Appendix A

Regular Variation

Since we use regular variation at several places in this book, we give a brief account here without proofs, collecting together the properties we need. Bingham, Goldie and Teugels (1989) is the definitive treatment; it contains everything here. Nice accounts also appear in Feller (1971) and Resnick (1987).

We say that a real-valued function f defined on the interval (c, ∞) for some $c > 0$ is *asymptotically equivalent* to another such function g (at infinity) and write

$$f(x) \sim g(x) \quad \text{as} \quad x \to \infty \tag{A.1}$$

if

$$f(x)/g(x) \to 1 \quad \text{as} \quad x \to \infty\ . \tag{A.2}$$

We say that the real-valued function f has a *power tail* of index α (at infinity) if

$$f(x) \sim Ax^{\alpha} \quad \text{as} \quad x \to \infty \tag{A.3}$$

for a non-zero constant A. Regular variation is a generalization of the power-tail property that captures just what is needed in many mathematical settings.

A positive, Lebesgue measurable real-valued function (on some interval (c, ∞)) L is said to be *slowly varying* (at infinity) if

$$L(\lambda x) \sim L(x) \quad \text{as} \quad x \to \infty \quad \text{for each} \quad \lambda > 0\,. \tag{A.4}$$

Examples of slowly varying functions are positive constants, functions that converge to positive constants, logarithms and iterated logarithms. (Note that $\log x$ is positive on $(1, \infty)$, while $L_2 x \equiv \log\log x$ is positive on (e, ∞).)

The positive functions $2 + \sin x$ and e^{-x} are not slowly varying. It can happen that a slowly varying function experiences infinite oscillation in the sense that

$$\liminf_{x \to \infty} L(x) = 0 \quad \text{and} \quad \limsup_{x \to \infty} L(x) = \infty \ ;$$

an example is

$$L(x) \equiv \exp\{(ln(1+x))^{1/2} \cos((ln(1+x))^{1/2})\} \,. \tag{A.5}$$

It is significant that there is local uniform convergence in (A.4).

Theorem A.1. (local uniform convergence) *If L is slowly varying, then $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ uniformly over compact λ sets.*

Theorem A.2. (representation theorem) *The function L is locally varying if and only if*

$$L(x) = c(x) \exp\left(\int_a^x (b(u)/u) du\right), \quad x \geq a \,, \tag{A.6}$$

for some $a > 0$, where c and b are measurable functions, $c(x) \to c(\infty) \in (0, \infty)$ and $b(x) \to 0$ as $x \to \infty$.

In general a slowly varying function need not be smooth, but it is always asymptotically equivalent to a smooth slowly varying function; see p. 15 of Bingham et al. (1989).

Theorem A.3. *If L is slowly varying function, then there exists a slowly varying function L_0 with continuous derivatives of all orders (in C^∞) such that*

$$L(x) \sim L_0(x) \quad \text{as} \quad x \to \infty \ .$$

If L is eventually monotone, then so is L_0.

We now turn to regular variation. A positive, Lebesgue measurable function (on some interval (c, ∞)) is said to be *regularly varying* of index α, and we write $h \in \mathcal{R}(\alpha)$, if

$$h(\lambda x) \sim \lambda^\alpha h(x) \quad \text{as} \quad x \to \infty \quad \text{for all} \quad \lambda > 0 \,. \tag{A.7}$$

Of course, (A.7) holds whenever h has a power tail with index α, but it also holds more generally.

The connection to slowly varying function is provided by the characterization theorem.

Theorem A.4. (characterization theorem) *If*

$$h(\lambda x) \sim g(\lambda) h(x) \quad as \quad x \to \infty \tag{A.8}$$

for all λ in a set of positive measure, then

(i) (A.8) *holds for all $\lambda > 0$,*

(ii) there exists a number α such that $g(\lambda) = \lambda^\alpha$ for all λ, and

(iii) $h(x) = x^\alpha L(x)$ for some slowly varying function L.

From Theorem A.4 (iii) we see that we could have defined a regularly varying function in terms of a slowly varying function L. On the other hand, (A.8) is an appealing alternative starting point, implying (A.7) and the representation in terms of slowly varying functions. As a consequence of Theorem A.4 (iii), we write $h \in \mathcal{R}(0)$ when h is slowly varying.

We now indicate how the local-uniform-convergence property of $\mathcal{R}(0)$ in Theorem A.1 extends to $\mathcal{R}$.

Theorem A.5. (local uniform convergence) *If $h \in \mathcal{R}(\alpha)$ and h is bounded on each interval $(0, c]$, then*

$$h(\lambda x)/h(x) \to \lambda^\alpha \quad as \quad x \to \infty$$

uniformly in λ:

$$\begin{array}{lll} on\ each\ [a,b], & 0 < a < b < \infty, & if\ \alpha = 0 \\ on\ each\ (0,b], & 0 < b < \infty, & if\ \alpha > 0 \\ on\ each\ [a,\infty), & 0 < a < \infty, & if\ \alpha < 0. \end{array}$$

The representation theorem for slowly varying functions in Theorem A.2 also extends to regularly varying functions.

Theorem A.6. (representation theorem) *We have $h \in \mathcal{R}(\alpha)$ if and only if*

$$h(x) = c(x) \exp\left\{\int_a^x (b(u)/u) du\right\}, \quad x \geq a ,$$

for some $a > 0$, where c and b are measurable functions, $c(x) \to c(\infty) \in (0, \infty)$ and $b(x) \to \alpha$ as $x \to \infty$.

We now present some closure properties.

Theorem A.7. (closure properties) *Suppose that* $h_1 \in \mathcal{R}(\alpha_1)$ *and* $h_2 \in \mathcal{R}(\alpha_2)$. *Then:*

(i) $h_1^\alpha \in \mathcal{R}(\alpha\alpha_1)$.

(ii) $h_1 + h_2 \in \mathcal{R}(\alpha)$ *for* $\alpha = \max\{\alpha_1, \alpha_2\}$.

(iii) $h_1 h_2 \in \mathcal{R}(\alpha_1 + \alpha_2)$.

(iv) If, in addition, $h_2(x) \to \infty$, *then* $(h_1 \circ h_2)(x) \equiv h_1(h_2(x)) \in \mathcal{R}(\alpha_1\alpha_2)$.

Regular variation turns out to imply local regularity conditions; see p. 18 of Bingham et al. (1989).

Theorem A.8. (local integrability theorem) *If* $h \in \mathcal{R}(\alpha)$ *for some* α, *then* h *and* $1/h$ *are both bounded and integrable over compact subintervals of* (c, ∞) *for suitably large* c.

It is significant that integrals of regularly varying functions are again regularly varying with the same slowly varying function (i.e., the slowly varying function passes through the integral).

Theorem A.9. (Karamata's integral theorem) *Suppose that* $L \in \mathcal{R}(0)$ *and that* L *is bounded on every compact subset of* $[c, \infty)$ *for some* $c \geq 0$. *Then*

(a) for $\alpha > -1$,

$$\int_c^x t^\alpha L(t)dt \sim \frac{x^{\alpha+1}}{\alpha+1} L(x) \quad \text{as} \quad x \to \infty \ ;$$

(b) for $\alpha < -1$,

$$\int_x^\infty t^\alpha L(t)dt \sim -\frac{x^{\alpha+1}}{\alpha+1} L(x) \quad \text{as} \quad x \to \infty \ .$$

It is also possible to deduce regular variation of functions from the regular-variation properties of the integrals, i.e., the converse half of the Karamata integral theorem; see p. 30 of Bingham et al. (1989).