0 < c < \infty$ induced by these processes:

$$N_n^i \equiv \frac{N^i(nt) - \mu_i nt}{n^{1/2}}, \qquad 1 \leqq i \leqq r. \tag{4.2}$$

Theorem 1 of [8] implies that $N_n^i \Rightarrow -\mu_i^{3/2} S^i$ if and only if $S_n^i \Rightarrow S^i$, where $P\{S^i \in C\} = 1$. It is also easy to show by the same argument that $(A_n, S_n, N_n) \Rightarrow (A, S, N)$ if $(A_n, S_n) \Rightarrow (A, S)$, where $N_n = (N_n^1, \cdots, N_n^r)$, $N = (N^1, \cdots, N^r)$, $N^i = -\mu_i^{3/2} S^i$, and $P\{(A,S) \in C^{2r}\} = 1$. The processes $N^i(t)$ are important because

$$\begin{aligned} D^r(t) &= N^r(t - I^r(t)), && 0 \leqq t \leqq 1, \text{ and} \\ D^i(t) &= N^i(I^{i+1}(t) - I^i(t)), && 0 \leqq t \leqq 1, \quad 1 \leqq i \leqq r-1. \end{aligned} \tag{4.3}$$

Again we shall consider the special case of $r = 2$, but the argument applies to any $r \geqq 2$.

Theorem 4.1. Let the assumptions of Theorem 3.2 be satisfied and let $\boldsymbol{Z}_n$ and $\boldsymbol{Z}$ be as in Theorem 3.2 in each case below.

(a) If $\rho_2 > 1$, if

$$\boldsymbol{B}_n(t) = \left(\boldsymbol{Z}_n(t), \frac{D^1(nt)}{n^{1/2}}, \frac{D^2(nt) - \mu_2 nt}{n^{1/2}}, \frac{D(nt) - \mu_2 nt}{n^{1/2}}, \frac{Q^1(nt) - \lambda_1 nt}{n^{1/2}}, \right.$$

$$\left. \frac{Q^2(nt) - (\lambda_2 - \mu_2)nt}{n^{1/2}}, \frac{Q(nt) - (\lambda_1 + \lambda_2 - \mu_2)nt}{n^{1/2}}\right), \qquad 0 \leqq t \leqq 1,$$

and if $\boldsymbol{B} = (\boldsymbol{Z}, \theta, N^2, N^2, A^1, A^2 - N^2, A^1 + A^2 - N^2)$, then $\boldsymbol{B}_n \Rightarrow \boldsymbol{B}$ in $D[0,1]^{3r+16}$.

(b) If $\rho_2 = 1$, if

$$B_n(t) = \left(Z_n(t), \frac{D^1(nt)}{n^{1/2}}, \frac{D^2(nt) - \mu_2 nt}{n^{1/2}}, \frac{D(nt) - \mu_2 nt}{n^{1/2}}, \frac{Q^1(nt) - \lambda_1 nt}{n^{1/2}}, \frac{Q^2(nt)}{n^{1/2}}, \frac{Q(nt) - \lambda_1 nt}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $B = (Z, \mu_1 g(X^2), N^2 - \mu_2 g(X^2), N^2 + (\mu_1 - \mu_2) g(X^2), A^1 - \mu_1 g(X^2), \mu_2 f(X^2), A^1 - \mu_1 g(X^2) + \mu_2 f(X^2)), 0 \leqq t \leqq 1$, then $B_n \Rightarrow B$ in $D[0,1]^{3r+16}$.

(c) If $\rho_2 < 1$ and $\rho_1 + \rho_2 > 1$, if

$$B_n(t) = \left(Z_n(t), \frac{D^1(nt) - \mu_1(1-\rho_2)nt}{n^{1/2}}, \frac{D^2(nt) - \lambda_2 nt}{n^{1/2}}, \frac{D(nt) - [\mu_1(1-\rho_2) + \lambda_2]nt}{n^{1/2}}, \frac{Q^1(nt) - [\lambda_1 - \mu_1(1-\rho_2)]nt}{n^{1/2}}, \frac{Q^2(nt)}{n^{1/2}}, \frac{Q(nt) - [\lambda_1 - \mu_1(1-\rho_2)]nt}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $B = (Z, (1-\rho_2)^{1/2} N^1 - \mu_1 X^2, A^2, (1-\rho_2)^{1/2} N^1 - \mu_1 X^2 + A^2, A^1 - (1-\rho_2) N^1 + \mu_1 X^2, \theta, A^1 - (1-\rho_2) N^1 + \mu_1 X^2)$, then $B_n \Rightarrow B$ in $D[0,1]^{3r+16}$.

(d) If $\rho_1 + \rho_2 = 1$, if

$$B_n(t) = \left(Z_n(t), \frac{D^1(nt) - \mu_1(1-\rho_2)nt}{n^{1/2}}, \frac{D^2(nt) - \lambda_2 nt}{n^{1/2}}, \frac{D(nt) - [\mu_1(1-\rho_2) + \lambda_2]nt}{n^{1/2}}, \frac{Q^1(nt) - [\lambda_1 - \mu_1(1-\rho_2)]nt}{n^{1/2}}, \frac{Q^2(nt)}{n^{1/2}}, \frac{Q(nt) - [\lambda_1 - \mu_1(1-\rho_2)]nt}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if

$$\begin{aligned} B = (&Z, (1-\rho_2)^{1/2} N^1 - \mu_1[X^2 + g(X^1 + X^2)], A^2, A^2 + (1-\rho_2)^{1/2} N^1 \\ &- \mu_1[X^2 + g(X^1 + X^2)], A^1 - (1-\rho_2)^{1/2} N^1 + \mu_1[X^2 + g(X^1 + X^2)], \\ &\theta, A^1 - (1-\rho_2)^{1/2} N^1 + \mu_1[X^2 + g(X^1 + X^2)]), \end{aligned}$$

then $B_n \Rightarrow B$ in $D[0,1]^{3r+16}$.

(e) If $\rho_1 + \rho_2 < 1$, if

$$B_n(t) = \left(Z_n(t), \frac{D^1(nt) - \lambda_1 nt}{n^{1/2}}, \frac{D^2(nt) - \lambda_2 nt}{n^{1/2}}, \frac{D(nt) - (\lambda_1 + \lambda_2)nt}{n^{1/2}}, \frac{Q^1(nt)}{n^{1/2}}, \frac{Q^2(nt)}{n^{1/2}}, \frac{Q(nt)}{n^{1/2}}\right), \quad 0 \leqq t \leqq 1,$$

and if $B = (Z, A^1, A^2, A^1 + A^2, \theta, \theta, \theta)$, then $B_n \Rightarrow B$ in $D[0,1]^{3r+16}$. (Recall that $X^i = S^i \circ \Phi^i + \mu_i^{-1} A^i$, $N^i = -\mu_i^{3/2} S^i$, $\Phi^i(t) = \lambda_i t$, and $\theta(t) = 0$, $0 \leqq t \leqq 1$. The functions f and g are defined in Theorem 3.1.)

Proof. (a) We have observed above that $(A_n, S_n, N_n) \Rightarrow (A, S, N)$ if $(A_n, S_n) \Rightarrow (A, S)$ where $P\{(A, S) \in C^4\} = 1$. Thus $(Z_n, N_n) \Rightarrow (Z, N)$ in $D[0,1]^{3r+11}$ in each case by Theorem 3.2. From Theorem 3.2 we note that $I_n^2 \Rightarrow \theta$, where

(4.4) $$I_n^2 \equiv \frac{I^2(nt)}{n^{1/2}}, \qquad 0 \leqq t \leqq 1,$$

and

(4.5) $$\theta(t) = 0, \qquad 0 \leqq t \leqq 1.$$

We now construct a random time change $\Phi_n^2 \in D_0[0,1]$, defined by

(4.6) $$\Phi_n^2 \equiv \frac{nt - I^2(nt)}{n}, \qquad 0 \leqq t \leqq 1$$

(cf. [2], Section 17). Clearly $\Phi_n^2 \Rightarrow \Phi$, where

(4.7) $$\Phi(t) = t, \qquad 0 \leqq t \leqq 1.$$

Theorem 4.4 of [2] and Theorem 3.2 imply that

$$(A_n, S_n, N_n, I_n^2, \Phi_n^2) \Rightarrow (A, S, N, \theta, \Phi)$$

and the continuous mapping theorem ([2], Theorem 5.1) implies that

$$(A_n, S_n, N_n, N_n^2 \circ \Phi_n^2 - \mu_2 I_n^2, A_n^2 - N_n^2 \circ \Phi_n^2 + \mu_2 I_n^2) \Rightarrow$$
$$(A, S, N, N^2 \circ \Phi^2 - \mu_2 \theta, A^2 - N^2 \circ \Phi^2 + \mu_2 \theta),$$

but since $N^2 \circ \Phi = N^2$, $D_n^2 = N_n^2 \circ \Phi_n^2 - \mu_2 I_n^2$, and $Q_n^2 = A_n^2 - D_n^2$, where D_n^2 and Q_n^2 are displayed in the theorem, we have shown that

$$(A_n, S_n, N_n, D_n^2, Q_n^2) \Rightarrow (A, S, N, N^2, A^2 - N^2).$$

Now recall that $D^1(t) = N^1(I^2(t) - I^1(t))$. We now define a random time change $\Phi_n \in D_0[0,1]$ by

(4.8) $$\Phi_n \equiv \frac{I^2(nt) - I^1(nt)}{n}, \qquad 0 \leqq t \leqq 1.$$

Observe that Φ_n is non-decreasing because increases in $I^1(t)$ can only occur when there are increases in $I^2(t)$. Moreover, $\Phi_n \Rightarrow \theta$, where θ is given in (4.5). Also define $\tilde{I}_n$ by

(4.9) $$\tilde{I}_n \equiv \frac{I^2(nt) - I^1(nt)}{n^{1/2}}, \qquad 0 \leqq t \leqq 1.$$

By Theorem 3.2, $\tilde{I}_n \Rightarrow \theta$ too. Hence, $D_n^1 = N_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n \Rightarrow \theta$. Again Theorem 4.4 of [2] implies that

$$(A_n, S_n, N_n, D_n^2, Q_n^2, \tilde{I}_n, \Phi_n) \Rightarrow (A, S, N, N^2, A^2 - N^2; \theta, \theta).$$

Finally Theorem 5.1 of [2] completes part (a), observing that $D_n^1 = N_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n$, $D_n = D_n^1 + D_n^2$, $Q_n^1 = A_n^1 - D_n^1, Q_n = Q_n^1 + Q_n^2$, where D_n^1, D_n^2, D_n, Q_n^1, Q_n^2, and Q_n are as described in B_n of (a).

(b) The argument for (b) is similar, but now we rely on the random sum theorem of Section 2. Now $I_n^2 \Rightarrow g(X^2)$, where I_n^2 is given in (4.4), so that we can use the random time change Φ_n^2 in (4.6). Again $\Phi_n^2 \Rightarrow \Phi$, where $\Phi(t) = t$, $0 \leqq t \leqq 1$. By Theorem 3.2 and [2], Theorem 4.4, we can conclude that

$$(A_n, S_n, N_n^2, I_n^2, \Phi_n^2) \Rightarrow (A, S, N^2, g(X^2), \Phi)$$

and by [2], Theorem 5.1,

$$\begin{aligned}(A_n, S_n, N_n^2 \circ \Phi_n^2 - \mu_2 I_n^2, A_n^2 - N_n^2 \circ \Phi_n^2 + \mu_2 I_n^2) \Rightarrow \\ (A, S, N^2 \circ \Phi^2 - \mu_2 g(X^2), A^2 - N^2 \circ \Phi^2 + \mu_2 g(X^2)),\end{aligned}$$

but $D_n^2 = N_n^2 \circ \Phi_n^2 - \mu_2 I_n^2$, $Q_n^2 = A_n^2 - D_n^2$, $N^2 \circ \Phi = N^2$, and $A^2 - N^2 + \mu_2 g(X^2) = \mu_2 f(X^2)$ so that

$$(A_n, S_n, D_n^2, Q_n^2) \Rightarrow (A, S, N^2 - \mu_2 g(X^2), \mu_2 f(X^2)).$$

Let Φ_n be as in (4.8). Once again $\Phi_n \Rightarrow \theta$, where θ is given in (4.5), but $\tilde{I}_n \Rightarrow g(X^2)$, where $\tilde{I}_n$ is defined in (4.9). Hence, $D_n^1 = N_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n \Rightarrow \mu_1 g(X^2)$, where D_n^1 is as described in B_n. The rest of (b) follows as usual from the continuous mapping theorem.

(c) Let I_n^2 and $\tilde{I}_n$ be defined by

$$(4.10)\qquad \begin{aligned} I_n^2 &\equiv \frac{I^2(nt) - (1 - \rho_2)nt}{n^{1/2}}, && 0 \leqq t \leqq 1, \quad \text{and} \\ \tilde{I}_n &\equiv \frac{I^2(nt) - I^1(nt) - (1 - \rho_2)nt}{n^{1/2}}, && 0 \leqq t \leqq 1, \end{aligned}$$

but define new random time changes Φ_n^2 and Φ_n by

$$(4.11)\qquad \begin{aligned} \Phi_n^2 &\equiv \left[\frac{nt - I^2(nt)}{\rho_2 n}\right] \wedge 1, && 0 \leqq t \leqq 1, \quad \text{and} \\ \Phi_n &\equiv \left[\frac{I^2(nt) - I^1(nt)}{(1 - \rho_2)n}\right] \wedge 1, && 0 \leqq t \leqq 1. \end{aligned}$$

By Theorem 3.2, $I_n^2 \Rightarrow -X^2$, $\tilde{I}_n \Rightarrow -X^2$, $\Phi_n^2 \Rightarrow \Phi$ and $\Phi_n \Rightarrow \Phi$, where $\Phi(t) = t$, $0 \leqq t \leqq 1$. Now let $\tilde{N}_n^2$ and $\tilde{N}_n^1$ be defined by $\tilde{N}_n^2 = N_{(\rho_2 n)}^2$ and $\tilde{N}_n^1 = N_{(1-\rho_2)n}^1$. It is easy to show that $D_n^2 = \rho_2^{1/2}\tilde{N}_n^2 \circ \Phi_n^2 - \mu_2 I_n^2$ if $\Phi_n^2 < 1$ and $d(D_n^2, \rho_2^{1/2}\tilde{N}_n^2 \circ \Phi_n^2 - \mu_2 I_n^2) \Rightarrow 0$. Similarly, $D_n^1 = (1-\rho_2)^{1/2}\tilde{N}_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n$ if $\Phi_n < 1$ and $d(D_n^1, (1-\rho_2)^{1/2}\tilde{N}_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n) \Rightarrow 0$. The rest of the argument is as before, observing that $\rho_2^{1/2}\tilde{N}_n^2 \circ \Phi_n^2 \Rightarrow \rho_2^{1/2}N^2$, $(1-\rho_2)^{1/2}\tilde{N}_n^1 \circ \Phi_n \Rightarrow (1-\rho_2)^{1/2}N^1$, and $\rho_2^{1/2}N^2 + \mu_2 X^2 = A^2$.

(d) The results for D_n^2 and Q_n^2 in parts (d) and (e) follow by the argument of (c) because $\rho_2 < 1$ in all three cases. Again we can use $\tilde{I}_n$ and Φ_n defined in (4.10) and (4.11), but now $\tilde{I}_n \Rightarrow -X^2 - g(X^1 + X^2)$, while $\Phi_n \Rightarrow \Phi$, where $\Phi(t) = t$, $0 \leqq t \leqq 1$, as before. Thus arguing as in (c), we obtain $d(D_n^1, \tilde{N}_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n) \Rightarrow 0$ and $D_n^1 \Rightarrow (1-\rho_2)^{1/2}N^1 - \mu_1[X^2 + g(X^1 + X^2)]$. The rest is straightforward.

(e) We have noted that the argument for D_n^2 and Q_n^2 is contained in (c). Let $\tilde{I}_n$ and Φ_n be redefined as

$$\tilde{I}_n \equiv \frac{I^2(nt) - I^1(nt) - \rho_1 nt}{n^{1/2}}, \qquad 0 \leqq t \leqq 1,$$

(4.12)

$$\Phi_n \equiv \left[\frac{I^2(nt) - I^1(nt)}{\rho_1 n}\right] \wedge 1.$$

Then $\tilde{I}_n \Rightarrow X^1$ and $\Phi_n \Rightarrow \Phi$, where $\Phi(t) = t$, $0 \leqq t \leqq 1$. Finally $d(D_n^1, \rho_1^{1/2}\tilde{N}_n^1 \circ \Phi_n + \mu_1 \tilde{I}_n) \Rightarrow 0$ so that $D_n^1 \Rightarrow \rho_1^{1/2}N^1 + \mu_1 X^1 = A^1$.

5. Embedded sequences

There are many embedded sequences associated with the processes of Sections 3 and 4. We could look at any of the processes only at the arrival points of a particular priority class, only at the arrival points of all customers, only at the departure points of a particular priority class, or only at the departure points of all customers. Weak convergence theorems are available for all of these new processes using the inverse random time change argument of [8], Section 7 and [7]. Our random time changes are based respectively on the processes $A^i(t)$, $A(t) = A^1(t) + \cdots + A^r(t)$, $D^i(t)$, and $D(t) = D^1(t) + \cdots + D^r(t)$.

We shall only carry out the argument for one process, but all the others can be treated in the same way. Note that the possibilities include studying the number of class i customers in the system at arrival points or departure points of class j customers, $i \neq j$.

We shall look at the number of class 2 customers in the system at arrival points of class 2 customers ($r = 2$). Let $\{\tilde{Q}_n^A\} = \{Q^2(t_n^2-), n \geqq 1\}$, $t_n^2 = u_1^2 + \cdots + u_n^2$, be this sequence of random variables and let $\{Q_n^A\}$ be the sequence of random functions induced in $D[0,c]$, $c \geqq 1 + \lambda_2$, by $\{\tilde{Q}_n^A\}$:

$$Q_n^A \equiv [\tilde{Q}_{[nt]}^A - \beta nt]/n^{1/2}, \qquad 0 \leqq t \leqq 1. \tag{5.1}$$

Theorem 5.1. Let the assumptions of Theorem 3.2 be satisfied ($c \geqq \lambda_2 + 1$).

(a) If $\rho_2 > 1$ and $\beta = \lambda_2^{-1}(\lambda_2 - \mu_2)$ in (5.1), then

$$Q_n^A \Rightarrow \lambda_2^{-1/2}(A^2 - N^2) + \lambda_2^{-3/2}(\lambda_2 - \mu_2)A^2;$$

(b) if $\rho_2 = 1$ and $\beta = \lambda_2^{-1}(\lambda_2 - \mu_2) = 0$ in (5.1), then

$$Q_n^A \Rightarrow \lambda_2^{-1/2}f(A^2 - N^2);$$

(c) if $\rho_2 < 1$ and $\beta = 0$ in (5.1), then

$$Q_n^A \Rightarrow \theta;$$

with the various quantities defined in Theorems 3.1, 3.2, and 4.1.

Proof. Let A_n^2 and Φ_n^2 be as in (2.2). By Theorems 2.1 and 4.1, $(A_n^2, \Phi_n^2, Q_n^2) \Rightarrow (A^2, \Phi^2, Q^2)$ in $D[0,c]^3$, where $\Phi^2(t) = \lambda_2 t$, $0 \leqq t \leqq 1$, and Q_n and Q^2 are displayed in each case of Theorem 4.1. Note that $Q_n = Q_n^A \circ \Phi_n^2 + \beta A_n^2$ in $D[0,1]$ if $\Phi_n^2 < c$ in parts (a) and (b). Hence, $d(Q_n, Q_n^A \circ \Phi_n^2 + \beta A_n^2) \Rightarrow 0$. Thus $Q_n^A \circ \Phi_n^2 \Rightarrow Q + \beta A^2 = (Q + \beta A^2) \circ \Phi^{-1} \circ \Phi$, where $\Phi^{-1}(t) = \lambda_2^{-1} t$, $0 \leqq t \leqq 1$. Then Theorem 7.1 of [8] implies that $Q_n^A \Rightarrow (Q^2 - \beta A^2) \circ \Phi^{-1}$, but $(Q^2 - \beta A^2) \circ \Phi^{-1} = Q^2 \circ \Phi^{-1} + \beta A^2 \circ \Phi^{-1} \sim \lambda_2^{-1/2} Q^2 + \lambda_2^{-1/2} \beta A^2$. In order to apply Theorem 7.1 of [8], we need C-tightness for $\{Q_n^A\}$, but we can obtain it with the relation

$$\{w_{Q_n^A}(3\lambda_2\delta) \geqq \varepsilon\} \subseteq \{w_{Q_n^A \circ \Phi_n^2}(\delta) \geqq \varepsilon\} \cup \{\rho(\Phi_n^2, \Phi) > \delta\},$$

(cf. [8], Theorem 8.1).

In (c) we have $d(Q_n^A \circ \Phi_n^2, Q_n) \Rightarrow 0$ so that $Q_n^A \circ \Phi_n^2 \Rightarrow \theta = \theta \circ \Phi^{-1} \circ \Phi$ which implies that $Q_n^A \Rightarrow \theta \circ \Phi^{-1} = \theta$ by Theorem 7.1 of [8] again. The same argument gets C-tightness for $\{Q_n^A\}$.

6. Special cases

Section 3 of [9] can be used to generate a number of different situations in which the hypotheses of Theorems 3.1, 3.2, 4.1, and 5.1 are satisfied. In [9] we studied sequences of queueing systems, but the analysis for a single queueing system is even easier. The most important special case is obtained from Theorem 2.1 in the proof of Corollary 2.1:

Corollary 6.1. Let $\{u_n^i, n \geqq 1\}$ and $\{v_n^i, n \geqq 1\}$ $(1 \leqq i \leqq r)$ be $2r$ independent sequences of i.i.d. random variables with $Eu_n^i = \lambda_i^{-1} > 0$, $Ev_n^i = \mu_i^{-1} > 0$,

$\text{Var}(u_n^i) = \sigma_{u_i}^2$, $0 < \sigma_{u_i}^2 < \infty$, $\text{Var}(v_n^i) = \sigma_{v_i}^2$, and $0 < \sigma_{v_i}^2 < \infty$, $1 \leqq i \leqq r$. Then Theorems 2.1, 3.1, 3.2, 4.1 and 5.1 hold with

$$A^i = (\lambda_i^3 \sigma_{u_i}^2)^{1/2} \xi^i \qquad (1 \leqq i \leqq r)$$

and

$$S^i = \sigma_{v_i} \xi^{r+i} \qquad (1 \leqq i \leqq r),$$

where ξ^i $(1 \leqq i \leqq 2r)$ are $2r$ independent Wiener processes. Then also

$$X^i = \lambda_i^{1/2} S^i + \mu_i^{-1} A^i = (\lambda_i \sigma_{v_i}^2)^{1/2} \xi^{r+i} + (\mu_i^{-2} \lambda_i^3 \sigma_{u_i}^2)^{1/2} \xi^i \sim \alpha_i \zeta^i,$$

$$N^i = -\mu_i^{3/2} S^i = -(\mu_i^3 \sigma_{v_i}^2)^{1/2} \xi^{r+i},$$

and

$$X = \alpha_1 \zeta^1 + \cdots + \alpha_r \zeta^r \sim \alpha \zeta,$$

where $\alpha_i = (\lambda_i \sigma_{v_i}^2 + \mu_i^{-2} \lambda_i^3 \sigma_{u_i}^2)^{1/2}$, $\alpha = (\alpha_1^2 + \cdots + \alpha_r^2)^{1/2}$, ξ^i $(1 \leqq i \leqq 2r)$ are $2r$ independent Wiener processes, ζ^i $(1 \leqq i \leqq r)$ are r independent Wiener processes, and ζ is a Wiener process. Moreover, $f(a\xi) \sim a|\xi|$ and $g(a\xi) \sim a|\xi|$, where f and g are the functions defined in Theorem 3.1, a is an arbitrary positive constant, and $|\xi|$ is the one-dimensional Bessel process.

Ordinary heavy traffic limit theorems for the processes studied in Sections 3, 4, and 5 under the assumptions of Corollary 6.1 are immediate, obtained in the same manner as Corollary 2.1. The projection for $t = 1$ ($\pi_1 : D[0,1] \to R$ defined by $\pi_1(x) = x(1)$) is a continuous function. Hence, we only need to apply Theorem 5.1 of [2]. We also replace n with a continuous variable going to infinity (cf. [2], page 16). We shall only spell out a few of these results.

Corollary 6.2. With the assumptions of Corollary 6.1,

(a) if $\rho < 1$, then for any $x > 0$

$$\lim_{t \to \infty} P\left\{\frac{L(t)}{t^{1/2}} \leqq x\right\} = 1;$$

(b) if $\rho = 1$, then

$$\lim_{t \to \infty} P\left\{\frac{L(t)}{\alpha t^{1/2}} \leqq x\right\} = \begin{cases} (2/\pi)^{1/2} \int_0^x e^{-y^2/2} dy, & x \geqq 0 \\ 0, & x < 0; \end{cases}$$

and

(c) if $\rho > 1$, then

$$\lim_{t \to \infty} P\left\{\frac{L(t) - (\rho - 1)t}{\alpha t^{1/2}} \leqq x\right\} = (2\pi)^{-1/2} \int_{-\infty}^x e^{-y^2/2} dy;$$

(d) if $\rho < 1$, then

$$\lim_{t\to\infty} P\left\{\frac{I(t)-(1-\rho)t}{\alpha t^{1/2}} \leqq x\right\} = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-y^2/2}\,dy;$$

(e) if $\rho = 1$, then

$$\lim_{t\to\infty} P\left\{\frac{I(t)}{\alpha t^{1/2}} \leqq x\right\} = \begin{cases} (2/\pi)^{1/2}\displaystyle\int_0^x e^{-y^2/2}\,dy, & x \geqq 0 \\ 0, & x < 0, \end{cases}$$

and for any $x_1 \geqq 0$ and $x_2 \geqq 0$

$$\lim_{t\to\infty} P\left\{\frac{L(t)}{\alpha t^{1/2}} \leqq x_1,\ \frac{I(t)}{\alpha t^{1/2}} \leqq x_2\right\} = \int_0^{x_2}\int_{-\infty}^{y\wedge(x_1-y)} \frac{2(2y-z)}{(2\pi)^{1/2}}\, e^{-(2y-z)^2/2}dz\,dy;$$

(f) if $\rho > 1$, then for any $x > 0$

$$\lim_{t\to\infty} P\left\{\frac{I(t)}{t^{1/2}} \leqq x\right\} = 1,$$

where $\alpha^2 = \sum_{i=1}^{r}(\lambda_i\sigma_{v_i}^2 + \mu_i^2\lambda_i^3\sigma_{u_i}^2)$ with $\alpha^2 = \lambda_i(\sigma_{v_i}^2 + \sigma_{u_i}^2)$ when $\rho = 1$.

Proof. As usual we apply the projection at $t = 1$ and the continuous mapping theorem ([2], Theorem 5.1). In (b) we have used the joint distribution of $\xi(1)$ and $g(\xi)(1)$ (cf. [11], page 281).

Corollary 6.3. With the assumptions of Corollary 6.1,

(a) if $\rho_2 > 1$, then

$$\lim_{t\to\infty} P\left\{\frac{L_1(t)-\rho_1 t}{\alpha_1 t^{1/2}} \leqq x_1,\ \frac{L^2(t)-(\rho_2-1)t}{\alpha_2 t^{1/2}} \leqq x_2\right\}$$

$$= (2\pi)^{-1}\int_{-\infty}^{x_1} e^{-y^2/2}dy \int_{-\infty}^{x_2} e^{-z^2/2}\,dz$$

(asymptotic independence);

(b) if $\rho_2 = 1$, then

$$\lim_{t\to\infty} P\left\{\frac{L^1(t)-\rho_1 t}{t^{1/2}} \leqq x_1,\ \frac{L^2(t)}{t^{1/2}} \leqq x_2\right\} =$$

$$\int_{-\infty}^{\infty}\int_{0\vee x}^{x_2-x}\int_{-\infty}^{x_1-y} \frac{(2z-y)}{\pi\alpha_1\alpha_2^3}\exp\left\{-\frac{(2z-y)^2}{2\alpha_2^2} - \frac{x^2}{2\alpha_1^2}\right\}dz\,dy\,dx\,;$$

(c) if $\rho_2 < 1$ and $\rho_1 + \rho_2 = \rho > 1$, then for any $x_2 > 0$

$$\lim_{t\to\infty} P\left\{\frac{L^1(t)-(\rho_1+\rho_2-1)t}{\alpha t^{1/2}} \leqq x_1, \frac{L^2(t)}{t^{1/2}} \leqq x_2\right\} = (2\pi)^{-1/2}\int_{-\infty}^{x_1} e^{-y^2/2}dy;$$

(d) if $\rho_1 + \rho_2 = \rho = 1$, then for any $x_2 > 0$

$$\lim_{t\to\infty} P\left\{\frac{L^1(t)}{\alpha t^{1/2}} \leqq x_1, \frac{L^2(t)}{t^{1/2}} \leqq x_2\right\} = \begin{cases} (2/\pi) \int_0^{x_1} e^{-y^2/2}\, dy, & x_1 \geqq 0 \\ 0, & x_1 < 0; \end{cases}$$

and

(e) if $\rho_1 + \rho_2 = \rho < 1$, then for any $x_1 > 0$ and $x_2 > 0$

$$\lim_{t\to\infty} P\left\{\frac{L^1(t)}{t^{1/2}} \leqq x_1, \frac{L^2(t)}{t^{1/2}} \leqq x^2\right\} = 1.$$

Proof. The standard projection argument applies, using $\pi_1: D[0,1] \times D[0,1] \to R^2$. In part (b) the weak convergence limit is $(\alpha_1\xi_1 + g(\alpha_2\xi_2), f(\alpha_2\xi_2))$ which equals $(\alpha_1\xi_1 + g(\alpha_2\xi_2), \alpha_2\xi_2 - g(\alpha_2\xi_2))$. We then express the joint density for $(\alpha_1\xi_1, \alpha_2\xi_2, g(\alpha_2\xi_2))(t)$ as in part (b) of Corollary 6.2.

Corollaries 6.2 and 6.3 are just samples of the ordinary limit theorems on R^1 that can be obtained with the assumptions of Corollary 6.1 by means of the projection at $t = 1$. Such corollaries could be stated for every theorem in Sections 3–5, but since the method is clear, we only give a few examples. In Corollary 6.1 we have mentioned the most important special case for which the hypotheses of Theorems 2.1, 3.1, 3.2, 4.1, and 5.1 are satisfied. Section 3.4 of [9] shows how to treat queues with batch processing and Section 3.5 of [9] shows how to treat queues in which customers of different priorities arrive in the same channel. This last situation would imply that $A_n \Rightarrow A$ even though A_n^i and A_n^j are not independent. Section 4 of [9] shows how we may obtain weak convergence theorems for networks of facilities such as the one we have analyzed. Lemma 1 of [14] illustrates how to treat multiple arrival channels for each priority class. Chapter 4 of [2] together with [7] shows how to treat dependent sequences.

It is also possible to obtain weak convergence theorems for many other related processes and functionals. The main tool is the continuous mapping theorem ([2], Theorem 5.1). For examples, see [8], Section 9.

Finally, it may be possible to obtain weak convergence theorems for other priority queues by using the processes studied in this paper as modified processes in the manner of [8] and [9].

References

[1] Beneš, V. (1963) *General Stochastic Processes in the Theory of Queues.* Addison-Wesley, Reading, Mass.

[2] Billingsley, P. (1968) *Convergence of Probability Measures.* John Wiley and Sons, New York.

[3] Hooke, J. and Prabhu, N. (1969) Priority queues in heavy traffic. *Technical Report* No. 83, Department of Operations Research, Cornell University.

[4] HOOKE, J. (1969) *Some Limit Theorems for Priority Queues*. Ph. D. thesis. Cornell University (*Technical Report* No. 91, Department of Operations Research, Cornell University.)

[5] IGLEHART, D. (1969) Multiple channel queues in heavy traffic, IV: law of the iterated logarithm. *Technical Report* No. 8, Department of Operations Research, Stanford University.

[6] IGLEHART, D. AND KENNEDY, D. (1970) Weak convergence of the average of flag processes. *J. Appl. Prob.* **7**, 747–753.

[7] IGLEHART, D. AND WHITT, W. (1969) The equivalence of functional central limit theorems for counting processes and associated partial sums. *Technical Report* No. 5, Department of Operations Research, Stanford University.

[8] IGLEHART, D. AND WHITT, W. (1970a) Multiple channel queues in heavy traffic. I. *Adv. Appl. Prob.* **2**, 150–177.

[9] IGLEHART, D. AND WHITT, W. (1970b) Multiple channel queues in heavy traffic. II: sequences, networks, and batches. *Adv. Appl. Prob.* **2**, 355–369.

[10] JAISWAL, N. (1968) *Priority Queues*. Academic Press, New York.

[11] KARLIN, S. (1966) *A First Course in Stochastic Processes*. Academic Press, New York.

[12] WHITT, W. (1969a) *Weak Convergence Theorems for Queues in Heavy Traffic*. Ph. D. thesis, Cornell University. (*Technical Report* No. 2, Department of Operations Research, Stanford University, 1968.)

[13] WHITT, W. (1969b) Weak convergence theorems for queues in heavy traffic. Paper submitted to *Adv. Appl. Prob.*

[14] WHITT, W. (1970) Multiple channel queues in heavy traffic. III: random server selection. *Adv. Appl. Prob.* **2**, 370–375.