J. Appl. Prob. **9**, 462–465 (1972)
Printed in Israel

LIMITS FOR THE SUPERPOSITION OF m-DIMENSIONAL POINT PROCESSES

WARD WHITT, *Yale University*

Abstract

To obtain a limit with independent components in the superposition of m-dimensional point processes, a condition corresponding to asymptotic independence must be included. When this condition is relaxed, convergence to limits with dependent components is possible. In either case, convergence of finite distributions alone implies tightness and thus weak convergence in the function space $D[0, \infty) \times \cdots \times D[0, \infty)$.

POINT PROCESSES; POISSON PROCESS; SUPERPOSITION OF POINT PROCESSES; SUPERPOSITION OF m-DIMENSIONAL POINT PROCESSES; SUM OF POINT PROCESSES; POISSON APPROXIMATIONS; RENEWAL PROCESSES; WEAK CONVERGENCE; TIGHTNESS; WEAK CONVERGENCE OF POINT PROCESSES

1. A plausible counterexample

When considering the superposition of m-dimensional point processes, Çinlar (1968) came up with the remarkable conclusion (top of p. 172) that the limiting process has independent components without assuming any independence among the components of the point processes being added. This result reappears in Sobel (1971). At first glance, the following would appear to be a counterexample.

Let $\{N_{1j}^n, j = 1, \cdots, n\}$, $n \geqq 1$, be an array of row-wise independent Poisson processes with the intensity of N_{1j}^n being λ/n, $\lambda > 0$. Let

(1) $$N_{2j}^n \equiv N_{2j}^n(t) = N_{1j}^n(t+1) - N_{1j}^n(1), \qquad t \geqq 0,$$

(2) $$N_j^n \equiv N_j^n(t) = N_{1j}^n(t) + N_{2j}^n(t), \qquad t \geqq 0,$$

(3) $$L_i^n \equiv L_i^n(t) = N_{i1}^n(t) + \cdots + N_{in}^n(t), \qquad t \geqq 0, \qquad (i = 1, 2),$$

(4) $$L^n \equiv L^n(t) = [L_1^n(t), L_2^n(t)], \qquad t \geqq 0.$$

We use (1) instead of making N_{2j}^n an exact copy of N_{1j}^n to insure unit jumps. It is well known that the sum of two i.i.d. Poisson processes is a Poisson process. Consequently, L_1^n and L_2^n are (highly) dependent Poisson processes each with intensity λ. In fact,

(5) $$L_2^n(t) = L_1^n(t+1) - L_1^n(1).$$

Received 26 July 1971. Research partially supported by National Science Foundation Grant GK-27866.

To put this in the setting of [8], assume the remaining processes are identically 0. It is easy to see that the component point processes N_{ij}^n being added satisfy the assumptions of [1].

2. Diagnosis

Although the component processes N_{ij}^n satisfy the conditions of [1], the processes N_i^n obtained by summing over the components fail to satisfy (7) of [1]. Let N_1 and N_2 be two Poisson processes connected by (1); let $N = N_1 + N_2$; and let $N(t_1, t_2) = N(t_2) - N(t_1)$. Then

$$\begin{aligned} P\{N(t) \geqq 1\} &= P\{N_1(0,t) + N_1(1, 1+t) \geqq 1\} \\ &= 1 - e^{-2\lambda t}, \qquad 0 \leqq t \leqq 1, \\ P\{N(t) \geqq 1\} &= P\{N_1(0,1) + 2N_1(1,t) + N_1(t, t+1) \geqq 1\} \\ &= P\{N_1(t+1) \geqq 1\} \\ &= 1 - e^{-\lambda(t+1)}, \qquad t \geqq 1. \end{aligned} \tag{6}$$

Consequently,

$$B(y) = \begin{cases} 1 - e^{-2\lambda y}, & 0 \leqq y < 1 \\ 1 - \frac{1}{2} e^{-\lambda(y+1)}, & y \geqq 1, \end{cases} \tag{7}$$

where B is the distribution function in (2) of [1]. Finally, use λ/n with $B_n(t)$ and note that $B_n(t) \to \frac{1}{2}$ for $t \geqq 1$ as $n \to \infty$, so that Condition (7) of [1] is violated. The reason for this is that $B(t)$ can be defined as

$$B(t) = \lim_{s \downarrow 0} P\{N(t+s) - N(s) \geqq 1, N(s) \geqq 1\} / P\{N(s) \geqq 1\}, \tag{8}$$

which we know to be $\frac{1}{2}$ by our construction.

The upshot of all this is that while [1] appears to be mathematically correct, some adjustment is needed in the interpretation. There is indeed a clearly identifiable condition relating to independence of the component processes. In fact, it is now clear from (8) here that, in the presence of the other assumptions, (7) of [1] is tantamount to asymptotic independence of the m component processes $N_{ij}^n(t), \cdots, N_{mj}^n(t)$ as $n \to \infty$. It is also apparent that Lemma (9) of [1] could just as well have been used as a starting point. In other words, (7) of [1], (9) of [1], and asymptotic independence are all equivalent.

3. Allowing for dependence in the limit

Our discussion above suggests that relaxing (7) of [1] might still lead to limit theorems for the superposition of point processes, but limit theorems for which the components of the limit process are dependent. We briefly indicate how this can be done.

The idea is to genuinely consider the multivariate problem, that is, consider an array of stationary point processes $\{N_j^n, j = 1, \cdots, n\}$, $n \geqq 1$, which take values in I^m (the product of m copies of the non-negative integers). Then use the multiple Poisson approximation for the multinomial distribution just as the Poisson approximation for the binomial distribution is used in the one-dimensional case, cf. [3], p. 162. For example, suppose all the marginal one-dimensional point processes are stationary and orderly, satisfying the conditions of [1]. We replace (7) of [1] with

$$\text{(9)} \qquad \text{for any } t > 0, \text{ and all } i,\ 1 \leqq i \leqq m,\ \sup_{1 \leqq j \leqq n} B_{ij}^n(t) \to 0 \text{ as } n \to \infty.$$

In addition, assume that

$$\text{(10)} \qquad \lim_{n \to \infty} nP\{N_j^n(t) = (i_1, \cdots, i_m)\} = \lambda(i_1, \cdots, i_m)t$$

and

$$\text{(11)} \qquad \begin{aligned} &\lim_{n \to \infty} P\{N_j^n(t) = (0, \cdots, 0)\} = 1 - t\Sigma\lambda(i_1, \cdots, i_m)/n + o(1/n), \\ &\qquad\qquad (i_1, \cdots, i_m) \in \{0,1\}^m,\ (i_1, \cdots, i_m) \neq (0, \cdots, 0) \end{aligned}$$

for each $t > 0$, where i_j is 0 or 1 so that $(i_1, \cdots, i_m) \in \{0,1\}^m$, $\lambda(i_1, \cdots, i_m) \geqq 0$. Then let $\{M_n(t), t \geqq 0\}$ be the counting process which records the number of times each $(i_1, \cdots, i_m)$ occurs in the n independent processes $N_1^n(t), \cdots, N_n^n(t)$. Then the arguments of [1] and [3] imply that $M_n \Rightarrow M$, where $\Rightarrow$ denotes weak convergence or convergence of the finite-dimensional distributions (see Section 4) and

$$\text{(12)} \qquad P\{M(t) = (k_1, \cdots, k_p)\} = \exp\left\{-t \sum_{j=1}^{p} \lambda_j\right\} \prod_{j=1}^{p} (\lambda_j t)^{k_j}/k_j!,$$

where j indexes $(i_1, \cdots, i_m)$ in $\{0,1\}^m$, cf. [3], p. 162. Since each N_{ij}^n is orderly, it is only necessary to consider $i_j = 0$ or 1 in (12), so that $p = 2^m$. Now $L_i^n(t) = N_{i1}^n(t) + \cdots + N_{in}^n(t)$ and $L^n(t) = N_1^n(t) + \cdots + N_n^n(t)$ can be obtained as functions of $M_n(t)$. Consequently, $L_i^n \Rightarrow L_i$ and $L^n \Rightarrow L$, where L_i is a Poisson process for each i, but L in general does not consist of m independent Poisson processes.

4. Weak convergence on function spaces

It is of interest to consider such superposition theorems in the context of weak convergence on function spaces, cf. Kennedy (1970) and Grigelionis (1971). Then many associated limit theorems are immediately implied. However, convergence of the finite-dimensional distributions of a sequence of point processes (to a limiting point process) automatically implies tightness and thus weak convergence, cf. Theorem 6.2.3 of Straf (1969), Section 3 of Jagers (1971), and Corollary 6.1 of Whitt (1971). Consequently, some of the assumptions in [4] and [6] can be relaxed.

5. Related literature

For a more complete picture of the literature, consult Çinlar (1971) and Grigelionis (1971).

References

[1] Çinlar, E. (1968) On the superposition of *m*-dimensional point processes. *J. Appl. Prob.* **5**, 169–176.

[2] ———. (1971) Superposition of point processes. *Proceedings of the Symposium on Stochastic Point Processes at Yorktown Heights.* To appear.

[3] Feller, W. (1957) *An Introduction to Probability Theory and Its Applications.* Vol. 1 (2nd Ed.) John Wiley and Sons, New York.

[4] Grigelionis, B. (1971) On weak convergence of the sums of multidimensional stochastic point processes. *Proceedings of the Symposium on Stochastic Point Processes at Yorktown Heights.* To appear.

[5] Jagers, P. (1971) On the weak convergence of superpositions of point processes. *Technical Report* No. 20, Department of Statistics, Stanford University.

[6] Kennedy, D. P. (1970) Weak convergence for the superposition and thinning of point processes. *Technical Report* No. 11, Department of Operations Research, Stanford University.

[7] Khintchine, A. Y. (1960) *Mathematical Methods in the Theory of Queueing.* Griffin, London.

[8] Sobel, M. J. (1971) On the aggregation of point processes. *CORE Discussion Paper* No. 7103, International Center for Management Sciences, Center for Operations Research and Econometrics, Université Catholique de Louvain, Belgium.

[9] Straf, M. (1969) *A General Skorokhod Space and its Application to the Weak Convergence of Stochastic Processes with Several Parameters.* Ph.D. Thesis, Department of Statistics, University of Chicago.

[10] ——— (1970) Weak convergence of stochastic processes with several parameters. *Proc. VI Berk. Symp. Math. Statist. Prob.* To appear.

[11] Whitt, W. (1971) Representation and convergence of point processes on the line· *Technical Report*, Department of Administrative Sciences, Yale University.