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THE CONTINUITY OF QUEUES

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Abstract

Kennedy (1972) showed that the standard single-server queueing model is continuous. These results are extended to the standard multi-server model here. Even when there is only one server, an additional condition is needed for the queue length process.

CONTINUITY OF QUEUES; APPROXIMATIONS FOR QUEUES; MULTI-SERVER QUEUES; $GI/G/s$ QUEUES; WEAK CONVERGENCE ON FUNCTION SPACES; THE CONTINUOUS MAPPING THEOREM

1. Introduction

Kennedy (1972) recently demonstrated that the standard single-server queueing model is continuous. Using the weak convergence theory for probability measures on function spaces, he showed that sequences of queueing processes such as queue length processes and virtual waiting time processes associated with a sequence of queueing systems converge if the sequences of underlying interarrival time and service time sequences converge. The purpose of this paper is to suggest slightly different proofs, to correct a minor error in Theorem 4.2, and to indicate how the results can be extended to multi-server queues.

Somewhat cleaner proofs result from eliminating probability measures from the discussion. We work with various deterministic queueing processes which are possible realizations (sample paths) of the corresponding stochastic processes. After continuity has been verified without any probability measures, it extends immediately to the stochastic setting by virtue of the continuous mapping theorem associated with weak convergence, cf. Section 5 of Billingsley (1968). An essential ingredient of our approach is the identification of appropriate subsets of the basic function spaces containing the sample paths of our queueing processes. For further discussion about this approach and the basic function spaces, see Whitt (1974a, b).

We now chart the way ahead. The representation of the initial data is discussed in Section 2. The single-server and multi-server systems are treated in Sections 3 and 4 respectively. Finally, applications are indicated in Section 5.

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We have also verified similar continuity properties for the single-server preemptive-resume priority model and the s-server rotation-server-selection model, but we shall not include the details. The continuity is closely related to other structural relationships such as monotonicity and stochastic bounds; see Brumelle (1971), (1972), Jacobs and Schach (1972), Ross (1973), Stoyan (1972), Yu (1973) and references there.

2. The initial data

The standard multi-server queueing model is typically specified by a sequence of ordered pairs of non-negative random variables, depicting the interarrival times and service times of successive customers. We call this sequence the initial data of the model. We begin by examining the possible sample paths of this basic sequence.

The function spaces here will all be subspaces of R^∞ or $D[0,\infty)$ or products of these spaces. In order to have no difficulty with the weak convergence theory, we shall require that all these subspaces be Polish spaces (topological spaces which are metrizable as CSMS — complete separable metric spaces). Let all subsets of topological spaces have the relative topology and let all products of topological spaces have the product topology. Recall that a subset of a Polish space is Polish if and only if it is a G_δ (countable intersection of open subsets) while a subset of a CSMS is a CSMS if and only if it is closed. Also recall that a countable product of Polish spaces is Polish.

Let R be the real line with the usual Euclidean metric: $d(x,y) = |x-y|$. Since the Euclidean metric makes R a CSMS, the product space R^∞ is Polish. Let R_+ be the subset of strictly positive (>0) elements in R. Since R_+ is open in R, R_+ is Polish but not a CSMS with the Euclidean metric. Since countable products of open proper subsets are neither open nor closed, R_+^∞ is Polish but neither open nor closed in R^∞. Let I_+^∞ be the subset of R_+^∞ in which $\sum_{k=0}^\infty x_k = \infty$ (let the index begin at 0). By Lemma 3.1 and Theorem 3.3 of Whitt (1974b), I_+^∞ is Polish (I_+^∞ here is $\hat{I}_+^\infty$ there).

We shall consider our queueing systems to be defined by a probability measure on the Polish space $I \equiv I_+^\infty \times R_+^\infty$ (I for initial data of the model). Let $\{(u_n, v_n), n \geqq 0\}$ be a generic element of I. We interpret u_n as the interarrival time between the nth and $(n+1)$th customers and we interpret v_n as the service time of the nth customer. We assume a 0th customer arrives at time $t = 0$ to find an empty system. (In this interpretation u_n and v_n are not random variables but possible realizations.)

Note that our interarrival times are constrained to satisfy $\sum_{k=0}^\infty u_k = \infty$. This guarantees that only finitely many arrivals can occur in any finite interval. Also note that the interarrival times and service times must be strictly positive.

There are several other equivalent representations of the initial data. For

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example, instead of interarrival times, we could specify the arrival epochs of successive customers ($U_n = u_0 + \cdots + u_n$, $n \geqq 0$) or the function $U(t)$ counting the number of arrivals in the interval $(0, t]$ for $t \geqq 0$:

$$(2.1) \qquad U(t) = \begin{cases} \max\{k \geqq 0 : u_0 + \cdots + u_{k-1} \leqq t\}, & u_0 \leqq t, \\ 0, & u_0 > t,\ t \geqq 0. \end{cases}$$

Instead of $\{v_n\}$, we could look at $\{V_n\}$ where $V_n = v_0 + \cdots + v_n$, $n \geqq 0$, but a counting function for $\{v_n\}$ might not be well defined (could be infinite) because $\sum_{k=0}^{\infty} v_k < \infty$ is possible.

It turns out that all these representations are equivalent in a very strong sense. In particular, they are homeomorphic if we regard the set of all counting functions as a subset of the function space $D[0, \infty)$ with Skorohod's J_1 topology, cf. Corollary 3.2 of Whitt (1974b). Since the interarrival times cannot be zero, the counting functions all have unit jumps. Consequently, the M_1 and J_1 topologies agree on this subset of $D[0, \infty)$, cf. Theorem 3.2 of Whitt (1974b). If we relax the strict positivity condition, then the homeomorphism result remains true with the M_1 topology on $D[0, \infty)$ but not with the J_1 topology, cf. Whitt (1974b).

3. The single server queue

We now introduce various continuous maps from I into R^∞ or $D[0, \infty)$ with the J_1 topology. The images of the sequence $\{(u_n, v_n)\}$ under these maps will be the sample paths of the associated stochastic processes. The associated stochastic processes themselves are the image measures of the measure on I under these mappings. We shall describe a map from I into $D[0, \infty)$ by writing $\{(u_n, v_n)\} \to \{X(t), t \geqq 0\}$, where $\{X(t), t \geqq 0\}$ is the image of the map in $D[0, \infty)$. Many maps on I will actually be the composition of several other maps. We frequently use the fact that the composition of two continuous maps is continuous. If the domain of any map is not I, it is understood to be the range of the preceding map. All the domains will be Polish spaces, but we will not give proofs.

We start with the sequence of waiting times $\{W_n\}$. The map $\{(u_n, v_n)\} \to \{X_n\}$, where $X_n = v_n - u_n$, is obviously continuous but not one-to-one. The map $\{X_n\} \to \{S_n\}$, where $S_0 = 0$ and $S_n = X_0 + \cdots + X_{n-1}$ for $n \geqq 1$, is obviously a homeomorphism. The map from R^∞ to $[0, \infty)^\infty$ defined by

$$(3.1) \qquad W_n = S_n - \min_{0 \leqq k \leqq n} S_k, \quad n \geqq 0,$$

is obviously continuous but not one-to-one. Thus, the map from I into $[0, \infty)^\infty$ mapping the basic data into $\{W_n\}$ is continuous but not one-to-one.

Let $L(t)$ represent the total workload in service time to enter the system during $[0, t]$. It can be defined by

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(3.2) $$L(t) = v_0 + \cdots + v_{U(t)} = V_{U(t)}, \quad t \geqq 0.$$

The net input, virtual waiting time, and accumulated idle time can then be defined by

(3.3) $$\begin{aligned} Y(t) &= L(t) - t, & t \geqq 0, \\ W(t) &= Y(t) - \inf_{0 \leqq s \leqq t} Y(s), & t \geqq 0, \\ I(t) &= -\inf_{0 \leqq s \leqq t} Y(s), & t \geqq 0. \end{aligned}$$

The following corresponds to Theorems 3.1–3.3 in Kennedy (1972).

Theorem 3.1. *Each of the maps*

$$\{(u_n, v_n)\} \xleftrightarrow[1]{} [\{U(t), t \geqq 0\}, \{V_n\}] \xrightarrow[2]{} \{L(t), t \geqq 0\}$$

$$\xrightarrow[3]{} \{Y(t), t \geqq 0\} \begin{array}{l} \xrightarrow{4} \{W(t), t \geqq 0\} \\ \xrightarrow{5} \{I(t), t \geqq 0\} \end{array}$$

is continuous and one-to-one.

Proof. The first map involves the equivalent representations of the initial data discussed in Section 2. The second map is a composition which is continuous by Theorem 3.1 (iii) of Whitt (1974a). In that context, we need to work with $\{V_{[t]}, t \geqq 0\}$ instead of $\{V_n\}$ but they are homeomorphic by Corollary 3.2 of Whitt (1974b). Alternatively, a direct argument can be given. The third map requires no comment. The fourth and fifth maps were shown to be continuous in proofs of heavy traffic limit theorems, cf. p. 62 of Whitt (1968). The one-to-one property follows because $u_n > 0$ and $v_n > 0$ for all n.

Remark 3.1. The maps

$$\{L(t), t \geqq 0\} \to \{Y(t), t \geqq 0\} \to \{W(t), t \geqq 0\}$$

are also homeomorphisms onto their range, but the map $\{(u_n, v_n)\} \to \{W(t), t \geqq 0\}$ is not. However, it is a homeomorphism on the G_δ subset of I in which $u_n > \varepsilon$ and $v_n > \varepsilon$ for all n for some $\varepsilon > 0$.

Let D_n denote the epoch of the nth departure, defined by

(3.4) $$D_n = U_n + W_n + v_n, \quad n \geqq 0.$$

Obviously, the map $\{(u_n, v_n)\} \to \{D_n\}$ is continuous. Since $D_{n+1} - D_n \geqq v_{n+1} > 0$, the map taking $\{D_n\}$ into the associated counting function

(3.5) $$D(t) = \begin{cases} \min\{k \geqq 1 : D_{k-1} \leqq t\}, & D_0 \leqq t, \\ 0, & D_0 > t, \ t \geqq 0, \end{cases}$$

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is also continuous by virtue of Lemma 3.4 and Corollary 3.1 of Whitt (1974b).

The queue length process $\{Q(t), t \geqq 0\}$ can now be defined by

$$Q(t) = U(t) - D(t), \quad t \geqq 0. \tag{3.6}$$

It is apparent that the map $\{(u_n, v_n)\} \to \{Q(t), t \geqq 0\}$ need not be continuous because addition is not continuous on $D[0,\infty) \times D[0,\infty)$, cf. Section 4 of Whitt (1974a). The following example demonstrates this, and thus shows that Theorem 4.2 of Kennedy (1972) needs an extra condition.

Example 3.1. Consider a sequence $\{(u_n, v_n)\}$ in which $u_0 = v_0$. Then the first customer arrives at the same instant the 0th customer departs. Thus $Q(t) = 1$ for all t, $0 \leqq t < u_0 + (u_1 \wedge v_1)$. Also consider a double sequence $\{(u_n^i, v_n^i)\}$ in which $u_n^i = u_n$ and $v_n^i = v_n$ for all $n \geqq 1$ and all $i \geqq 1$, while

$$u_0^i = u_0 - 2^{-i} \tag{3.7}$$

and

$$v_0^i = v_0 + 2^{-i}, \quad i \geqq 1.$$

Let i be sufficiently large so that $u_0^i > 0$. Obviously, $\{(u_n^i, v_n^i)\} \to \{(u_n, v_n)\}$ in I as $i \to \infty$, but $Q^i \not\to Q$ in $D[0,\infty)$ with the J_1 topology. Note that $Q^i(t)$ assumes the value 2 at $t = u_0$ for all i, while $Q(t)$ is 1 in a neighborhood of u_0.

Since the mapping $\{(u_n, v_n)\} \to U_j - D_k$ is obviously continuous for each j and k, the subset $\{|U_j - D_k| > 0\}$ is open in I and the subset

$$A = \bigcap_{j=0}^{\infty} \bigcap_{k=0}^{\infty} \{|U_j - D_k| > 0\} \tag{3.8}$$

is a G_δ.

Theorem 3.2. *The map* $\{(u_n, v_n)\} \to \{Q(t), t \geqq 0\}$ *is measurable on* I *and continuous at all limit points in* A.

Proof. The processes $\{U(t), t \geqq 0\}$ and $\{D(t), t \geqq 0\}$ have no common discontinuities if $\{(u_n, v_n)\} \in A$. Thus, we can apply Theorem 4.1(i) of Whitt (1974a). Addition is known to be measurable on $D[0,\infty) \times D[0,\infty)$.

Remark 3.2. It is easy to impose conditions on the measure P on I in order to obtain $P(A) = 1$. For example, it suffices for $\{u_n\}$ and $\{v_n\}$ to be independent sequences of independent random variables with the distributions of the random variables in one of the two sequences being nonatomic.

4. The multi-server queue

We now consider the standard s-server model in which customers are served in order of arrival by the first available server, with some unspecified procedure

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to break ties. The starting point is again an element of I, that is, a deterministic sequence $\{(u_n, v_n),\ n \geqq 0\}$. Let $\{\mathbf{W}_n\}$ be the vector-valued waiting time sequence introduced by Kiefer and Wolfowitz (1955). The map $\{(u_n, v_n)\} \to \{\mathbf{W}_n\}$ is obviously the composition of several continuous maps and is thus continuous. The actual waiting time W_n is of course just the first component of $\mathbf{W}_n$. Let D_n denote the epoch of departure for the nth arriving customer, defined in (3.4). Obviously, the map $\{(u_n, v_n)\} \to \{D_n\}$ is continuous. Note that $\{D_n\}$ is not necessarily an increasing sequence, but the map which rearranges $\{D_n\}$ into ascending order is continuous. The ascending sequence, say $\{E_n\}$, depicts the epochs of successive departures. Continuity is easy to verify for the rearrangement because only $D_0, \cdots, D_{n-1}$ could be less than U_n for each n.

The map $\{E_n\} \to \{D(t), t \geqq 0\}$, where $D(t)$ counts the number of departures in $[0, t]$, is not necessarily continuous because $\{E_n\}$ need not be strictly increasing. Let B be the subset of I defined by

$$(4.1) \qquad \bigcap_{j=0}^{\infty} \bigcap_{k \neq j} \{|D_j - D_k| > 0\}.$$

Just as with A in (3.8), B is a G_δ subset of I. Corollary 3.2 of Whitt (1974b) implies that the map $\{E_n\} \to \{D(t), t \geqq 0\}$ is continuous on I at limits in B (of course, the domain is actually $f(I)$ where $f(\{(u_n, v_n)\}) = \{E_n\})$. Again in that context we should use $\{E_{[t]}, t \geqq 0\}$ but it is homeomorphic with $\{E_n\}$.

Since the first passage time map is continuous on the subset of $D[0, \infty)$ containing $\{E_{[t]},\ t \geqq 0\}$ for all of I in the M_1 topology, it is measurable in the Borel field associated with the M_1 topology. Since the Borel fields on this subset of $D[0, \infty)$ associated with the M_1 and J_1 topologies coincide, cf. Theorem 2.3 of Whitt (1974a), measurability of the map $\{E_n\} \to \{D(t), t \geqq 0\}$ is established for all I.

We now define the queue length just as in (3.6). Again, the map $\{(u_n, v_n)\} \to \{Q(t), t \geqq 0\}$ is measurable but not continuous. However, the argument of Theorem 3.2 applies again to yield continuity for this map on the G_δ subset $A \cap B$, where A is defined in (3.8) and B is defined in (4.1). Just as in Remark 3.2, it is easy to impose conditions on the measure P on I so that $P(A \cap B) = 1$, which is sufficient for the continuous mapping theorem associated with weak convergence. It suffices for $\{u_n\}$ and $\{v_n\}$ to be independent sequences of independent random variables with the distributions of the random variables in one sequence being nonatomic.

The output in completed service time can then be represented by the continuous map

$$(4.2) \qquad 0(t) = \int_0^t Q(s)ds, \quad t \geqq 0.$$

The total workload to enter the system in $[0, t]$ can be defined just as in (3.2).

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Then the function $W(t)$ recording the total workload in service time still in the system at time t can be defined by

$$W(t) = L(t) - 0(t), \quad t \geqq 0. \tag{4.3}$$

Since $0(t)$ is a continuous function, the addition in (4.3) is continuous, cf. Theorem 4.1 of Whitt (1974a).

5. Applications

First, we illustrate a weak convergence consequence. Consider a sequence of $GI/G/s$ queues indexed by n with interarrival time c.d.f.'s F_n and service time c.d.f.'s G_n. Also consider a prospective limiting $GI/G/s$ system with corresponding c.d.f.'s F and G. Let $Q_n \equiv \{Q_n(t),\ t \geqq 0\}$ and $Q \equiv \{Q(t),\ t \geqq 0\}$ denote the associated queue length stochastic processes. Let $\Rightarrow$ denote weak convergence.

Corollary 5.1. *If* $F_n \Rightarrow F$, $G_n \Rightarrow G$, *and* F *or* G *is continuous, then*

$$Q_n \Rightarrow Q \text{ in } (D[0,\infty), J_1).$$

Proof. Since the systems are $GI/G/s$ systems, the basic sequences $\{u_k^n, k \geqq 0\}$ and $\{v_k^n, k \geqq 0\}$ are independent sequences of i.i.d. random variables for each n. The independence means that weak convergence on I is characterized by weak convergence $u_k^n \Rightarrow u_k$ and $v_k^n \Rightarrow v_k$ in R for each k separately, cf. Theorem 3.2 of Billingsley (1968). Since F or G is continuous too, the map $\{(u_n, v_n)\} \to \{Q(t), t \geqq 0\}$ is continuous almost surely with respect to the limit measure.

Remark 5.1. At the outset we assumed that all interarrival and service times are strictly positive. However, in this context it suffices to have $F(0) = 0$ and $G(0) = 0$. The first passage time function will then be continuous almost surely with respect to the limit, cf. Theorem 3.2 of Whitt (1974b).

Continuity results serve at least two purposes. First, they help justify the application of queueing models. We now know that if the distributions of the interarrival times and services times are close to those of a standard $M/M/s$ model, then related stochastic processes such as the queue length process will also be close to the corresponding stochastic processes in a standard $M/M/s$ model.

Second, continuity suggests various approximations. For example, the class of $E_N/E_N/s$ or general Erlang queues is dense in the class of all $GI/G/s$ queues. (A general Erlang distribution is an Erlang distribution with random shape parameter.) This can be seen from an approximation scheme used by Schassberger (1970), (1972). For any given c.d.f. $F(t)$ concentrating on $[0,\infty)$, a sequence of general Erlang c.d.f.'s $\{F_n(t),\ n \geqq 1\}$ can be constructed so that $F_n \Rightarrow F$. This is done by setting

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$$(5.1) \qquad F_n(t) = \sum_{k=0}^{\infty} [F(k/n) - F((k-1)/n)]E_n^k(t), \quad t \geqq 0,$$

where $E_n^k(t)$ is the c.d.f. of the k-fold convolution of an exponential distribution with mean n^{-1}. Since the mean and variance of $E_n^k(t)$ are k/n and k/n^2 respectively, it is easy to see that

$$(5.2) \qquad \lim_{t \to \infty} F_n(t) = \tfrac{1}{2}[F(t+) - F(t-)]$$

for each $t \geqq 0$. This implies that $F_n(t)$ converges to $F(t)$ at all continuity points of F, which is the condition for weak convergence. We can thus construct a sequence of general Erlang models in which the interarrival time and service time c.d.f.'s are constructed from those associated with a given $GI/G/s$ model by means of (5.1). Since we have $GI/G/s$ systems, convergence of these two sequences of c.d.f.'s implies weak convergence of the corresponding sequence of measures on I. Corollary 5.1 can then be applied to show that the queue length process in the $GI/G/s$ system is the weak convergence limit of the sequence of queue length processes associated with the sequence of $E_N/E_N/s$ systems.

The possibility of approximating an arbitrary distribution concentrating on $[0, \infty)$ by general Erlang distributions is apparently widely known, cf. p. 1230 of Kingman (1963) and pp. 114–116 of Cox and Smith (1961), but it has not yet been extensively exploited. Queues with general Erlang components were studied by Luchak (1958) and Prabhu and Lalchandani (1966). Erlang models are naturally structured for an application of the method of stages and phases, cf. p. 110 of Cox and Smith (1961), and this is the procedure used by Schassberger (1970), (1972) to determine Laplace transforms of the virtual waiting time processes in the Erlang systems. Convergence of these Laplace transforms to the corresponding Laplace transforms of the general independent queue is a consequence of Theorem 3.2 of Kennedy (1972) or Theorem 3.1 here and corresponding results for the priority model. A few additional steps are necessary to go from convergence of the processes to convergence of the transforms in Schassberger (1970), (1972), but these steps are not difficult.

Finally, observe that (5.1) yields a sequence of approximating distribution functions for many other $E_n^k(t)$ besides the one given there. For example, if $E_n^k(t)$ is the distribution function corresponding to a unit mass at the point k/n, then $F_n(t)$ is a sequence of discrete distribution functions converging weakly to $F(t)$. In this way, we see that all discrete queueing models, as well as all general Erlang models, are dense in the class of general independent queues. Another approximating system is obtained by letting $E_n^k(t)$ be the k-fold convolution of a Bernoulli distribution which assumes the values 0 and $2/n$ each with probability $\frac{1}{2}$. Of course, $E_n^k(t)$ need not be a convolution at all. It is only necessary that

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the mean be k/n and the variance go to 0 appropriately. Since the resulting classes are each dense in the general independent queues, they are potential approximations and warrant further study.

References

BILLINGSLEY, P. (1968) *Convergence of Probability Measures*. John Wiley, New York.

BRUMELLE, S. L. (1971) Some inequalities for parallel-server queues. *Operat. Res.* **19,** 402–413.

BRUMELLE, S. L. (1972) Bounds on the wait in a $GI/M/k$ queue. *Management Sci.* **19,** 773–777.

COX, D. R. AND SMITH, W. L. (1961) *Queues*. Methuen, London.

JACOBS, D. R., JR. AND SCHACH, S. (1972) Stochastic order relationships between $GI/G/k$ systems. *Ann. Math. Statist.* **43,** 1623–1633.

KEIFER, J. AND WOLFOWITZ, J. (1955) On the theory of queues with many servers. *Trans. Amer. Math. Soc.* **78,** 1–18.

KENNEDY, D. P. (1972) The continuity of the single server queue. *J. Appl. Prob.* **9,** 370–381.

KINGMAN, J. F. C. (1963) Poisson counts for random sequences of events. *Ann. Math. Statist.* **34,** 1217–1232.

LUCHAK, G. (1958) The continuous time solution of the equations of the single channel queue with a general class of service time distributions by the method of generating functions. *J. Roy. Statist. Soc.* B **20,** 176–181.

PRABHU, N. U. AND LALCHANDANI, A. P. (1966) Markov process analysis of Luchak's queueing model. *Technical Report* No. 16, Department of Industrial Engineering and Operations Research, Cornell University.

ROSS, S. M. (1973) Bounds on the delay distribution in $GI/G/1$ queues. Operations Research Center, University of California, Berkeley.

SCHASSBERGER, R. (1970) On the waiting time in the queueing system $GI/G/1$. *Ann. Math. Statist.* **41,** 182–187.

SCHASSBERGER, R. (1972) On the work load process in a general preemptive resume priority queue. *J. Appl. Prob.* **9,** 588–603.

STOYAN, D. (1972) Monotonicity of stochastic models. *ZAMM* **52,** 23–30 (in German).

WHITT, W. (1968) *Weak convergence theorems for queues in heavy traffic* Ph.D. Thesis, Department of Operations Research, Cornell University. (*Technical Report* No. 2, Department of Operations Research, Stanford University.)

WHITT, W. (1974a) Continuity of several functions on the function space *D*. *Ann. Probability,* **2.** To appear.

WHITT, W. (1974b) Representation and convergence of point processes on the line. *Ann. Probability,* **2.** To appear.

YU, O. S. (1973) Stochastic bounding relations for a heterogeneous-server queue with Erlang service times. Stanford Research Institute, Menlo Park, California.

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